Abstract

Inferences Over Fields: A Preliminary Investigation into the Deductive Capabilities of Field-Theoretically Defined Logical Connectives

by

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In this paper, we will be concerned with developing an inferential structure over the field with four elements in characteristic 2. We begin by discussing the historical context in which this research occurs. In subsequent sections, we will construct the field, called $\mathbb{F}_4$ and describing the algebraic structure over $\mathbb{F}_4$. We then define the connectives $\wedge$, $\vee$, and $\neg$ over $\mathbb{F}_4$ by extending their standard definition over $\mathbb{F}_2$. We define the basic syntax and semantics of $\mathbb{F}_4$. We show that $\wedge$ and $\vee$ are dual over $\mathbb{F}_4$ with respect to $\neg$ and that $\mathbb{F}_4$ is functionally complete over $\{\wedge, \vee, \neg\} \cup \mathbb{F}_4$. We develop a notion of inferences over $\mathbb{F}_4$ by imbuing it with a partial order, defining validity and the material implication, and defining a proof. Upon completing this, we prove the Deduction, Soundness, and Completeness Theorems, thereby showing that inferences over $\mathbb{F}_4$ behaves in ways comparable to, but not equivalent to, those over a field of two values in characteristic 2.

Inferences Over Fields: A Preliminary Investigation into the Deductive Capabilities of Field-Theoretically Defined Logical Connectives

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Contents

1 Introduction

1.1 A Brief History of Multi-Valued Logic

For most of its history, logic has dealt with propositions that can take on one of two values. The values are usually interpreted as denoting the concepts of *true* and *false*. While treatises on logic were often concerned with the philosophical meaning of these terms (i.e. "What does it mean for a sentence to be true or false?"), they rarely considered the formal character of *true* and *false* except insofar as it related to the valuation of sentences. However, with the advent of symbolic logic, the groundwork was laid for considering logics with values other than *true* and *false*. These new logics typically act as extensions of two-valued logic, with the new value(s) taking intermediate positions between *true* and *false*.

Several logicians developed systems of multi-valued logic independently. The earliest known discussion of multi-valued logic was carried out by the American logician C. S. Peirce in three pages of his notebook. Pierce uses three values, called V, F, and L, to denote sentences with true, false, and indeterminate truth values respectively. It is likely that Peirce viewed L as a designation of *possible* (as oppossed to strictly *true* or *false*). Though Peirce's work anticipated later work in multi-valued logic, it was not published during his lifetime and remained largely unknown until the pages were discovered sometime in the 1960s (Fisch and Turquette, 1966).

Several decades after Peirce, the Polish logician Jan Łukasiewicz and the American mathematician Emil Post independently developed distinct systems of multi-valued logic. Like Peirce, Łukasiewicz used an intermediate truth value (in his case, $\frac{1}{2}$) to denote an indeterminate state of veracity between *true* or 1 and *false* or 0, claiming that his logical system might be justified "when the consequences of the indeterministic philosophy . . . can be compared with empirical data" (Łukasiewicz, 1920, p. 88). Łukasiewicz's work anticipated later developments in probablistic and fuzzy logic, though his interpretation of multi-valued logic as a model for "indeterministic philosophy" was largely left behind with the advent of quantified modal logic.

Post's system of multi-valued divererges sharply from the course charted by Peirce and Łukasiewicz. Post was concerened with showing that, relative to a formal language $\mathcal{L}$, every truth table has a formula that generates it, a property now known as functional completeness. After proving functional completeness in the two-valued setting, Post developed a system with m truth values. The two primitive connectives, called $\neg_m$ and $\vee_m$, are defined as follows:

$$\neg_m t_i = t_{i+1} \quad \text{and} \quad t_i \wedge_m t_j = t_{\min\{i,j\}}$$

for $i \in \mathbb{Z}_m$ and t_i a truth value. Post proved that, under this generalization of two-valued logic, the set $\{\neg_m, \vee_m\}$ is functionally complete (1921, pp. 180 - 181).

In the mid-1960's, the Azerbaijani computer scientist Lofti Zadeh reinvigorated the study of multi-valued logic by introducing fuzzy logic. Zadeh (1965) was principally concerned with fuzzy sets, which he defined as a set with a membership function that can take on values in the interval $[0, 1]$ (ordinary set theory is interpreted as a restriction membership functions that only take on 0 and 1). Zadeh implicitly developed fuzzy logic through his definitions of fuzzy complements (the correlate of negation), fuzzy union (the correlate of disjunction), and fuzzy intersection (the correlate of conjunction). The operations are defined as follows: Let f_X be the membership function for the fuzzy set X. Then, for sets A and B,

1. Complements: $f_{A'}(x) = 1 - f_A(x) = \neg f_A(x)$,

2. Unions: $f_{A\cup B}(x) = \max\{f_A(x), f_B(x)\} = f_A(x) \vee f_B(x)$,

3. Intersection: $f_{A\cap B}(x) = \min\{f_A(x), f_B(x)\} = f_A(x) \wedge f_B(x)$.

for all fuzzy sets A, B, and C. Using these definitions, Zadeh proved many standard identities of classical logic, such ad De Morgan's Law and the Distributive Law.

1.2 Arithmetic with Additional Structure

A well-trod program in mathematics is to consider a structured set of objects and imbue it with additional operations or relations. As a simple example, consider Peano arithmetic $\mathbb{N} = \langle N, +, \cdot, S, 1\rangle$, where $N = \{1, 2, 3, \dots\}$ and S is the successor function with the standard Peano axioms. We can give $\mathbb{N}$ an ordered structure with the following definition: for $x, y \in \mathbb{N}$, we say $x < y$ if there exists an $n \in \mathbb{N}$ such that $S^n(x) = y$, where $S^n(x)$ means n applications of S to x[1]. In fact, this gives the standard order on the natural numbers.

Often, arithmetics are given additional structure as a means of proving something about the arithmetic. An example is the theory of real closed fields, which has proved useful in computation, geometry, and algebraic topology. A real closed field is some set F such that a first-order sentence is true over F if, and only if, it is true over $\mathbb{R}$. In effect, this means that a structure $\mathbb{F} = \langle F, +, \cdot, 0, 1, \leq\rangle$ is a real closed field if it satisfies the following axioms:

1. The (finitely many) axioms of ordered fields,
2. $\forall x\, \exists y\, [\,(0 \leq x) \to (y \cdot y = x)\,]$, and

3_n. the axiom scheme $\forall(a_0, \dots, a_{2n+1})\exists x(a_{2n+1} \neq 0 \to a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + \cdots + a_1 x + a_0 = 0$.

Alfred Tarski proved that the first-order theory of real closed fields, denoted $\mathcal{T}_{rcf}$, admits decidable quantifier elimination. On that basis, he showed $\mathcal{T}_{rcf}$ is complete and decidable.

[1]While $\mathbb{N}$ is not, strictly speaking, a set, we will use the notation $n \in \mathbb{N}$ to denote that $n \in N$ and propositions involving n obey the Peano axioms. We will adopt this convention for other structures as well.

In collaboration with Abraham Seidenberg, Tarski applied these results to the theory of algebraic geometry (van den Dries 1988).

The research herein rests at the intersection of the two traditions discussed above. It describes a four-valued logic with two incomparable values between *true* and *false*. To develop this logic, we make use of a standard construction in ring theory which produces a field of four elements in characteristic 2 (hense the name $\mathbb{F}_4$). Thus, when we call $\mathbb{F}_4$ an extension of $\mathbb{F}_2$, we mean this in two senses. In one sense, it is an extension in the colloquial sense; that is, we have added truth values to $\mathbb{F}_2$ to produce $\mathbb{F}_4$. In the other sense, $\mathbb{F}_4$ is an algebraic extension of $\mathbb{F}_2$. However, the construction of $\mathbb{F}_4$ differs from the usual way $\mathbb{F}_2$ is extended to produce multi-valued logics. To understand this difference, we must briefly discuss Boolean algebra.

1.3 Boolean Algebra

Let $\mathbb{B} = \langle B, \cup, \cap, ', 0, 1\rangle$ be a structure with a (possible infinite) set B, two binary operations $\cup$ and $\cap$, a unary operation $'$, and two constant symbols 0 and 1. We will call $\mathbb{B}$ a Boolean algebra with $|B| = n$ (possibly infinite) if it satisfies the following axioms[2]: for all $x, y, z \in B$

1.	Idempotency	$(i)\ x \cup x = x$	$(ii)\ x \cap x = x$
2.	Commutativity	$(i)\ x \cup y = y \cup x$	$(ii)\ x \cap y = y \cap x$
3.	Associativity	$(i)\ x \cup (y \cup z) = (x \cup y) \cup z$	$(ii)\ x \cap (y \cap z) = (x \cap y) \cap z$
4.	Absorption	$(i)\ (x \cup y) \cap y = y$	$(ii)\ (x \cap y) \cup y = y$
5.	Distributivity	$(i)\ x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$	$(ii)\ x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$
6.	Complements	$(i)\ \exists x'\ (x \cup x') = 1$	$(ii)\ \exists x'\ (x \cap x') = 0$
7.	Uniquness of Elements	$0 \neq 1$	

[2]The set of axioms presented is neither unique nor the most parcimonious set of axioms for Boolean algebra.

