CLASSIFICATION OF KUGA FIBER VARIETIES

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ABSTRACT. We complete Satake's classification of Kuga fiber varieties by showing that if a representation ρ of a hermitian algebraic group satisfies Satake's necessary conditions, then some multiple of ρ defines a Kuga fiber variety.

1. Introduction

Kuga fiber varieties [20,23] are families of abelian varieties $\mathcal{A} \to \mathcal{V}$, where $\mathcal{V} = \Gamma \backslash G(\mathbb{R})^0/K$ is an arithmetic variety, and $\mathcal{A}$ is the pullback of the universal family of abelian varieties over a Siegel modular variety. Here, G is a semisimple algebraic group over $\mathbb{Q}$ such that $G(\mathbb{R})$ is of hermitian type, K is a maximal compact subgroup of $G(\mathbb{R})^0$, and Γ is an arithmetic subgroup of $G(\mathbb{Q})$. A Kuga fiber variety is constructed from a symplectic representation $\rho\colon G \to Sp(2n, \mathbb{Q})$ which is equivariant with a holomorphic map $\tau\colon X \to \mathfrak{S}_n$, where $X = G(\mathbb{R})^0/K$ is the symmetric domain belonging to G, and $\mathfrak{S}_n$ is the Siegel space of degree n. Kuga assumed that $\mathcal{V}$ is compact; we do not make this assumption.

Kuga's original motivation was to prove the Ramanujan conjecture, a goal achieved by Deligne [9, 11]. Kuga fiber varieties, which include Shimura's PEL-families [41], have played a central role in the arithmetic theory of automorphic forms [26,30,31]. These varieties are also key to the study of algebraic cycles on abelian varieties and abelian schemes [4,5,14,15,18,21,29,42]; indeed, the concept of the Hodge group (or Mumford-Tate group) of an abelian variety arose in the context of Kuga fiber varieties [28]. Another area in which Kuga fiber varieties play a key role is in the study of $K3$-surfaces, via the Kuga-Satake construction of abelian varieties associated to $K3$-surfaces [25], as in Deligne's proof of the Weil conjectures for these surfaces [10].

We consider the following problem in this paper: Given an arithmetic variety $\mathcal{V}$, classify all Kuga fiber varieties over it. Equivalently, given the group G, find all representations of it into a symplectic group which are equivariant with holomorphic maps of the corresponding symmetric domains. From another point of view, this problem is equivalent to the classification (up to isogeny) of the semisimple parts of the Hodge groups of abelian varieties, together with their action on the first cohomology of the abelian variety. This problem was raised by Kuga [20] in the 1960's, and partially answered by

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Satake [33–40]. Addington [6] completed Satake's classification for $\mathbb{Q}$-simple groups of type II (orthogonal groups) and type III (symplectic groups). This was partially extended to non-simple groups of type III by Abdulali [1, 2].

Deligne [12] and Milne [27] considered this problem from a somewhat different point of view. Their results are similar to those of Satake. Milne states the theorem for all suitable representations of a $\mathbb{Q}$-simple group, though he proves it only in the situations where Satake proved it, and he does not deal fully with non-simple groups. It is important to deal with non-simple groups because the semisimple part of the Hodge group of a simple abelian variety need not be simple. We give an example of such an abelian variety in Section 6. Further examples may be found in Satake [36, Remark 2, p. 356], Kuga [22, §5], and Abdulali [2, §4; 3, §2.4].

Green, Griffiths, and Kerr [16, 17] and Patrikis [32] have considered the more general problem of classifying the Hodge groups of Hodge structures of higher weight. They completely classify the reductive groups which are Mumford-Tate groups of polarizable Hodge structures of arbitrary weight; however the representations of the groups on the Hodge structures have not been classified.

The key to our classification is a reduction to the *rigid* case, which is much easier. In the proof of our main theorem (Theorem 4) we reduce the general case to the rigid case, which is proved in Theorem 3. The inspiration for this strategy comes from the construction of families of families of abelian varieties by Kuga and Ihara [24], and the related concept of "sharing" in Kuga [22, §5, p. 277].

Notations and conventions. All representations are finite-dimensional and algebraic. For a finite field extension E of a field F, we let $\operatorname{Res}_{E/F}$ be the restriction of scalars functor, from schemes over E to schemes over F. For an algebraic or topological group G, we denote by G^0 the connected component of the identity.

2. Kuga fiber varieties

In this section we give an overview of the construction of Kuga fiber varieties. Our primary purpose is to fix the notations and terminology; for details we refer to Satake [39].

2.1. **Groups of hermitian type.** Let G be a group of hermitian type. This means that G is a semisimple real Lie group, and $X := G^0/K$ is a bounded symmetric domain for a maximal compact subgroup K of G^0. Denote by 1 the identity element of G. Let $\mathfrak{g} := \operatorname{Lie} G$ be the Lie algebra of G, $\mathfrak{k} := \operatorname{Lie} K$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Differentiating the natural map $\nu \colon G^0 \to X$ induces an isomorphism of $\mathfrak{p}$ with $T_o(X)$, the tangent space of X at the base point $o = \nu(1)$, and there exists a unique $H_0 \in Z(\mathfrak{k})$, called the *H-element* at o, such that $\operatorname{ad} H_0|\mathfrak{p}$ is the complex structure on $T_o(X)$.