Note that the first four pairs of axioms (idempotency, commutativity, associativity, and absorption) are the foundational axioms of lattice theory. Since Boolean algebras admit distribution of $\cap$ over $\cup$, $\cup$ over $\cap$, and complements (in the sense described above), we say that Boolean algebras are complemented distributive lattices. If we interpret $\cup$ as "supremum" and $\cap$ as "infimum", then the axioms of a Boolean algebra induce a partial order over B. (Note: the order will only be strict when $|B| = 2$.)

In fact, the order of a Boolean algebra $\mathbb{B}$ is defined to be the cardinality of B. By the Stone Representation Theorem, every Boolean algebra B is isomorphic to a family F of subsets of a given set S with $|S| = |B|$ such that F is closed under $\cup$, $\cap$, and $'$. The family F is partially ordered by $\subset$ (Stone, 1936). When B is finite, the atoms (or least non-zero elements) are sent to the singleton sets under the isomorphism. A natural consequence of the Stone Representation Theorem is that any finite Boolean algebra with n atoms will have 2^n elements. As such, we shall call a finite Boolean algebra $\mathbb{B}_{2^n}$. The smallest four non-degenerate Boolean Algebras are drawn below, making use of their Hasse diagrams:

$$\begin{tikzcd} \{1\} \\ \emptyset \arrow[u] \end{tikzcd}$$

Fig. 1a: $\mathbb{B}_2$

$$\begin{tikzcd} & \{1,2\} & \\ \{1\} \arrow[ur] & & \{2\} \arrow[ul] \\ & \emptyset \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Fig. 1b: $\mathbb{B}_4$

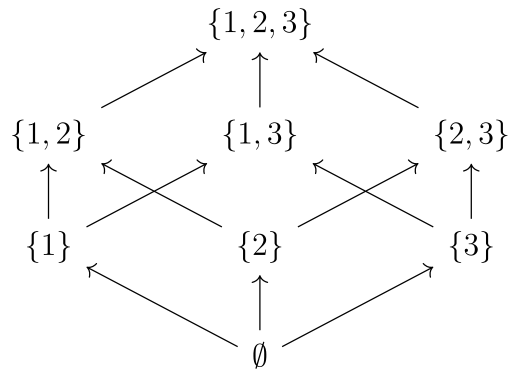

Fig 1c: $\mathbb{B}_8$

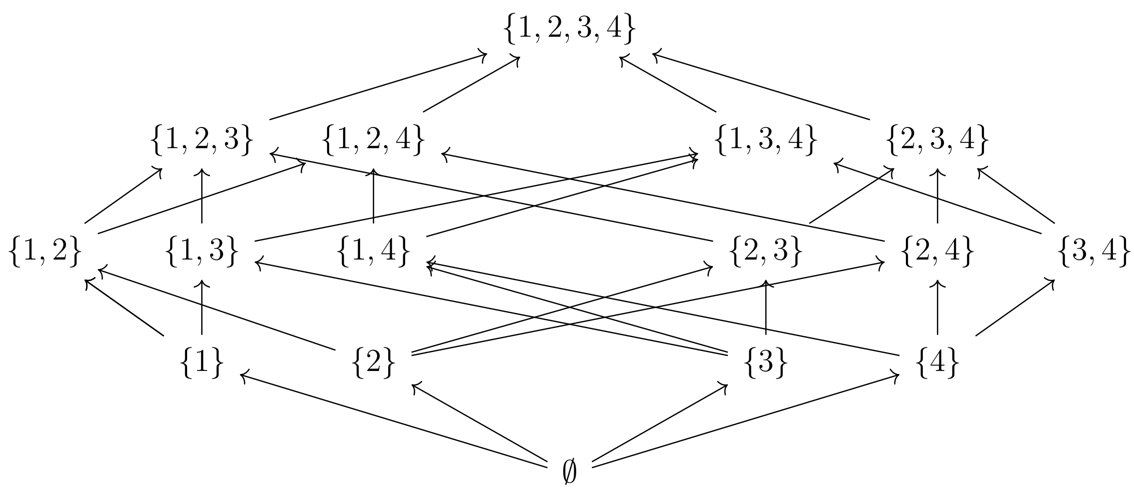

Fig 1d: $\mathbb{B}_{16}$

To avoid clutter, the arrows in a lattice are understood to be transitive. Thus, $\emptyset \subset \{1,2\}$, even though there is no arrow directly connecting $\emptyset$ to $\{1,2\}$.

Boolean algebras are the standard extensions of the two-valued logic. The Boolean algebra $\mathbb{B}_2$ is the standard two-valued logic with $\cup$ corresponding to $\wedge$, $\cap$ corresponding to $\vee$, and $'$ corresponding to $\neg$. For a set of formulas Γ taking values in $\mathbb{B}_n$ and another formula ϕ, we say Γ semantically implies ϕ, denoted $\Gamma \models \phi$, if $\bigwedge_{\psi\in\Gamma}[[\psi]]^V \leq [[\phi]]^V$ for every Boolean valuation $[[\,\cdot\,]]^V$. If we let $\Gamma \vdash \phi$ denote the existence of a proof of ϕ from Γ, then the axioms of Boolean algebra imply that $\Gamma \vdash \phi$ if, and only if, $\Gamma \models \phi$.

The right-directed implication, known as the Soundness theorem, is a standard feature of most commonly used logics. Over $\mathbb{B}_2$, we can interpret Soundness theorem as "If there is a proof of ϕ with true premises, then ϕ is true." For Boolean algebras of higher cardinality, the Soundness theorem says that the valuation of the conclusion of a proof will neither move closer to 0 nor will it move off a chain. Generally speaking, the Soundness theorem for any given logic is a direct consequence of the definition of a proof over that logic. The left-directed implication, known as the Completeness theorem, is a sought after feature of a logic. Over $\mathbb{B}_2$, the Completeness theorem states "There exists a proof for every true formula." This translation, while true, is inadequate over any Boolean algebra with more than 2 elements. For $n > 2$, the Completeness theorem over $\mathbb{B}_n$ states "If the valuation of a formula ϕ is greater than that of the conjunction over a set of formulas Γ, then there exists a proof of ϕ from Γ." Not all logics admit the Completeness theorem. As proved by Gödel, any logic sufficiently strong to handle the axioms of Peano arithmetic is incomplete, meaning that there are true sentences that cannot be proved. For a review of the Completeness theorem over Boolean algebras, see Akiba (2022).

When a logic admit both the Soundness and Completeness theorems, there is a deep link between syntax, or the structure of formulas, and semantics, the means of evaluating formulas. If the Deduction theorem (which states that if $\Gamma, \phi \vdash \psi$ then $\Gamma \vdash \phi \Rightarrow \psi$) and its converse hold in a sound and complete logic, then the notions of semantic implication, proof, and the material implication (the connective that models "if ... then" statements) are equivalent notions. We will show that these theorems hold in $\mathbb{F}_4$ under suitable conditions. Peculiarly, the Completeness theorem holds without any conditions over $\mathbb{F}_4$, but the Soundness theorem requires some modification.

2 The Syntax and Semantics of $\mathbb{F}_4$

2.1 The Algebraic Structure of $\mathbb{F}_4$

Let $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$; that is $\mathbb{F}_2 = \{[0],[1]\}$.[3] Define the operations $+$ and $\cdot$ over $\mathbb{F}_2$ as the standard addition and multiplication over equivalence classes. Then, the tables form $+$ and $\cdot$ are as follows:

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

A useful observation that follows from this construction is that $\mathbb{F}_2$ is of characteristic 2.

We will define the operations $\wedge$, $\vee$, and $\neg$ as follows:

$$\begin{aligned} \wedge : \; & \mathbb{F}_2 \times \mathbb{F}_2 \longrightarrow \mathbb{F}_2 \\ & (x,y) \longmapsto x \cdot y \end{aligned} \qquad \begin{aligned} \vee : \; & \mathbb{F}_2 \times \mathbb{F}_2 \longrightarrow \mathbb{F}_2 \\ & (x,y) \longmapsto x + y + (x \cdot y). \end{aligned}$$

$$\begin{aligned} \neg : \; & \mathbb{F}_2 \longrightarrow \mathbb{F}_2 \\ & x \longmapsto x + 1 \end{aligned}$$

The tables for $\wedge$, $\vee$, $\neg$ are

$$\begin{array}{c|cc} \wedge & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \qquad \begin{array}{c|cc} \vee & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{c|cc} \neg & 0 & 1 \\ \hline & 1 & 0 \end{array}$$

Note that the tables for $\wedge$, $\vee$, $\neg$ give the truth tables for the logical connectives representing *"and"*, *"or"*, and *"not"* if we interpret 0 as $false$ and 1 as $true$. Said another way, sentences in the language of $\langle \mathbb{F}_2, \wedge, \vee, \neg \rangle$ make up the language of the standard two-valued logic.

Consider $\mathbb{F}_2[x]/\langle x^2 + x + 1 \rangle$. By Euclidean Division, it is clear that a system of representatives for the equivalence classes are $\mathbb{F}_2/\langle x^2 + x + 1 \rangle = \{\ 0, 1, x, x+1\ \}$. We will call this field $\mathbb{F}_4$. It is clear that

[3]Since $\mathbb{F}_2 \cong \mathbb{Z}_2$, we will drop the equivalence class notation and only write 0 and 1.

$$\mathbb{F}_4 = \{\alpha + \beta x \mid \alpha, \beta \in \mathbb{F}_2\}.$$

That is, $\mathbb{F}_4$ is an $\mathbb{F}_2$-vector space. Note that x is a solution to $x^2 + x + 1 = 0$. Thus, $x^2 = x + 1$. It can be easily verified that $(x + 1)^2 = x$. In general, computations in $\mathbb{F}_4$ are done using the identity $x^2 + x + 1 = 0$. For convenience, we will call $a = x$ and $b = x + 1$. Since $\mathbb{F}_4$ has characteristic 2, we get the following tables for $+$ and $\cdot$ (defined in the usual ways for polynomials)

$+$	0	a	b	1
0	0	a	b	1
a	a	0	1	b
b	b	1	0	a
1	1	b	a	0

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	b	1	a
b	0	1	a	b
1	0	a	b	1

If we define $\wedge : \mathbb{F}_4 \times \mathbb{F}_4 \longrightarrow \mathbb{F}_4$, $\vee : \mathbb{F}_4 \times \mathbb{F}_4 \longrightarrow \mathbb{F}_4$, and $\neg : \mathbb{F}_4 \times \mathbb{F}_4 \longrightarrow \mathbb{F}_4$ as we did over $\mathbb{F}_2$, we get the tables

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	b	1	a
b	0	1	a	b
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	b	0	1
b	b	0	a	1
1	1	1	1	1

$\neg$	0	a	b	1
	1	b	a	0

We will use these tables as the semantic foundation for $\mathbb{F}_4$.

We will pause here to make some observations about $\wedge$, $\vee$ and $\neg$ over $\mathbb{F}_4$. First, $\wedge$ and $\vee$ are commutative and associative, conforming to our usual understanding of *and* and *or*. However, since $a \wedge a = a \vee a = b$, idempotency does not hold for all elements in $\mathbb{F}_4$. Furthermore, since

$$a \wedge (a \vee b) = 0 \neq b = (a \wedge a) \vee (a \wedge b)$$

and

$$a \vee (a \wedge b) = a \neq 0 = (a \vee a) \wedge (a \vee b),$$

$\mathbb{F}_4$ does not admit distribution of $\wedge$ over $\vee$ and vice versa. Thus, the behavior of $\wedge$ and $\vee$ over $\mathbb{F}_4$ diverges radically from their behavior over any Boolean algebra.

2.2 The Syntax of Propositional Calculus over $\mathbb{F}_4$

We will now develop the syntax of logic over $\mathbb{F}_4$. This is the standard syntax of logic over $\mathbb{F}_2$, augmented only by the addition of symbols for the new elements of $\mathbb{F}_4$:

1. The elements of $\mathbb{F}_4$;

2. A countably infinite set of propositional variables $\mathfrak{X} = \{p, q, r, \ldots\}$; and

3. The set of logical connectives $\mathfrak{C} = \{\neg, \wedge, \vee\}$.

To develop an inferential structure over $\mathbb{F}_4$, we must first define what a formula over $\mathbb{F}_4$ is.

Definition 1. *We will call a string ϕ of symbols over $\mathbb{F}_4$ a formula if it satisfies any of the following conditions:*

1. $\phi \in \mathbb{F}_4$;

2. $\phi \in \mathfrak{X}$; *and*

3. *For formulas ψ and π, $\neg\psi$, $(\psi \wedge \pi)$, and $(\psi \vee \pi)$ are also formulas.*

No other strings will be called formulas. We will let $P(\mathbb{F}_4, \mathfrak{C})$ represent the set of formulas over $\mathbb{F}_4$ with connectives in $\mathfrak{C}$.

Note that every formula in $P(\mathbb{F}_4, \mathfrak{C})$ is also a polynomial. We will exploit this fact.

We have inherited this definition of a formula from two-valued logic. Since this definition is purely syntactic[4], every theorem derivable solely from the definition of a formula holds for the syntax of $\mathbb{F}_4$. In particular, the Unique Parsing Lemma, which

[4] That is, it makes no reference to evaluations of formulas.

states that the constituent non-connective elements of a formula are uniquely determined by the structure of the formula, holds in $\mathbb{F}_4$.[5] It is conventional to drop the outermost pair of parenthesis from formulas. We will adopt this convention.

2.3 The Semantics of Propositional Calculus over $\mathbb{F}_4$

When we speak of the semantics of $\mathbb{F}_4$, we mean a system by which we may assign formulas a value in $\mathbb{F}_4$. The most basic notion in semantics is the truth function.

Definition 2. *We call $\tau : P(\mathbb{F}_4, \mathfrak{C}) \longrightarrow \mathbb{F}_4$ a truth function if, for $\phi, \psi \in P(\mathbb{F}_4, \mathfrak{C})$,*

$$\tau(\neg\phi) = \neg\tau(\phi) \qquad \tau(\phi \wedge \psi) = \tau(\phi) \wedge \tau(\psi) \qquad \tau(\phi \vee \psi) = \tau(\phi) \vee \tau(\psi).$$

We have already made use of truth functions; the tables for $\wedge$, $\vee$ and $\neg$ display the valuations of the connectives. As an example, the entry in the second row and third column of the $p \wedge q$ truth table represents a truth function that takes p to a and q to b. We will call tables displaying the valuations of formulas containing $\wedge$, $\vee$, and $\neg$ truth tables. A truth table will be represented by n dimensional arrays, with n corresponding to the number of variables in the formula. We will use $T(V)$ to denote the set of truth tables over a set of values V. Thus, the set of truth tables over $\mathbb{F}_4$ is $T(\mathbb{F}_4)$.

We can use the truth function to define the meta-symbol $\equiv$ (pronounced "equivalence") as follows:

Definition 3. *Let $\phi, \psi \in P(\mathbb{F}_4, \mathfrak{C})$. We will say $\phi \equiv \psi$ if, and only if, $\tau(\phi) = \tau(\psi)$ for all truth functions τ.*

From the definition of equivalence, the following lemmas hold:

[5]For a proof of the Unique Parsing Lemma, see Ebbinghaus, Flum, Thomas (2021) p. 22 Theorem 4.4.

Lemma 1. *Let $\phi, \psi \in P(\mathbb{F}_4, \mathfrak{C})$ and $\hat{x} = (x_1, x_2, \ldots, x_n) \in \mathfrak{X}^n$. Then $\phi \equiv \psi$ if, and only if, $\phi(\hat{x}) = \psi(\hat{x})$ for all values $\hat{x}$.*

Proof. Suppose $\phi \equiv \psi$. Then $\tau(\phi) = \tau(\psi)$ for all τ. Since τ is a valuation of the variables in ϕ and ψ, it follows immediately that $\phi(\hat{x}) = \psi(\hat{x})$ for all values of $\hat{x}$. The converse holds for similar reasons.

□

Lemma 2. *The $\equiv$ relation defines an equivalence class on $P(\mathbb{F}_4, \mathfrak{C})$.*

Proof. The lemma follows immediately from the fact that $=$ is an equivalence relation.

□

We can use truth functions to define several important concepts. The first will be the notion of an α-tautology.

Definition 4. *Let $\phi \in P(\mathbb{F}_4, \mathfrak{C})$. We will say ϕ is an α-tautology if $\tau(\phi) = \alpha$ for all truth functions τ.*

If we consider this definition over $\mathbb{F}_2$, then a 1-tautology corresponds to the notion of a tautology while a 0-tautology corresponds to the notion of a contradiction. To preserve clarity, we will largely avoid this simplification here. Occasionally, it will be necessary to talk about formulas that only take on values in $\mathbb{F}_2$. We will call these formulas absolute. Put formally

Definition 5. *Let $\phi \in P(\mathbb{F}_4, \mathfrak{C})$. We say ϕ is absolute if $\tau(\phi) \in \mathbb{F}_2$ for all truth functions τ.*

We will now demonstrate some of the algebraic properties of absolute formulas. First, any absolute formula can be made non-absolute by conjoining a or b with it. For example, the formula $p \wedge p \wedge p$ is absolute since

	0	a	b	1
$p \wedge p \wedge p$	0	1	1	$1.$

If we conjoin a, we get

	0	a	b	1
$a \wedge p \wedge p \wedge p$	0	a	a	$a.$

Given that $\wedge$ is just polynomial multiplication, this is not a surprising result. We will exploit this fact in a future proof. The set of absolute formulas is also closed under $\wedge$, $\vee$, and $\neg$. This follows immediately from the fact that the table for each connective over $\mathbb{F}_2$ is a subtable of the table for the connective over $\mathbb{F}_4$.

2.4 Basic Properties of $\mathbb{F}_4$

An interesting feature of $\mathbb{F}_4$ is displayed in the following truth table

p	0	a	b	1
$p \wedge \neg p$	0	1	1	0
$p \vee \neg p$	1	0	0	1

Under $\mathbb{F}_2$, $p \vee \neg p$ always evaluates to 1 ($''true''$) and $p \wedge \neg p$ always evaluates to 0 ($''false''$). In fact, the formulas are often used as stand-ins for the notion of a tautology and a contradiction respectively. By contrast, $\mathbb{F}_4$ does not allow this simple equivalence.

We have already seen that $\mathbb{F}_4$ is neither distributive nor idempotent, implying that $\mathbb{F}_4$ is not a Boolean algebra. Thus, by moving to $\mathbb{F}_4$, we must define our logical notions to accomodate these peculiarities. However, before moving to those logical notions, we will discuss certain familiar relationships that hold in $\mathbb{F}_4$.