2.2. **Equivariant holomorphic maps.** Let G_1 and G_2 be groups of hermitian type, with symmetric spaces X_1 and X_2 respectively. Let H_0 and H_0' be H-elements at base points $o_1 \in X_1$ and $o_2 \in X_2$ respectively. Let $\rho\colon G_1 \to G_2$ be a homomorphism of Lie groups. We say that ρ satisfies the H_1-*condition* relative to the H-elements H_0 and H_0' if

$$[d\rho(H_0) - H_0', d\rho(g)] = 0 \qquad \text{for all } g \in \mathfrak{g}. \tag{2.2.1}$$

The stronger condition

$$d\rho(H_0) = H_0' \tag{2.2.2}$$

is called the H_2-*condition*. If either of these is satisfied, then there exists a unique holomorphic map $\tau\colon X_1 \to X_2$ such that $\tau(o_1) = o_2$, and the pair (ρ, τ) is equivariant in the sense that

$$\tau(g \cdot x) = \rho(g) \cdot \tau(x) \qquad \text{for all } g \in G^0,\ x \in X.$$

In fact, Clozel [8] has shown that if G_2 has no exceptional factors, then the H_1-condition is equivalent to the existence of an equivariant holomorphic map.

2.3. **The Siegel space.** Let E be a nondegenerate alternating form on a finite-dimensional real vector space V. The symplectic group $Sp(V, E)$ is a Lie group of hermitian type; the associated symmetric domain is the *Siegel space*

$$\mathfrak{S}(V, E) = \{\, J \in GL(V) \mid J^2 = -I \text{ and } E(x, Jy) \text{ is symmetric, positive definite} \,\}.$$

$Sp(V, E)$ acts on $\mathfrak{S}(V, E)$ by conjugation. The H-element at $J \in \mathfrak{S}(V, E)$ is $J/2$.

Lemma 1. *Let G be a group of hermitian type with symmetric domain X, and let E be a nondegenerate alternating form on a finite-dimensional real vector space V. Let $\rho\colon G \to Sp(V, E)$ satisfy the H_2-condition with respect to H-elements H_0 and $H_0' = J_0/2$ at base points $o \in X$ and $J_0 \in \mathfrak{S}(V, E)$, respectively. Then $J_0 \in \rho(G)$.*

Proof. Since J_0 is a complex structure on V, there exists a basis of V with respect to which $J_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, where $2n = \dim V$. Then, using the H_2-condition, we calculate that $J_0 = \exp(\frac{\pi}{2} J_0) = \exp(\pi H_0') = \exp(d\rho(\pi H_0)) = \rho(\exp(\pi H_0))$. □

2.4. **The fiber varieties.** We shall say that an algebraic group G over a subfield of $\mathbb{R}$ is of hermitian type if the Lie group $G(\mathbb{R})$ is of hermitian type. Now let G be a connected, semisimple algebraic group of hermitian type over $\mathbb{Q}$. Assume that G has no nontrivial, connected, normal subgroup H such that $H(\mathbb{R})$ is compact.

Let E be a nondegenerate alternating form on a finite-dimensional rational vector space V. The symplectic group $Sp(V,E)$ is then a $\mathbb{Q}$-algebraic group of hermitian type. We write $\mathfrak{S}(V,E)$ for $\mathfrak{S}(V_{\mathbb{R}},E_{\mathbb{R}})$. Let $\rho\colon G \to Sp(V,E)$ be a representation defined over $\mathbb{Q}$, which satisfies the H_1 condition with respect to the H-elements H_0 and $H_0' = J/2$. Let $\tau\colon X \to \mathfrak{S}(V,E)$ be the corresponding equivariant holomorphic map. Let Γ be a torsion-free arithmetic subgroup of $G(\mathbb{Q})$, and L a lattice in V such that $\rho(\Gamma)L = L$. Then the natural map

$$\mathcal{A} = (\Gamma \ltimes_\rho L)\backslash(X \times V_{\mathbb{R}}) \longrightarrow \mathcal{V} := \Gamma\backslash X$$

is a morphism of smooth quasiprojective algebraic varieties (Borel [7, Theorem 3.10, p. 559] and Deligne [13, p. 74]), so that $\mathcal{A}$ is a fiber variety over $\mathcal{V}$ called a *Kuga fiber variety.* The fiber $\mathcal{A}_P$ over any point $P \in \mathcal{V}$ is an abelian variety isomorphic to the torus $V_{\mathbb{R}}/L$ with the complex structure $\tau(x)$, where x is a point in X lying over P.

We say that a representation $\rho\colon G \to GL(V)$ defines a Kuga fiber variety if $\rho(G)$ is contained in a symplectic group $Sp(V,E)$, and ρ satisfies the H_1-condition with respect to some H-elements.