Theorem 1. *Let $\phi, \psi \in P(\mathbb{F}_4, \mathfrak{C})$. Then the following equivalences hold*

1. $\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi$,

2. $\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$,

3. $\phi \equiv \neg(\neg\phi)$.

Proof. Observe that $\neg(\phi \wedge \psi) = \phi\psi + 1$ and $\neg(\phi \vee \psi) = \phi + \psi + \phi\psi + 1$. For Part 1, we note that

$$\begin{aligned}
\phi\psi + 1 &= \phi + 1 + \psi + 1 + \phi + \psi + \phi\psi + 1 \\
&= (\phi + 1) + (\psi + 1) + \phi(\psi + 1) + \psi + 1 \\
&= (\phi + 1) + (\psi + 1) + (\phi + 1)(\psi + 1) \\
&= \neg\phi \vee \neg\psi.
\end{aligned}$$

For Part 2, we note that

$$\begin{aligned}
\phi + \psi + \phi\psi + 1 &= \phi(\psi + 1) + \psi + 1 \\
&= (\phi + 1)(\psi + 1) \\
&= \neg\phi \wedge \neg\psi.
\end{aligned}$$

Thus, $\neg(\phi \wedge \psi) = \neg\phi \vee \neg\psi$ and $\neg(\phi \vee \psi) = \neg\phi \wedge \neg\psi$. Thus, by Lemma 1, $\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi$ and $\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$.

For Part 3, it is sufficient to note that $\neg\phi = \phi + 1$ and $\mathbb{F}_4$ is in characteristic 2.

□

The first two items in Theorem 1 are jointly known as DeMorgan's Law, while the third item is known as Double Negation. These are standard features of many logical systems. From DeMorgan's Law, we can prove the following theorem.

Theorem 2. *The Duality Theorem: The connectives $\wedge$ and $\vee$ are dual to one another with respect to $\neg$. That is, if $\phi \in P(\mathbb{F}_4, \mathfrak{C})$ and we define ϕ' to be that element of $P(\mathbb{F}_4, \mathfrak{C})$ such that every propositional variable x_i appearing ϕ is replaced with $\neg x_i$, every instance of $\wedge$ is replaced with $\vee$, and every instance of $\vee$ is replaced with $\wedge$, then $\phi' = \neg\phi$.*

Proof. The Duality Theorem is an immediate consequence of Theorem 1 by induction on the complexity of terms in ϕ.

$\square$

An important relationship between the syntactic and semantic structures of many logics is the notion of functional completeness. We say that a set of connectives F is functionally complete over a set of values V if there is a function $f : P(V, F) \to T(V)$ that is surjective. That is, every truth table corresponds to at least one proposition in $P(V, F)$. We will be concerned with functional completeness of $\mathfrak{C} \cup \mathbb{F}_4$ over $\mathbb{F}_4$. We first prove a lemma.

Lemma 3. *Every absolute truth table with only one* 1 *has a formula.*

Proof. The following table proves the lemma for the single variable case.

p	$\chi_0(p) = \neg(p \wedge p \wedge p)$	$\chi_a(p) = (p \wedge \neg p) \wedge \neg(a \wedge p) \wedge \neg a$
0	1	0
a	0	1
b	0	0
1	0	0
p	$\chi_b(p) = \neg[\chi_0(p) \vee \chi_a(p) \vee \chi_1(p)]$	$\chi_1(p) = p \vee p \vee p$
0	0	0
a	0	0
b	1	0
1	0	1

Note that each entry in a truth table corresponds to a valuation of the variables defining the table. Let $\mathcal{T}$ be a truth table in n variables as in the statement of Lemma 3. We claim that the formula for the truth table $\mathcal{T}$ is $\phi(p_1, p_2, \ldots, p_n) = \bigwedge_{i \in \{1,\ldots,n\}} \chi_{\alpha_i}(p_i)$, where α_i is the valuation of the variable p_i that give 1. The table above proves the single variable case. Suppose the truth table $\mathcal{T}$ in n variables has value 1 at $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then

$$\tau\left(\chi_{\alpha_1}(\alpha_1) \wedge \chi_{\alpha_2}(\alpha_2) \wedge \cdots \wedge \chi_{\alpha_n}(\alpha_n)\right) = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n = 1$$

Since each non-zero element has a unique inverse under $\wedge$, changing the value of any of the α_i will make $\tau(\chi_{\alpha_i}(p_i)) = 0$, making $\tau(\phi(p_1, p_2, \ldots, p_n)) = 0$. Thus, $\phi(p_1, p_2, \ldots, p_n)$ gives a formula for any absolute $\mathcal{T}$ with exactly one entry of 1.

$\square$

Corollary 1. *Every truth table over $\mathbb{F}_4$ with exactly one non-zero entry is given by a proposition.*

Proof. The absolute case is covered in Lemma 3. For the non-absolute case, suppose the truth table $\mathcal{T}$ has value α at position $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. We can construct the truth table $\mathcal{T}'$ with value 1 at position $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ as in Lemma 3. Call the proposition for $\mathcal{T}'$ ϕ. Then, $\mathcal{T}$ is determined by the formula $\alpha \wedge \phi$.

$\square$

Theorem 3. *The set $\mathfrak{C} \cup \mathbb{F}_4$ is functionally complete over $\mathbb{F}_4$.*

Proof. Let $\mathcal{T}$ be a truth table over $\mathbb{F}_4$. Let $\phi_{(\alpha_1,\ldots,\alpha_n)}(p_1, \ldots, p_n)$ be the formula evaluating to α at $(\alpha_1, \ldots, \alpha_n)$ and 0 under all other evaluations. Construct 4^n such formulas with each $\phi_{(\alpha_1,\ldots,\alpha_n)}(p_1, \ldots, p_n)$ having the same value as $\mathcal{T}$ at $(\alpha_1, \ldots, \alpha_n)$. Note that, for $\alpha \in \mathbb{F}_4$, $\alpha \vee 0 = \alpha$. Let τ be the truth function such that $\tau(\phi_{(\iota_1,\ldots,\iota_n)}(p_1, \ldots, p_n)) = \alpha$. Then, by construction, $\tau(\phi_{(\alpha_1,\ldots,\alpha_n)}(p_1, \ldots, p_n)) = 0$ for all $(\alpha_1, \ldots, \alpha_n) \neq (\iota_1, \ldots, \iota_n)$. Then,

$$\tau\left(\bigvee_{(\alpha_1,\ldots,\alpha_n)\in\mathbb{F}_4^n} \phi_{(\alpha_1,\ldots,\alpha_n)}(p_1, \ldots, p_n)\right) = 0 \vee 0 \vee \cdots \vee \alpha \vee \cdots \vee 0 = \alpha.$$

Therefore, $\bigvee_{(\alpha_1,\ldots,\alpha_n)\in\mathbb{F}_4^n} \phi_{(\alpha_1,\ldots,\alpha_n)}(p_1, \ldots, p_n)$ is a formula for $\mathcal{T}$.

$\square$

An important consequence of functional completeness is that we can define a connective not in $\mathfrak{C}$ over $\mathbb{F}_4$ simply by giving its truth table. For example, given the definition of equivalence, it is clear that its truth table is

$\equiv$	0	a	b	1
0	1	0	0	0
a	0	1	0	0
b	0	0	1	0
1	0	0	0	1

if we let 1 mean the formulas are equivalent and 0 mean they are not. In fact, we can encode any metalogical concept into the language of $\mathbb{F}_4$. We will exploit this in the next section. Before concluding, we will show an interesting equivalence.

Corollary 2. $P(\mathbb{F}_4, \mathfrak{C}) = \mathbb{F}_2[\hat{x}]$.

Proof. Since $p \wedge q := pq$ and $p \vee q := p + q + pq$, it is clear that $P(\mathbb{F}_4, \mathfrak{C}) \subset \mathbb{F}_2[\hat{x}]$. By functional completeness, $p + q$ can be written in terms of $\wedge$, $\vee$, and $\neg$. Since 0, 1, and $\mathfrak{X}$ are elements of $P(\mathbb{F}_4, \mathfrak{C})$, the elements of $\mathbb{F}_2[\hat{x}] \subset P(\mathbb{F}_4, \mathfrak{C})$, as needed.

□

One oft-followed line of inquiry in classical symbolic logic is to ask what connectives are strictly necessary. It has been proved that $\{\vee, \neg\}$, $\{\wedge, \neg\}$, and $\{\rightarrow, \neg\}$ are functionally complete in $\mathbb{F}_2$.[6] We can carry out a similar reduction over $\mathfrak{C} \cup \mathbb{F}_4$ via a few observations. First, by Theorem 1, $\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$. We can use this equivalence to give a definition of $\vee$ in terms of $\wedge$, namely $\phi \vee \psi := \neg(\neg\phi \wedge \neg\psi)$. We also do not need every element of $\mathbb{F}_4$. Since $0 \equiv \neg 1$ and $a \equiv \neg b$, we only need to include one element from $\mathbb{F}_2$ and one element from $\mathbb{F}_4 \setminus \mathbb{F}_2$ to generate all truth tables. Thus, for example, the set $\{\wedge, \neg, 0, a\}$ is functionally complete by an argument almost identical to the proof of Theorem 3. However, while there are philosophical and aesthetic advantages to this

[6]Here, $\rightarrow$ is the implication symbol, typically defined as $\phi \rightarrow \psi := \neg\phi \vee \psi$. If $\rightarrow$ is taken as a primitive, it is usually defined axiomatically.

reduction, we will largely ignore it since it decreases the number of logical symbols we have at hand.

3 Inferential Structures on $\mathbb{F}_4$

3.1 Validity, Semantic Implication, and Proofs

Up to this point, we have defined the semantic notions of $\mathbb{F}_4$ in a similar way as we defined them for $\mathbb{F}_2$. However, there are several semantic notions in $\mathbb{F}_2$ that require more care to extend to $\mathbb{F}_4$. Principal among these are the related notions of validity and semantic implication. Over $\mathbb{F}_2$, for a set of formulas Γ and a formula ϕ, we say Γ semantically implies ϕ, denoted $\Gamma \models \phi$, if, and only if, the truth of the conjunction of the formulas in Γ guarantees the truth of ϕ. We say an argument is valid if the premises semantically imply the conclusion. We will adopt this definition of a valid argument. Thus, to determine how to make inferences over $\mathbb{F}_4$, it suffices to determine how validity ought to be defined.

Since $\mathbb{F}_2$ only contains two elements, the order of those elements is rarely discussed explicitly. However, as we saw implicitly with Peirce's and Łukasiewicz's "intermediate" truth value and explicitly with Post's multiples truth values, applying an order to the set of truth values can formalize many logical intuitions. In fact, the tendency among mathematicians to prefer truth over falsity (that is, 1 over 0) implies an order on $\mathbb{F}_2$. If we let $0 < 1$, the standard definition of a valid argument can be rephrased in terms of order as follows:

Definition 6. *Let Γ be a set of formulas and ϕ be a formula (possibly in Γ). Then the argument from Γ to ϕ is valid, written as $\Gamma \models \phi$, if, and only if, $\tau\left(\bigwedge \Gamma\right) \leq \tau(\phi)$, where $\bigwedge \Gamma$ is the conjunction of all formulas in Γ.*

We can extend this notion of semantic implication to $\mathbb{F}_4$ by defining a suitable partial order. Note the following two facts:

1. There is a natural Boolean homomorphism from $\mathbb{B}_2$ to $\mathbb{B}_4$ (when interpreted as algebras of sets) that takes $\{1\}$ to $\{1, 2\}$ and $\emptyset$ to $\emptyset$,

2. There is a natural field homomorphism from $\mathbb{F}_2$ to $\mathbb{F}_4$ that takes 1 to 1 and 0 to 0.

Call these homomorphisms f and g respectively. It is clear that there is an isomorphism $h : \mathbb{B}_2 \longrightarrow \mathbb{F}_2$ that takes $\cap$ to $\wedge$ and $\cap$ to $\vee$. Thus, we seek some morphism σ such that

$$\begin{tikzcd} \mathbb{B}_2 \arrow[r, "f"] \arrow[d, "h"] & \mathbb{B}_4 \arrow[d, "\sigma"] \\ \mathbb{F}_2 \arrow[r, "g"] & \mathbb{F}_4 \end{tikzcd}$$

commutes. We will not be concerened with the construction of σ. It will suffice to note that there are two possibilities, defined as follows:

	x	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
1.	$\sigma(x)$	0	a	b	1
2.	$\sigma(x)$	0	b	a	1

Since $\{1\}$ and $\{2\}$ are incomparable under $\subset$, the choice of definition for σ is inconsequential. We can define the partial order of $\mathbb{F}_4$ using σ. If $x, y \in \mathbb{F}_4$, we will say that $x \leq y$ if, and only if, $\sigma^{-1}(x) \subset \sigma^{-1}(y)$. We define other order symbols (e.g. $<$ and $\not>$) in the usual way. Under this definition, 1 is the greatest element of $\mathbb{F}_4$, 0 is the least element, and a and b are incomparable intermediate values. That is, the $\mathbb{F}_4$ lattice induced by σ is as follows:

$$\begin{tikzcd} & 1 & \\ a \arrow[ur] & & b \arrow[ul] \\ & 0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

We can now begin to define validity. We will call a finite ordered sequence of formulas $\Phi = \{\phi_i\}_{1 \leq i \leq n}$ an argument for ϕ if, and only if, $\phi_n = \phi$; that is, if ϕ is the last formula in the sequence. Fix some number $k < n$. We will call $\Gamma = \{\ \phi_i \in \Phi \mid i \leq k\ \}$ a set of premises. Since any collection of formulas can be put in sequence, it is clear from the definition that not all arguments are equally useful. Thus, we need some means by which to distinguish the quality of arguments. As in $\mathbb{F}_2$, validity will serve this function.

To begin, we will define three notions of semantic implication. Let Γ be a set of premises and ϕ be a formula in some argument with Γ as its inital subset. Then, we define semantic implication in one of the three following ways.

1. Γ semantically implies ϕ classically, denoted $\Gamma \models_{clas} \phi$, iff $\tau\left(\bigwedge \Gamma\right) \leq \tau\left(\phi\right)$ for all truth functions τ, or
2. Γ semantically implies ϕ cardinally, denoted $\Gamma \models_{card} \phi$, iff $\#\left(\bigwedge \Gamma\right) \leq \#(\phi)$, or
3. Γ semantically implies ϕ bi-conversely, denoted $\Gamma \models_{bc} \phi$, iff $\tau\left(\bigwedge \Gamma\right) \not> \tau(\phi)$ for all truth functions τ,

where $\#(\phi)$ is the cardinality of $\tau(\phi)$ under σ^{-1}.[7] The definition given in 1 is the standard definition of semantic implication over Boolean algebras, hence the description "classical". The definition given in 3 is the converse of the converse of 1, hense the description "bi-conversely".

Note that, in $\mathbb{F}_4$, $\Gamma \models_{bc} \phi$ and $\Gamma \models_{card} \phi$ are equivalent. This is clear in the case where $\tau\left(\bigwedge \Gamma\right) \in \mathbb{F}_2$. Suppose $\tau\left(\bigwedge \Gamma\right) = \alpha \in \mathbb{F}_4 \setminus \mathbb{F}_2$. Then, $\#\left(\bigwedge \Gamma\right) = 1$. For any given value of ϕ, we have

$\#$	$\bigwedge \Gamma = a$	$\bigwedge \Gamma = b$	τ	$\bigwedge \Gamma = a$	$\bigwedge \Gamma = b$
$\phi = 0$	I	I	$\phi = 0$	I	I
$\phi = a$	V	V	$\phi = a$	V	V
$\phi = b$	V	V	$\phi = b$	V	V
$\phi = 1$	V	V	$\phi = 1$	V	V

where V represents validity and I represents invalidity. Thus, any difference between the possible definitions of semantic implication in $\mathbb{F}_4$ can be reduced to discussion of

[7]Note that $\tau(\phi)$ is guaranteed to have a cardinality by the Stone Representation Theorem.

the differences between 1 and 3. To adjudicate between the two, we must develop the inferential system of $\mathbb{F}_4$ further.

Suppose we fix a definition of semantic implication and let $\models$ denote this relationship. We define validity as follows:

Definition 7. *Let $\Phi = \{\phi_i\}_{i \leq n}$ be an argument of length n for ϕ and $\Gamma \subset \Phi$ be the set of premises. We will say that Φ is a valid argument if, and only if, $\Gamma \models \phi$.*

To define the notion of proof, we must define some way of "moving" between formulas in the argument. We will call these locomotive objects rules of inference, formally defined as:

Definition 8. *A rule of inference is a function $i : Q^n \subset P(\mathbb{F}_4, \mathfrak{C})^n \longrightarrow P(\mathbb{F}_4, \mathfrak{C})$, where each formula in the tuple $(\phi_1, \phi_2, \ldots, \phi_n) \in Q^n$ is of a certain specified form. If Φ is a set of formulas, we will define $i(\Phi)$ as the application of i to the tuple $(\phi_1, \phi_2, \ldots, \phi_n)$ where $\bigcup_{i=1}^{n} \{\phi_i\} = \Phi$ and each ϕ_i is in a place appropriate for the given i.*

For two rules of inference i_0 and i_1 and sets of formulas Φ and Ψ, we will call $i_1(i_0(\Phi) \cup \Psi)$ a composition of rules of inferences. We will use the notation $i_1 \circ i_0$ to indicate the composition of inference rules with suitably many arguments. When stating a rule of inference, we will write it in the following form: for a set of premises Γ and a conclusion ϕ

$$\frac{\Gamma}{\therefore \phi}$$

We will call a rule of inference sound if, and only if, it is valid.

We can now begin to adjudicate between the competing definitions of semantic implication. A basic property of most inferential systems is the soundness of the following inference rule:

$$\begin{array}{r} \phi \\ \hline \therefore \phi \mathbin{\&} \phi \end{array}$$

where & is the conjunction operation in the inferential system. We can see from the table for $\wedge$, if $\tau(\phi) = a$ for some truth function τ, definition 1 fails while definition 3 holds. Thus, we will accept definition 3 as out definition of validity.

Having defined validity, we can prove the following results:

Lemma 4. *The inference rule*

$$\begin{array}{r} \phi \\ \psi \\ \hline \therefore \phi \wedge \psi \end{array}$$

is a sound.