3. Satake's Classification

3.1. **Necessary Conditions.** In a series of papers [33–40] Satake classified the H_1-representations of a given hermitian group into a symplectic group. We summarize his results below. Let G be a connected, semisimple, linear algebraic group over $\mathbb{Q}$. Assume that $G(\mathbb{R})^0$ is of hermitian type, and has no nontrivial, connected, normal $\mathbb{Q}$-subgroup H with $H(\mathbb{R})$ compact. After replacing G by a finite covering, if necessary, we may write

$$\mathfrak{g}_{\mathbb{R}} = \bigoplus_{j=0}^{s} \mathfrak{g}_j, \qquad G(\mathbb{R}) = G_0 \times G_1 \times \cdots \times G_s,$$

where G_0 is compact, each G_j is a noncompact absolutely simple Lie group for $j > 0$, and, each $\mathfrak{g}_j = \mathrm{Lie}(G_j)$. Suppose $\rho\colon G \to Sp(V,E)$ is a symplectic representation satisfying the H_1-condition. Then,

(1) For $j = 1, \ldots s$, we have that G_j is one of the following:
   (a) Type I: $SU(p,q)$ with $p \geq q \geq 1$;
   (b) Type II: $SU^-(n,\mathbb{H})$ with $n \geq 5$ (this is the group that Helgason [19, p. 445] calls $SO^\star(2n)$);
   (c) Type III: $Sp(2n,\mathbb{R})$ with $n \geq 1$;
   (d) Type IV: $Spin(p,2)$ with $p \geq 1, p \neq 2$.

(2) Let ρ' be a nontrivial $\mathbb{C}$-irreducible subrepresentation of $\rho_{\mathbb{C}}$. Then, for some index j $(1 \leq j \leq s)$, we have that ρ' is equivalent to $\rho_0 \otimes \rho_j$, where ρ_0 is a representation of $G_{0,\mathbb{C}}$, and, ρ_j is a representation of $G_{j,\mathbb{C}}$. We call this the *stability* condition.

(3) Fix an index j with $1 \leq j \leq s$. Let ρ_j be an irreducible subrepresentation of $V_{\mathbb{R}}$ considered as a G_j-module. Then ρ_j is either trivial or given by one of the following:
   (a) If $G_j = SU(p,q)$ with $p \geq q \geq 2$, then $\rho_{j,\mathbb{C}}$ is the direct sum of the standard representation of $G_{j,\mathbb{C}} = SL_{p+q}(\mathbb{C})$ and its contragredient; it satisfies the H_2-condition if and only if $p = q$.
   (b) If $G_j = SU(p,1)$, then ρ_j is one of the following:
      (i) $\bigwedge^k \oplus \bigwedge^{p+1-k}$, for some k with $1 \leq k < \frac{p+1}{2}$;
      (ii) $\bigwedge^k$ with $k = \frac{p+1}{2}$, and $p \equiv 1 \pmod 4$;
      (iii) the direct sum of two copies of $\bigwedge^k$ with $k = \frac{p+1}{2}$, and $p \equiv 3 \pmod 4$.

      The H_2-condition is satisfied if and only if $k = \frac{p+1}{2}$.
   (c) If $G_j = SU^-(n,\mathbb{H})$ with $n \geq 5$, then $\rho_{j,\mathbb{C}}$ is the direct sum of two copies of the standard representation. The H_2-condition is satisfied in this case.
   (d) If $G_j = Sp(2n,\mathbb{R})$, then ρ_j is the standard representation, and satisfies the H_2-condition.
   (e) If $G_j = Spin(p,2)$ with $p \geq 1$ and p odd, then
      (i) ρ_j is the spin representation if $p \equiv 1, 3 \pmod 8$;
      (ii) $\rho_{j,\mathbb{C}}$ is the direct sum of two copies of the spin representation if $p \equiv 5, 7 \pmod 8$.

      In both cases, ρ_j satisfies the H_2-condition.
   (f) If $G_j = Spin(p,2)$ with $p \geq 4$, and p even, then ρ_j is
      (i) one of the two spin representations if $p \equiv 2 \pmod 8$;
      (ii) the direct sum of two copies of a spin representation if $p \equiv 6 \pmod 8$;
      (iii) the direct sum of the two spin representations if $p \equiv 0 \pmod 4$.

      In each case, ρ_j satisfies the H_2-condition.

We note that the above conditions imply that ρ is self-dual.

3.2. **A sufficient condition.** Satake showed that the necessary conditions listed in §3.1 are sufficient if we make an additional assumption.

Theorem 2 (Satake [38]). *Let G be a $\mathbb{Q}$-simple hermitian group, and write $G(\mathbb{R}) = \prod_{\alpha \in S} G_\alpha$ where each G_α is an absolutely simple real algebraic group. Let ρ be a representation of G satisfying the conditions of* §3.1*. Assume further that each irreducible subrepresentation of $\rho_{\mathbb{C}}$ is nontrivial on G_α for exactly one α. Then some multiple of ρ defines a Kuga fiber variety.*

3.3. **More on type I.** We now take a closer look at the H_1-representation $\rho\colon SU(p,q) \to Sp(V,E)$ given by item (3a) of the list in §3.1. We recall the matrix representation of this given by Satake. Let $J_0 \in \mathfrak{S}(V,E)$ be the base point. The eigenvalues of J_0 on $V_{\mathbb{C}}$ are i and $-i$, and we take a basis of $V_{\mathbb{C}}$ with respect to which the matrix of J_0 is $\begin{pmatrix} iI_m & 0 \\ 0 & -iI_m \end{pmatrix}$. The Lie algebra of

$Sp(V,E)$ with respect to this basis is given by

$$\mathfrak{sp}(V,E)=\left\{\begin{pmatrix}A & B\\ C & -{}^tA\end{pmatrix}\;\middle|\; B,C \text{ symmetric}\right\},$$

and the H-element is $H_0'=\begin{pmatrix}\frac{i}{2}I_m & 0\\ 0 & -\frac{i}{2}I_m\end{pmatrix}$.