Proof. Since $\phi \wedge \psi \equiv \phi \wedge \psi$, it is clear that $\tau(\phi \wedge \psi) \not> \tau(\phi \wedge \psi)$, as needed.

□

Lemma 5. *The composition of two sound rules of inference is a sound rule of inference.*

Proof. Suppose i_0 and i_1 are sound rules of inference. We claim $i_1 \circ i_0$ is a sound rule of inference. Let ϕ be a formula such that $i_0(\phi) = \phi_0$ is a formula and $i_1(\phi_0) = \phi_1$ is a formula. (Note: since conjunction of finitely many premises is a valid argument, the proof may proceed from one formula without loss of generality). It is sufficient to show that $\tau(\phi) \not\geq \tau(\phi_1)$ for all truth functions τ. Let $\tau(\phi) = \alpha$. By the definition of a rule of inference, $\alpha \not\geq \tau(\phi_0) = \beta$. Similarly, $\beta \not\geq \tau(\phi_1) = \gamma$. Therefore, $\alpha \not\geq \gamma$, as needed.

□

We will now define what a proof is in $\mathbb{F}_4$.

Definition 9. *Let Γ be a set of formulas. A proof of a formula ϕ from Γ is a finite sequence of ordered pairs of the form (ϕ_i, ρ_i), where ϕ_i is a formula satisfying one of the following conditions*

1. *ϕ_i is an axiom, or*

2. *ϕ_i is a formula in Γ, or*

3. *ϕ_i is derivable from a collection of axioms or formulas in Γ by a sound rule of inference,*

and $\rho_i \subset \Gamma$ with (ϕ, ρ_n) as the last pair in the sequence. We denote the existence of a proof of ϕ from Γ by $\Gamma \vdash \phi$. We will call a formula ϕ a theorem if there exists a proof for ϕ.

Note that ρ_n is not necessarily equal to Γ. We will use $\Sigma_i(\Gamma, \phi)$ to denote a subset of Γ for which every element of $\Sigma_i(\Gamma, \phi)$ is necessary for a proof of ϕ. Since Γ is finite and each proof is of finite length, there are at most countably many proofs of ϕ from a subset of Γ. If we index these proofs to $\mathbb{N}$, we will say $\Sigma_i(\Gamma, \phi)$ is the set of premises for the i^{th} proof.

There are several immediate consequences of the definition of a proof. First, since proofs only make use of sound rules of inference, if every premise of a proof is a 1-tautology, then the conclusion is also a 1-tautology. This recovers the standard notion of proof over $\mathbb{F}_2$. Another immediate consequence of the definition is that $\vdash$ is transitive. Suppose $\Gamma \vdash \Delta$ and $\Delta \vdash \phi$ for Γ and Δ sets of formulas and ϕ a formula. Since a proof of Δ exists from Γ, it follows that $\Gamma \vdash \phi$.

3.2 Substituion

A core feature of any inferential system is substitution. Many non-trivial proofs in two-valued logic requires the substituion of formulas in a rule of inference with the formulas present in the proof. Thus, verifying that substitution is sound of great importance. We will be concerned with two forms of substitution. For a formula $\phi(\hat{x}, y)$, we will use the notation $\phi(\hat{x}, \psi|y)$ to denote a substitution of the variable y with the formula ψ. The first kind of substitution, which we will call tautological substitution, claims that, if some formula $\phi(\hat{x}, y)$ takes the truth value α under all truth functions τ, then $\tau(\phi(\hat{x}, \psi|y)) = \alpha$. The second, which we will call equivalent substitution, claims that if $\psi \equiv \pi$ then $\phi(\hat{x}, \psi) = \phi(\hat{x}, \pi)$.

Theorem 4. *Let $\phi(\hat{x}, y)$ be a formula with variables $\hat{x} \in \mathfrak{X}^n$ and y and ψ be another formula. If $\tau(\phi(\hat{x}, y)) = \alpha$ for all truth functions τ then $\tau(\phi(\hat{x}, \psi|y)) = \alpha$.*

Proof. We will proceed by induction on the complexity of terms. The theorem vacuously holds for variables since they take on any value in $\mathbb{F}_4$. Fix n such that every formula of length less than n satisfies the theorem. Then there are three cases.

1. Case 1: Let $\phi(\hat{x}, y) = \neg\psi(\hat{x}, y)$ for formula $\psi(\hat{x}, y)$. It is clear that $\psi(\hat{x}, y)$ has length less than n and it admits tautological substitution. Thus, $\psi(\hat{x}, y) \equiv \psi(\hat{x}, p|y)$. Let $\tau(\phi(\hat{x}, y)) = \alpha$. Then, $\tau(\neg\psi(\hat{x}, y)) = \alpha$. We can see that

$$\begin{aligned}
\tau(\neg\psi(\hat{x}, y)) &= \neg\tau(\psi(\hat{x}, y)) \\
&= \neg\alpha \\
&= \neg\tau(\psi(\hat{x}, \psi|y)) \\
&= \tau(\neg\psi(\hat{x}, \psi|y))
\end{aligned}$$

   Thus, $\phi(\hat{x}, y) \equiv \phi(\hat{x}, p|y)$.

2. Case 2: Let $\phi(\hat{x}, y) = \psi(\hat{x}, y) \wedge \pi(\hat{x}, y)$. Since $\psi(\hat{x}, y)$ and $\pi(\hat{x}, y)$ are both of length less than n, they admit tautological substituion. Let $\tau(\psi(\hat{x}, y)) = \beta$ and $\tau(\pi(\hat{x}, y)) = \gamma$. Then

$$\begin{aligned}\tau(\psi(\hat{x},y)\wedge\pi(\hat{x},y)) &= \tau(\psi(\hat{x},y))\wedge\tau(\pi(\hat{x},y))\\ &= \beta\wedge\gamma\\ &= \tau(\psi(\hat{x},\psi|y))\wedge\tau(\pi(\hat{x},\psi|y))\\ &= \tau(\psi(\hat{x},\psi|y)\wedge\pi(\hat{x},\psi|y))\end{aligned}$$

Thus, $\phi(\hat{x},y)\equiv\phi(\hat{x},p|y)$.

3. Case 3: Let $\phi(\hat{x},y)=\psi(\hat{x},y)\vee\pi(\hat{x},y)$. This case is identical to Case 2 with every $\wedge$ replaced with $\vee$

Therefore, tautological substituion holds, as needed.

$\square$

Theorem 5. *Suppose $\tau(\phi(\hat{x},\gamma))=\alpha$ for all $\hat{x}\in\mathfrak{X}^n$ and π. Let $\gamma\equiv\pi$. Then $\tau(\phi(\hat{x},\pi))=\alpha$*

Proof. We will proceed by induction on the complexity of ϕ. If $\phi(\hat{x},\gamma)=x_i$, then substitution holds trivially. Suppose substitution holds for all formulas of length less than n and $\phi(\hat{x},\gamma)$ has length n. There are three cases (of which the $\wedge$ and the $\vee$ cases are equivalent up to renaming of symbols)

1. Case 1: Let $\phi(\hat{x},\gamma)=\neg\psi(\hat{x},\gamma)$. Note that $\psi(\hat{x},\gamma)$ has length less than n. Thus, $\psi(\hat{x},p)\equiv\psi(\hat{x},q)$, implying that

$$\begin{aligned}\tau(\psi(\hat{x},\gamma))=\tau(\psi(\hat{x},\pi)) &\Rightarrow \neg\tau(\psi(\hat{x},\gamma))=\neg\tau(\psi(\hat{x},\pi))\\ &\Rightarrow \tau(\neg\psi(\hat{x},\gamma))=\tau(\neg\psi(\hat{x},\pi))\\ &\Rightarrow \phi(\hat{x},\gamma)\equiv\phi(\hat{x},\pi)\end{aligned}$$

as needed.

2. Case 2: Let $\phi(\hat{x},\gamma) = \psi(\hat{x},\gamma) \wedge \rho(\hat{x},\gamma)$. Note that ψ and ρ have lenght less than n. Thus, $\psi(\hat{x},\gamma) \equiv \psi(\hat{x},\pi)$ and $\rho(\hat{x},\gamma) \equiv \rho(\hat{x},\pi)$. It follows that $\psi(\hat{x},\gamma) \wedge \rho(\hat{x},\gamma) \equiv \psi(\hat{x},\pi) \wedge \rho(\hat{x},\pi)$, as needed.

3. Case 3: Let $\phi(\hat{x},\gamma) = \psi(\hat{x},\gamma) \vee \rho(\hat{x},\gamma$. This case is identical to Case 2 with every $\wedge$ replaced with $\vee$

Thus, equivalent substituion holds in $\mathbb{F}_4$, as needed.

$\square$

To give substition its full deductive force, we must show first that equivalent formulas imply one another. The soundness of this deduction follows immediately from the definition of validity. We will express this observation in the following lemma:

Lemma 6. *Let ϕ and ψ be equivalent formulas. Then*

$$\frac{\phi}{\therefore \psi}$$

is sound.

Corollary 3. *The deductions*

$$\frac{\phi(\hat{x},y)}{\therefore \phi(\hat{x},\gamma|y)} \quad \textit{and} \quad \frac{\psi(\hat{x},\pi)}{\therefore \psi(\hat{x},\gamma)}$$

are sound when $\tau(\phi(\hat{x},y)) = \alpha$ for all τ and $\pi \equiv \gamma$.