With respect to a suitable basis the Lie algebra of $SU(p,q)$ is given by

$$\mathfrak{su}(p,q)=\left\{\begin{pmatrix}X_1 & X_{12}\\ {}^t\overline{X}_{12} & X_2\end{pmatrix}\in\mathfrak{sl}_{p+q}(\mathbb{C})\;\middle|\;\begin{matrix}X_1\in M_p(\mathbb{C}),X_2\in M_q(\mathbb{C})\\ {}^t\overline{X}_j=-X_j(j=1,2)\end{matrix}\right\},$$

and an H-element is given by

$$H_0=\begin{pmatrix}\frac{qi}{p+q}I_p & 0\\ 0 & -\frac{pi}{p+q}I_q\end{pmatrix}.$$

Then, $d\rho\colon\mathfrak{su}(p,q)\to\mathfrak{sp}_{2p+2q}$ is given by

$$\begin{pmatrix}X_1 & X_{12}\\ {}^t\overline{X}_{12} & X_2\end{pmatrix}\mapsto\begin{pmatrix}\overline{X}_2 & 0 & 0 & {}^tX_{12}\\ 0 & X_1 & X_{12} & 0\\ 0 & {}^t\overline{X}_{12} & X_2 & 0\\ \overline{X}_{12} & 0 & 0 & \overline{X}_1\end{pmatrix}.$$

We extend $d\rho$ to $\mathfrak{u}(p,q)$, and denote by

$$\bar\rho\colon U(p,q)\to Sp(2p+2q,\mathbb{R})$$

the corresponding map of Lie groups which extends ρ. Let

$$\bar H_0^{p,q}=\begin{pmatrix}\frac{i}{2}I_p & 0\\ 0 & -\frac{i}{2}I_q\end{pmatrix}.$$

Then $d\bar\rho(\bar H_0)=H_0'$. It follows, as in the proof of Lemma 1, that $J_0\in\bar\rho(U(p,q))$.

Consider, next, the H_1-representation $\rho\colon SU(p,1)\to Sp(2m,\mathbb{R})$ given in item (3(b)i) of the list in §3.1. We extend it to a representation $\bar\rho\colon U(p,1)\to Sp(2m,\mathbb{R})$. Let

$$\tilde H_0=\begin{pmatrix}\frac{i}{2k}I_p & 0\\ 0 & \frac{1-2k}{2k}i\end{pmatrix}.$$

Then

$$\bigwedge^k(\tilde H_0)=\begin{pmatrix}\frac{i}{2}I_p' & 0\\ 0 & -\frac{i}{2}I_q'\end{pmatrix}=\bar H_0^{p',q'},$$

where $p'=\binom{p}{k}$ and $q'=\binom{p}{k-1}$. From this we see that $d\bar\rho(\tilde H_0)=H_0'$, the H-element of $Sp(2m,\mathbb{R})$. It follows, as in the proof of Lemma 1, that $J_0\in\bar\rho(U(p,1))$.

4. The rigid case

4.1. **Statement of the theorem.** Let G be an algebraic group over $\mathbb{Q}$ of hermitian type, and ρ a representation of G satisfying Satake's conditions in §3.1. By the stability condition (2) of §3.1, every irreducible subrepresentation of $\rho_{\mathbb{C}}$ is nontrivial on at most one noncompact factor of $G(\mathbb{R})$. We say that ρ is *rigid*, if every irreducible subrepresentation of $\rho_{\mathbb{C}}$ is nontrivial on exactly one noncompact factor of $G(\mathbb{R})$. We begin by classifying the rigid representations which define Kuga fiber varieties.

Theorem 3. *Let G be a semisimple connected algebraic group over $\mathbb{Q}$ such that $G(\mathbb{R})^0$ is of hermitian type and has no compact factors defined over $\mathbb{Q}$. Let ρ be a representation of G satisfying Satake's conditions in* §3.1*. If ρ is rigid, then some multiple of ρ defines a Kuga fiber variety.*

The rest of this section is devoted to the proof of this theorem.

4.2. **Beginning of the proof.** Without loss of generality we assume that G is simply connected, and ρ is nontrivial and a multiple of a $\mathbb{Q}$-irreducible representation (see Satake [39, p. 189]).

Write $G = \prod_{j=1}^{t} G_j$, where each G_j is a simple group of hermitian type. Then there are totally real number fields F_j, and absolutely simple groups $\widetilde{G}_j$ over F_j, such that $G_j = \mathrm{Res}_{F_j/\mathbb{Q}} \widetilde{G}_j$ for $1 \le j \le t$. Let F be the smallest Galois extension of $\mathbb{Q}$ containing all the F_j, and $\mathcal{G} = \mathrm{Gal}(F/\mathbb{Q})$. Let S_j be the set of embeddings of F_j into $\mathbb{R}$, and let S be the disjoint union of the S_j's. For $\alpha \in S$, we let $j(\alpha)$ be the unique index such that $\alpha \in S_{j(\alpha)}$. We note that F is a totally real field, $\mathcal{G}$ acts on S, and the orbits of this action are the sets S_j. For $\alpha \in S$, we let $G_\alpha = \widetilde{G}_{j(\alpha)} \otimes_{F_{j(\alpha)},\alpha} F$. Then $G_F = \prod_{\alpha \in S} G_\alpha$.

Let $S_0 = \{\alpha \in S \mid G_{\alpha,\mathbb{R}} \text{ is not compact}\}$. An H-element of $G_{\mathbb{R}}$ is given by $H_0 = \sum_{\alpha \in S_0} H_{0,\alpha}$, where $H_{0,\alpha}$ is an H-element of $G_{\alpha,\mathbb{R}}$.

Let $\rho_0 \colon G_{\mathbb{C}} \to GL(V_0)$ be a $\mathbb{C}$-irreducible subrepresentation of $\rho_{\mathbb{C}}$. Let

$$M = \{\alpha \in S \mid \rho_0 \text{ is nontrivial on } G_{\alpha,\mathbb{C}}\},$$

and let α_0 be the unique element of M such that $G_{\alpha_0,\mathbb{R}}$ is not compact. Write $\rho_0 = \otimes_{\alpha \in M} \rho'_\alpha$, where ρ'_α is an irreducible representation of $G_{\alpha,\mathbb{C}}$. Either ρ'_α, or the sum of two copies of ρ'_α, or, the direct sum of ρ'_α and its complex conjugate is defined over $\mathbb{R}$. Let ρ_α be the real representation such that ρ'_α equals $\rho_{\alpha,\mathbb{C}}$, the direct sum of two copies of $\rho_{\alpha,\mathbb{C}}$, or the direct sum of $\rho_{\alpha,\mathbb{C}}$ and its complex conjugate, respectively.

For each $\alpha \in M$, let $\hat{\rho}_\alpha = \sum_{\sigma \in \mathcal{G}} \rho_\alpha^\sigma$. Then $\hat{\rho}_\alpha$ is a representation of $G_{j(\alpha)}$ satisfying the hypotheses of Theorem 2, so some multiple of it defines a Kuga fiber variety. By Satake's construction (see [39, §IV.6, Theorems 6.1, 6.2, 6.3]), this representation is defined over $F_{j(\alpha)}$ in the sense that it is the restriction from $F_{j(\alpha)}$ to $\mathbb{Q}$ of a symplectic representation

$$\tilde{\rho}_\alpha \colon \widetilde{G}_{j(\alpha)} \to Sp(\widetilde{V}_\alpha, \widetilde{E}_\alpha).$$

Here, $\widetilde{E}_\alpha$ is an $F_{j(\alpha)}$-bilinear alternating form on $\widetilde{V}_\alpha$, and $E_\alpha = \mathrm{Tr}_{F_{j(\alpha)}/\mathbb{Q}} \widetilde{E}_\alpha$ is a $G_{j(\alpha)}$-invariant $\mathbb{Q}$-bilinear alternating form on $V_\alpha = \mathrm{Res}_{F_{j(\alpha)}/\mathbb{Q}} \widetilde{V}_\alpha$. Let $V_\alpha = \mathrm{Res}_{F_{j(\alpha)}/\mathbb{Q}} \widetilde{V}_\alpha$. Then $V_\alpha \otimes F_{j(\alpha)} = \oplus_{\sigma\in\mathcal{G}} \widetilde{V}_\alpha^\sigma$ (up to multiplicity), and $\widetilde{V}_\alpha^\sigma$ is the representation space of ρ_α^σ. For each $\alpha \in S_{j(\alpha)}$, there is a complex structure J_α on $\widetilde{V}_{\alpha,\mathbb{R}}$ such that $\widetilde{E}_\alpha(x, J_\alpha y)$ is a symmetric, positive definite form. Moreover, $\rho_{\alpha_0} \colon G_{\alpha_0,\mathbb{R}} \to Sp(\widetilde{V}_{\alpha_0,\mathbb{R}}, \widetilde{E}_{\alpha_0})$ satisfies the H_1-condition with respect to the H-elements H_{0,α_0} and $\frac{1}{2} J_{\alpha_0}$.

4.3. **Construction of a symmetric form.** We claim that for each $\alpha \in M$ there exists a G_α-invariant, $F_{j(\alpha)}$-bilinear positive definite symmetric form γ_α on $\widetilde{V}_\alpha$. The space of symmetric positive definite $G_{\alpha,\mathbb{R}}$-invariant forms on $\widetilde{V}_{\alpha,\mathbb{R}}$ is an open subset of the space of symmetric forms; it is nonempty because $\widetilde{E}_\alpha(x, J_\alpha y)$ is such a form. Therefore it contains an $F_{j(\alpha)}$-rational point γ_α. This proves the claim.

We next claim that when $\alpha = \alpha_0$ we have

$$\gamma_{\alpha_0}(J_{\alpha_0} x, J_{\alpha_0} y) = \gamma_{\alpha_0}(x, y). \tag{4.3.1}$$

If ρ_{α_0} satisfies the H_2-condition, then Lemma 1 shows that J_{α_0} belongs to the image of $G_{\alpha_0}(\mathbb{R})$ under ρ_{α_0}, so (4.3.1) is a consequence of γ_{α_0} being G_{α_0}-invariant. Finally, we consider the situation when the H_2-condition is not satisfied. Then we are in either case (3a) or case (3(b)i) of §3.1. In both cases, G_α is a special unitary group $SU(W, h)$. We can extend $\hat{\rho}_\alpha$ to a representation of the full unitary group $U(W, h)$. We have seen in §3.3 that J_{α_0} belongs to the image of $U(W, h)$, so now we can argue as before to prove (4.3.1) in this situation.