3.3 Material Implication

Formally, we define material implication, denoted here by $\circ$, according to the following axiom scheme[8]:

[8] The axioms presented are a simplification of a system first proposed in Frege (1879) presented in Łukasiewic and Tarski (1930).

1. $\tau\left([s \circ [p \circ q]] \circ [[s \circ p] \circ [s \circ q]]\right) = 1$

2. $\tau\left(p \circ (q \circ p)\right) = 1$

3. $\tau\left([\neg q \circ \neg p] \circ [p \circ q]\right) = 1$

4. For all formulas ϕ and ψ, the rule of inference

$$\begin{array}{r} \phi \\ \phi \circ \psi \\ \hline \therefore \psi \end{array}$$

is sound. We will call this deduction modus ponens (abbreviated MP).

We will say that any connective that satisfies the scheme above models the axioms of the material implication.

Before defining the semantics of the material implication over $\mathbb{F}_4$, we will derive some consequences of the axiom scheme. From the definition of a proof, an axiom can be used at any line of a proof. Thus, we can make the following deduction,

$$\begin{array}{lccc}
1. & [s \circ [p \circ q]] \circ [[s \circ p] \circ [p \circ q]] & \{1\} & P \\
2. & p \circ (q \circ p) & \{2\} & P \\
3. & [p \circ [q \circ p]] \circ [[p \circ q] \circ [p \circ p]] & \{1\} & 1, Sub \\
4. & [[p \circ q] \circ [p \circ p]] & \{1,2\} & 2, 3, MP \\
5. & [[p \circ (q \circ p)] \circ [p \circ p]] & \{1,2\} & 4, Sub \\
6. & p \circ p & \{1,2\} & 5, MP
\end{array}$$

Note that, in lines 1 and 2, since variables are formulas, we may substituted the variables s, p, and q into the axioms. Since the axioms are definitionally 1-tautologies, it follows that $\tau(p \circ p) = 1$. We will use $p \circ p$ as our standard, non-trivial 1-tautology.

In logics over $\mathbb{F}_2$, the connective $\rightarrow$ defined as

$$p \to q := \neg p \vee q$$

for formulas p and q models the axioms of the material implication. Let $\tau(p) = a$ and $\tau(q) = 0$. Then $p \wedge (\neg p \vee q) = a \wedge (\neg a \vee 0) = 1 > 0$. Thus, MP is not sound with $\to$ and $\to$ does not model the axioms of the material implication.

Let's pause to consider the semantics of $\to$ over $\mathbb{F}_2$.[9] The truth table for $\to$ is

$p \to q$	$q = 0$	1
$p = 0$	1	1
1	0	1

From the table, we can see that $\to$ encodes the notion of semantic implication over $\mathbb{F}_2$ by letting an entry of 1 denote "p semantically implies q" while an entry of 0 denotes "p does not semantically implies q". This encoding is guaranteed to have a formula in terms of $\{0, \neg, \wedge, \vee\}$ by the functional completeness of the connectives over $\mathbb{F}_2$. We will encode the notion of validity into $\mathbb{F}_4$ using the connective $\Rightarrow$. Put formally, if $\phi, \psi \in P(\mathbb{F}_4, \mathfrak{C})$, we will let $\tau(\phi \Rightarrow \psi) = 1$ if, and only if, $\tau(\phi) \not> \tau(\psi)$; otherwise $\tau(\phi \Rightarrow \psi) = 0$. Thus, the truth table for $p \Rightarrow q$ is

$p \Rightarrow q$	$q = 0$	a	b	1
$p = 0$	1	1	1	1
a	0	1	1	1
b	0	1	1	1
1	0	0	0	1

By the functional completeness of $\mathfrak{C} \cup \mathbb{F}_4$, there is a formula for $\Rightarrow$ in terms of the connectives over $\mathbb{F}_4$. We will show that $\Rightarrow$ models the axioms of the material implication.

Theorem 6. *The connective $\Rightarrow$ models the axioms of the material implication.*

Proof. The first three axioms can be verified with truth tables as follows:

[9]Equivalently $\mathbb{B}_2$.

$[0 \Rightarrow [p \Rightarrow q]] \Rightarrow [[0 \Rightarrow p] \Rightarrow [p \Rightarrow q]]$	0	a	b	1
0	1	1	1	1
a	1	1	1	1
b	1	1	1	1
1	1	1	1	1

$[a \Rightarrow [p \Rightarrow q]] \Rightarrow [[a \Rightarrow p] \Rightarrow [p \Rightarrow q]]$	0	a	b	1
0	1	1	1	1
a	1	1	1	1
b	1	1	1	1
1	1	1	1	1

$[b \Rightarrow [p \Rightarrow q]] \Rightarrow [[b \Rightarrow p] \Rightarrow [p \Rightarrow q]]$	0	a	b	1
0	1	1	1	1
a	1	1	1	1
b	1	1	1	1
1	1	1	1	1

$[1 \Rightarrow [p \Rightarrow q]] \Rightarrow [[1 \Rightarrow p] \Rightarrow [p \Rightarrow q]]$	0	a	b	1
0	1	1	1	1
a	1	1	1	1
b	1	1	1	1
1	1	1	1	1

$p \Rightarrow (q \Rightarrow p)$	0	a	b	1
0	1	1	1	1
a	1	1	1	1
b	1	1	1	1
1	1	1	1	1

$[\neg q \Rightarrow \neg p] \Rightarrow [p \Rightarrow q]$	0	a	b	1
0	1	1	1	1
a	1	1	1	1
b	1	1	1	1
1	1	1	1	1

We must now show that MP is sound. Consider the truth table:

$p \wedge (p \Rightarrow q)$	$q = 0$	a	b	1
$p = 0$	0	0	0	0
a	0	a	a	a
b	0	b	b	b
1	0	0	0	1

Since, for all p, q, $\tau(p \wedge (p \Rightarrow q)) \not> \tau(q)$, MP is sound over $\Rightarrow$, as needed.

□

Since $\mathbb{F}_4$ has a material implication, we can begin to determine the rules of inference over $\mathbb{F}_4$.

3.4 Derivations for Standard Rules of Inference

We now have a sufficiently developed inferential structure to produce some basic rules of inference over $\mathbb{F}_4$. Many of the inference rules covered here will be familiar. However, we will begin with a rule that differs subtly from its $\mathbb{F}_2$ counterpart.

Theorem 7. *Let $\phi, \psi \in P$ such that $\phi \equiv \psi$. Then the inference rules*

$$\frac{\phi \equiv \psi}{\therefore \phi \Rightarrow \psi} \quad \textit{and} \quad \frac{\phi \equiv \psi}{\therefore \psi \Rightarrow \phi}$$

are sound.

Proof. Since $\phi \equiv \psi$, it follows that $\tau(\phi \equiv \psi) = 1$ and that $\tau(\phi) = \tau(\psi)$ for all τ. Since $\tau(p \Rightarrow p) = 1$ for all τ, it follows that

$$\tau(\phi \Rightarrow \psi) = (\tau(\phi) \Rightarrow \tau(\psi)) = (\tau(\phi) \Rightarrow \tau(\phi)) = 1.$$

Thus, the rule of inference is sound. The converse holds by similar reasoning.

□

Theorem 7 is true in $\mathbb{F}_2$. However, we cannot infer $\phi \equiv \psi$ from $\phi \Rightarrow \psi$ and $\psi \Rightarrow \phi$. Suppose $\tau(\phi) = a$ and $\tau(\psi) = b$ for all τ. Then, $\phi \Rightarrow \psi$ and $\psi \Rightarrow \phi$. But, it is clear that $\phi \not\equiv \psi$.

Since $\phi \wedge \psi = \phi \cdot \psi$ and $\phi \vee \psi = \phi + \psi + \phi \cdot \psi$, it follows that the empty conjunction equals the empty product and the empty disjunction is the sum of the empty sum and the empty product. By convention, the empty product is 1 and the empty sum is 0. Thus, $\bigwedge \emptyset = 1$ and $\bigvee \emptyset = 0 + 1 = 1$.

Theorem 8. *Suppose $\tau(\phi) = 1$ for all τ. Then the inference rule*

$$\frac{}{\therefore \phi}$$

is sound. Put another way, we may deduce a 1*-tautology from the empty set of premises.*

Proof. Since $\tau(\phi) = 1$, it follows that $\phi \equiv (p \Rightarrow p)$. Thus, by Theorem 7, $(p \Rightarrow p) \Rightarrow \phi$. Since $p \Rightarrow p$ is deducible from the empty set of premises, we can deduce ϕ from the empty set of premises by MP.

□

We can now prove two important rules of inference, namely Double Negation and DeMorgan's Law. We will omit the proof for DeMorgan's Law, as it is qualitatively identical to the proof of Double Negation.

Theorem 9. *Let* $\phi \in P$*. Then*

$$\frac{\phi}{\therefore \neg(\neg\phi)} \quad \textit{and} \quad \frac{\neg(\neg\phi)}{\therefore \phi}$$

Proof. By Theorem 1, $\phi \equiv \neg(\neg\phi)$. Thus, by Theorem 8, we can deduce $\phi \equiv \neg(\neg\phi)$ from the empty set of premises. By Theorem 7, we can deduce $\phi \Rightarrow \neg(\neg\phi)$ and $\neg(\neg\phi) \Rightarrow \phi$. Suppose we have ϕ. Then, by MP, we can deduce $\neg(\neg\phi)$. On the other hand, suppose we have $\neg(\neg\phi)$. By MP, we can deduce ϕ. By Lemma 5, the deductions are sound rules of inference, as needed.