Observe that (4.3.1) is equivalent to

$$\gamma_{\alpha_0}(x, J_{\alpha_0} y) = -\gamma_{\alpha_0}(y, J_{\alpha_0} x), \tag{4.3.2}$$

since J_{α_0} is a complex structure.

4.4. **Construction of an alternating form.** Let $\widetilde{V} = \otimes_{\alpha\in M}(\widetilde{V}_\alpha \otimes_{F_{j(\alpha)}} F)$. Define an F-bilinear alternating form $\widetilde{E}$ on $\widetilde{V}$ by

$$\widetilde{E}(\otimes_\alpha x_\alpha, \otimes_\alpha y_\alpha) = \sum_{\alpha\in M} \left(\widetilde{E}_\alpha(x_\alpha, y_\alpha) \prod_{\substack{\beta\in M \\ \beta\neq\alpha}} \gamma_\beta(x_\beta, y_\beta) \right).$$

Then $\widetilde{E}$ is $\widetilde{G}$-invariant, where $\widetilde{G} = \prod_{j=1}^t \widetilde{G}_j$. Next, we define a $\mathbb{Q}$-bilinear alternating form E on $V = \mathrm{Res}_{F/\mathbb{Q}} \widetilde{V}$ by

$$E(x, y) = \mathrm{Tr}_{F/\mathbb{Q}} \widetilde{E}(x, y).$$

Then E is G-invariant.

4.5. **Construction of a complex structure.** We next construct a complex structure $\widetilde{J}$ on $\widetilde{V}_{\mathbb{R}} = \otimes_{\alpha\in M}(\widetilde{V}_\alpha \otimes_{F_{j(\alpha)}} \mathbb{R})$. We have seen that we have a complex structure J_{α_0} on $\widetilde{V}_{\alpha_0,\mathbb{R}}$ such that $\widetilde{E}_{\alpha_0}(x, J_{\alpha_0}y)$ is symmetric and positive definite. Define $\widetilde{J}$ on $\widetilde{V}_{\mathbb{R}}$ by $\widetilde{J}(\otimes x_\alpha) = \otimes I_\alpha x_\alpha$, where $I_{\alpha_0} = J_{\alpha_0}$, and I_α is the identity for $\alpha \neq \alpha_0$. Then $\widetilde{J}$ is a complex structure.

For $x = \otimes x_\alpha, y = \otimes y_\alpha \in \widetilde{V}_{\mathbb{R}}$, we have

$$\begin{aligned}
\widetilde{E}(x, \widetilde{J}y) &= \sum_{\alpha\in M}\left(\widetilde{E}_\alpha(x_\alpha, I_\alpha y_\alpha)\prod_{\substack{\beta\in M\\ \beta\neq\alpha}}\gamma_\beta(x_\beta, I_\beta y_\beta)\right)\\
&= \widetilde{E}_{\alpha_0}(x_{\alpha_0}, J_{\alpha_0}y_{\alpha_0})\prod_{\substack{\beta\in M\\ \beta\neq\alpha_0}}\gamma_\beta(x_\beta, y_\beta)\\
&\qquad + \sum_{\substack{\alpha\in M\\ \alpha\neq\alpha_0}}\left(\widetilde{E}_\alpha(x_\alpha, y_\alpha)\gamma_{\alpha_0}(x_{\alpha_0}, J_{\alpha_0}y_{\alpha_0})\prod_{\substack{\beta\in M\\ \beta\neq\alpha\\ \beta\neq\alpha_0}}\gamma_\beta(x_\beta, y_\beta)\right)\\
&= \widetilde{E}_{\alpha_0}(y_{\alpha_0}, J_{\alpha_0}x_{\alpha_0})\prod_{\substack{\beta\in M\\ \beta\neq\alpha_0}}\gamma_\beta(y_\beta, x_\beta)\\
&\qquad + \sum_{\substack{\alpha\in M\\ \alpha\neq\alpha_0}}\left(\widetilde{E}_\alpha(y_\alpha, x_\alpha)\gamma_{\alpha_0}(y_{\alpha_0}, J_{\alpha_0}x_{\alpha_0})\prod_{\substack{\beta\in M\\ \beta\neq\alpha\\ \beta\neq\alpha_0}}\gamma_\beta(y_\beta, x_\beta)\right)\\
&= \sum_{\alpha\in M}\left(\widetilde{E}_\alpha(y_\alpha, I_\alpha x_\alpha)\prod_{\substack{\beta\in M\\ \beta\neq\alpha}}\gamma_\beta(y_\beta, I_\beta x_\beta)\right)\\
&= \widetilde{E}(y, \widetilde{J}x),
\end{aligned}$$

because $\widetilde{E}_{\alpha_0}(x, J_{\alpha_0}y)$ and γ_α are symmetric, I_α is the identity for $\alpha \neq \alpha_0$, $\widetilde{E}_{\alpha_0}(x_{\alpha_0}, J_{\alpha_0}y_{\alpha_0})$ and $\gamma_{\alpha_0}(x_{\alpha_0}, J_{\alpha_0}y_{\alpha_0})$ are alternating, and using (4.3.2). Thus $\widetilde{E}(x, \widetilde{J}y)$ is a symmetric form on $\widetilde{V}_{\mathbb{R}}$. It follows that $E(x, \widetilde{J}y)$ is symmetric.

If $x = y$ we have

$$\begin{aligned}
\widetilde{E}(x, \widetilde{J}x) &= \sum_{\alpha\in M}\left(\widetilde{E}_\alpha(x_\alpha, I_\alpha x_\alpha)\prod_{\substack{\beta\in M\\ \beta\neq\alpha}}\gamma_\beta(x_\beta, I_\beta x_\beta)\right)\\
&= \widetilde{E}_{\alpha_0}(x_{\alpha_0}, J_{\alpha_0}x_{\alpha_0})\prod_{\substack{\beta\in M\\ \beta\neq\alpha_0}}\gamma_\beta(x_\beta, x_\beta),
\end{aligned}$$

so $\widetilde{E}(x, \widetilde{J}y)$ is positive definite.

Our next task is to define a complex structure J on $V_{\mathbb{R}}$, where $V = \operatorname{Res}_{F/\mathbb{Q}} \widetilde{V}$. Now, $V_{\mathbb{R}} = \bigoplus_{\sigma\in\mathcal{G}} \widetilde{V}^{\sigma}_{\mathbb{R}}$, so it is sufficient to define a complex structure $\widetilde{J}^{\sigma}$ on $\widetilde{V}^{\sigma}_{\mathbb{R}}$ for each $\sigma \in \mathcal{G}$. When σ is the identity, we have already defined $\widetilde{J}$ on $\widetilde{V}_{\mathbb{R}}$. In the same manner we can define $\widetilde{J}^{\sigma}$ for each $\sigma \in \mathcal{G}$, such that $\widetilde{E}^{\sigma}(x, \widetilde{J}^{\sigma} y)$ is symmetric and positive definite on $\widetilde{V}^{\sigma}_{\mathbb{R}}$. Then $J = \sum_{\sigma\in\mathcal{G}} \widetilde{J}^{\sigma}$ is a complex structure on $V_{\mathbb{R}}$.

4.6. **Conclusion of the proof.** Since each $\widetilde{E}^{\sigma}(x, \widetilde{J}^{\sigma} y)$ is symmetric, so is $E(x, Jy)$. It remains to show that $E(x, Jy)$ is positive definite. For each $\sigma \in \mathcal{G}$, let $\alpha(\sigma)$ be the unique element of $M^{\sigma} \cap S_0$. Then, we have

$$\begin{aligned} E(x, Jx) &= \sum_{\sigma\in\mathcal{G}} \widetilde{E}^{\sigma}(x, \widetilde{J}^{\sigma} x) \\ &= \sum_{\sigma\in\mathcal{G}} \left(\widetilde{E}_{\alpha(\sigma)}(x_{\alpha(\sigma)}, \widetilde{J}^{\alpha(\sigma)} x_{\alpha(\sigma)}) \prod_{\substack{\beta\in M^{\sigma} \\ \beta\neq\alpha(\sigma)}} \gamma_{\beta}(x_{\beta}, x_{\beta}) \right) \\ &= \sum_{\sigma\in\mathcal{G}} Q^{\sigma}(x), \end{aligned}$$

where

$$Q^{\sigma}(x) = \widetilde{E}_{\alpha(\sigma)}(x_{\alpha(\sigma)}, \widetilde{J}^{\alpha(\sigma)} x_{\alpha(\sigma)}) \prod_{\substack{\beta\in M^{\sigma} \\ \beta\neq\alpha(\sigma)}} \gamma_{\beta}(x_{\beta}, x_{\beta})$$

is a symmetric form on $V_{\mathbb{R}}$. For σ equal to the identity, we know that $Q(x) = \widetilde{E}(x, Jx)$ is positive definite. Therefore there exists a positive integer N such that

$$Q'(x) = NQ(x) + \sum_{\substack{\sigma\in\mathcal{G} \\ \sigma\neq\mathrm{id}}} Q^{\sigma}(x)$$

is positive definite (see Addington [6, Lemma 4.9, p. 80]). For each j $(1 \leq j \leq t)$, let $c_j \in F_j$ be such that $\alpha(c_j) > N$ if $\alpha \in S_0$, and $0 < \alpha(c_j) < 1$ if $\alpha \notin S_0$. Replace each $\widetilde{E}_{\alpha}$ by $c_{j(\alpha)} \widetilde{E}_{\alpha}$. Then $E(x, Jy)$ is positive definite.

An H-element for $Sp(V, E)$ is given by $\frac{1}{2}J$. Since $H_0 = \sum_{\alpha\in S_0} H_{0,\alpha}$ is an H-element of G, and ρ_{α} satisfies the H_1-condition with respect to $H_{0,\alpha}$ and $\frac{1}{2}J_{\alpha}$ whenever $\alpha \in S_0$, it follows from our construction that ρ satisfies the H_1-condition, and therefore defines a Kuga fiber variety.

5. The general case

We next derive the general case from the rigid case.

Theorem 4. *Let ρ be a representation of G satisfying Satake's conditions in §3.1. Then some multiple of ρ defines a Kuga fiber variety.*

Proof. We keep the notations used in the proof of Theorem 3. Without loss of generality we assume that ρ is a primary representation, i.e., a multiple of an irreducible representation. An irreducible subrepresentation ρ_0 of $\rho_{\mathbb{C}}$ is said to be *rigid* if it is nontrivial on some noncompact factor of $G(\mathbb{R})$. We define the *index of rigidity* of ρ to be the cardinality of the set $\{\sigma \in \mathcal{G} \mid \mu^\sigma \text{ is rigid}\}$, where μ is an irreducible subrepresentation of $\rho_{\mathbb{C}}$. It depends only on ρ, and not on the choice of μ.

Suppose ρ is not rigid. Then there exists a subrepresentation μ of $\rho_{\mathbb{C}}$ such that μ is trivial on all noncompact factors of $G_{\mathbb{R}}$. Let $\mathcal{G}_1 = \{\sigma \in \mathcal{G} \mid \mu^\sigma = \mu\}$. Then $\rho_{\mathbb{C}}$ is equivalent to a multiple of $\sum_{\sigma\in\mathcal{G}/\mathcal{G}_1} \mu^\sigma$.

Let $\alpha_0 \in S_j$ be such that μ is nontrivial on G_{α_0}. Extend α_0 to an embedding of F into $\mathbb{R}$, and denote it again by α_0. Let B be a quaternion algebra over F which splits at α_0 and ramifies at all other infinite places. Let $SL_1(B)$ be the group of norm 1 units of B, and $H = \operatorname{Res}_{F/\mathbb{Q}} SL_1(B)$. Then $H(\mathbb{R}) = \prod_{\alpha\in\overline{S}} H_\alpha$, where $\overline{S}$ is the set of embeddings of F into $\mathbb{R}$, and $H_\alpha = H \otimes_{F,\alpha} \mathbb{R}$.

Define a representation of $G \times H$ by

$$\rho_1 = \sum_{\sigma\in\mathcal{G}} \mu^\sigma \otimes p^\sigma_{\alpha_0},$$

where $p_{\alpha_0} \colon H(\mathbb{R}) \to H_{\alpha_0} = SL_2(\mathbb{R})$ is the projection. Since ρ_1 is invariant under any automorphism of $\mathbb{C}$, some multiple $n\rho_1$ of ρ_1 is defined over $\mathbb{Q}$. Now $p^\sigma_{\alpha_0} = p_{\alpha_0^\sigma}$. If σ is not the identity then $\alpha_0^\sigma \neq \alpha_0$. Since H_α is compact for $\alpha \neq \alpha_0$, we see that $n\rho_1$ satisfies the stability condition. We verify that $n\rho_1$ satisfies all of Satake's conditions.

Now, $\mu^\sigma \otimes p^\sigma_\alpha$ is rigid whenever μ^σ is rigid, and it is also rigid when σ is the identity. Hence the index of rigidity of $n\rho_1$ is greater than the index of rigidity of ρ. Continuing this process, if necessary, we will eventually get a representation $\tilde{\rho}$ whose index of rigidity is the cardinality of $\mathcal{G}$, i.e., one which is rigid. Then Theorem 3 implies that some multiple of $\tilde{\rho}$ defines a Kuga fiber variety. Since the restriction of $\tilde{\rho}$ to G is a multiple of ρ, this completes the proof. □

6. An example

Let $F = \mathbb{Q}(\sqrt{3})$. Let α_1, α_2 be the embeddings of F into $\mathbb{R}$. Let B be a quaternion algebra over F which splits at α_1 and ramifies at α_2. Let $\widetilde{G}_1$ be the group of norm 1 units in B, and, $G_1 = \operatorname{Res}_{F/\mathbb{Q}} \widetilde{G}_1$. Let $\widetilde{G}_2$ be a group over F such that $\widetilde{G}_2 \otimes_{F,\alpha_1} \mathbb{R} = SU(5,1)$ and $\widetilde{G}_2 \otimes_{F,\alpha_2} \mathbb{R} = SU(6,0)$. Let $G_2 = \operatorname{Res}_{F/\mathbb{Q}} \widetilde{G}_2$. Let $G = G_1 \times G_2$. Then

$$G(\mathbb{R}) = SL_2(\mathbb{R}) \times SU(2) \times SU(5,1) \times SU(6,0).$$

We shall classify all representations of G which define Kuga fiber varieties. Let

$$p_1 \colon G(\mathbb{R}) \to SL_2(\mathbb{R}),$$
$$p_2 \colon G(\mathbb{R}) \to SU(2),$$

$$p_3\colon G(\mathbb{R}) \to SU(5,1),$$
$$p_4\colon G(\mathbb{R}) \to SU(6,0),$$

be the projections. Then, the representations of G defining Kuga fiber varieties are equivalent over $\mathbb{R}$ to linear combinations of the following:

(1) $p_1 \oplus p_2$,
(2) $p_1 \otimes p_2$,
(3) $\bigwedge^k p_3 \oplus \bigwedge^k p_4$,
(4) $\left(\bigwedge^j p_3 \otimes \bigwedge^k p_4\right) \oplus \left(\bigwedge^k p_3 \otimes \bigwedge^j p_4\right)$,
(5) $\left(p_1 \otimes \bigwedge^k p_4\right) \oplus \left(p_2 \otimes \bigwedge^k p_3\right)$.

Of these, the first four are direct sums of representations of either G_1 alone, or G_2 alone. The last one is a representation of the product group in an essential manner. A general fiber of the corresponding Kuga fiber variety is a simple abelian variety; the semisimple part of its Hodge group is isogenous to $G = G_1 \times G_2$.

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