□

Theorem 10. *Let* $\phi, \psi \in P$*. Then*

$$\frac{\neg(\phi \wedge \psi)}{\therefore \neg\phi \vee \neg\psi} \quad \textit{and} \quad \frac{\neg\phi \vee \neg\psi}{\therefore \neg(\phi \wedge \psi)}$$

$$\frac{\neg(\phi \vee \psi)}{\therefore \neg\phi \wedge \neg\psi} \qquad \textit{and} \qquad \frac{\neg\phi \wedge \neg\psi}{\therefore \neg(\phi \vee \psi)}$$

A standard rule of deduction in $\mathbb{F}_2$ is simplification which states that from a conjunction we can deduce the conjuncts. This fails over $\mathbb{F}_4$. Suppose, for some truth function τ, we have $\tau(\phi) = a$ and $\tau(\psi) = b$. Then

$$\frac{\phi \wedge \psi}{\therefore \phi}$$

is not sound since $\tau(\phi \wedge \psi) > \tau(\phi)$. Thus, $\mathbb{F}_4$ does not admit simplification. However, $\mathbb{F}_4$ does admit what we will call pseudo-simplification.

Corollary 4. *Suppose* $\Gamma, \phi \wedge \psi \vdash \lambda$*, for* Γ *a set of formulas and* ϕ*,* ψ*, and* λ *formulas. Then* $\Gamma, \phi, \psi \vdash \lambda$.

Proof. By Lemma 4, $\phi, \psi \vdash \phi \wedge \psi$. The corollary follows immediately.

□

We are now in the position to prove the Deduction Theorem, which demonstrates an important property of the $\vdash$ relationship. The proof given here is heavily inspired by the proof given in Church (1970), but is changed to fit our inferential structure.

Theorem 11. *Let* Γ *be a set of formulas and* $\phi, \psi \in P$*. Suppose* $\Gamma, \phi \vdash \psi$*. Then* $\Gamma \vdash \phi \Rightarrow \psi$.

Proof. Let $\pi_1, \pi_2, \pi_3, \ldots, \pi_n$ be a proof of ψ. By the definition of a proof, we may conclude

1. $\pi_n = \psi$,

2. There is some $k \leq n$ such that $\{\pi_1, \pi_2, \ldots, \pi_k\} = \Gamma \cup \{\phi\}$.

Construct the sequence $\{\phi \Rightarrow \pi_i\}_{1\leq i\leq n}$. We will now prove that every element of the sequence can be proved from the elements of Γ. As a consequence, $\phi \Rightarrow \psi$.

Suppose $\psi \in \Gamma$. Then, by substitution of axiom 2, $\psi \Rightarrow (\phi \Rightarrow \psi)$. Thus, by MP, we can deduce $\phi \Rightarrow \psi$. That is, for any premise ψ, we can prove $\phi \Rightarrow \psi$.

Fix $j \in \mathbb{N}$ such that every formula in $\{\phi \Rightarrow \pi_i\}_{i<j}$ has been proved. We will prove $\phi \Rightarrow \pi_k$. Suppose $\phi = \pi_j$. Then, $\phi \Rightarrow \pi_j$. Suppose π_j is an axiom under some substitution of its variables. Then, $\tau(\pi_j) = 1$ and π_j can be deduced from the empty set of premises. By substitution of axiom 2, $\pi_j \Rightarrow (\phi \Rightarrow \pi_j)$; by MP, we deduce $\phi \Rightarrow \pi_j$.

Suppose π_j is deducible by MP from π_m and π_n, for $m, n < j$. Then, without loss of generality, assume π_m is of the form $\pi_n \Rightarrow \pi_j$. Note that, by assumption, we have shown $\phi \Rightarrow \pi_n$ and $\phi \Rightarrow (\pi_n \Rightarrow \pi_j)$. By several substitutions in axiom 1, we deduce $(\phi \Rightarrow (\pi_n \Rightarrow \pi_j)) \Rightarrow ((\phi \Rightarrow \pi_n) \Rightarrow (\phi \Rightarrow \pi_j))$. Thus, by two applications of MP, we deduce $\phi \Rightarrow \pi_j$.

Suppose π_j is deducible from π_m, for $m < j$, by substitution. By assumption, we have deduced $\phi \Rightarrow \pi_m$. Thus, by equivalent substitution, we deduce $\phi \Rightarrow \pi_j$.

Since every element of $\{\phi \Rightarrow \pi_i\}_{1\leq i\leq n}$ can be proved from the premises in Γ, it follows that $\phi \Rightarrow \psi$ can be proved from the premises in Γ. Thus, $\Gamma \vdash \phi \Rightarrow \psi$, as needed.

□

Under $\mathbb{F}_2$, the converse of the Deduction Theorem holds; that is, if $\Gamma \vdash \phi \to \psi$, then $\Gamma, \phi \vdash \psi$. This relation does not necessarily hold in $\mathbb{F}_4$. Suppose $\Gamma \vdash \phi \Rightarrow \psi$ and $\tau\left(\bigwedge \Gamma\right) = \tau(\psi) = a$ and $\tau(\phi)$ for some truth function τ. Then, $\tau\left(\bigwedge \Gamma \wedge \phi\right) = 1$ and $\tau\left(\bigwedge \Gamma \wedge \phi\right) > \tau(\psi)$. Since an argument is a proof only if it is valid, it follows that $\Gamma, \phi \nvdash \psi$. We can avoid this complication by restricting the formula that admit the converse of the deduction theorem to those bearing the relation $\Gamma \cup \{\phi\} \models_{bc} \psi$. We summarize this result with the following corollary

Corollary 5. *The converse of the deduction theorem holds when* $\Gamma \cup \{\phi\} \models_{bc} \psi$.

We will conclude this section by discussing how the Deduction Theorem can be used in a proof. Since a proof is just a valid argument, if we have $\Gamma, \phi \vdash \psi$, it follows that $\tau(\wedge\Gamma) \not> \tau(\phi \Rightarrow \psi)$. Thus, we can use the Deduction Theorem in a way similar to a rule of inference.[10] The procedure for using the Deduction Theorem in a proof will be as follows:

1. List the premises of the argument as usual.

2. Enter in however many antecedents are required to prove the theorem.

3. Once the consequent has been proven, use the Deduction Theorem, cited as DT, to infer successive implications.

We will use this method to deduce transitivity of the material implication. That is, for $\phi, \psi, \gamma \in P$, that the formula $(\phi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \gamma) \Rightarrow (\phi \Rightarrow \gamma))$ is deducible from the empty set of premises.

1.	$\phi \Rightarrow \psi$	$\{1\}$	P (for use of DT)
2.	$\psi \Rightarrow \gamma$	$\{2\}$	P (for use of DT)
3.	ϕ	$\{3\}$	P (for use of DT)
4.	ψ	$\{1, 3\}$	$1, 3, MP$
5.	γ	$\{1, 2, 3\}$	$2, 4, MP$
6.	$\phi \Rightarrow \gamma$	$\{1, 2\}$	$3, 5, DT$
7.	$(\psi \Rightarrow \gamma) \Rightarrow (\phi \Rightarrow \gamma)$	$\{1\}$	$2, 6, DT$
8.	$(\phi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \gamma) \Rightarrow (\phi \Rightarrow \gamma)$	$\{\ \}$	$1, 7, DT$

There are two important obeservations here. First that $\tau((\phi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \gamma) \Rightarrow (\phi \Rightarrow \gamma))) = 1$ for all τ. Note that $\tau((\phi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \gamma) \Rightarrow (\phi \Rightarrow \gamma))) \in \{0, 1\}$ for all τ. For $\tau((\phi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \gamma) \Rightarrow (\phi \Rightarrow \gamma))) = 0$, it must be the case that $\tau(\phi \Rightarrow \psi) = 1$

[10] Strictly speaking, the Deduction Theorem is not a rule of inference since $\Gamma, \phi \vdash \psi$ is not a sentence in the language of $\mathbb{F}_4$.

and $\tau((\psi \Rightarrow \gamma) \Rightarrow (\phi \Rightarrow \gamma)) = 0$. By the same token, for $\tau((\psi \Rightarrow \gamma) \Rightarrow (\phi \Rightarrow \gamma)) = 0$, it must be the case that $\tau(\psi \Rightarrow \gamma) = 1$ and $\tau(\phi \Rightarrow \gamma) = 0$. In short, this means that there is some valuation τ such that $\#(\phi) \leq \#(\psi)$ and $\#(\psi) \leq \#(\gamma)$ but $\#(\phi) > \#(\gamma)$. This is a contradiciton. Therefore, $\tau((\phi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \gamma) \Rightarrow (\phi \Rightarrow \gamma))) = 1$ for all τ.

Second, notice that $\phi \Rightarrow \psi, \psi \Rightarrow \gamma \vdash \phi \Rightarrow \gamma$. It follows by two applications of the Deduction Theorem, $\emptyset \vdash (\phi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \gamma) \Rightarrow (\phi \Rightarrow \gamma))$. Since $\tau(p \Rightarrow q) \in \{0, 1\}$ for all τ and $\tau(\bigwedge \emptyset) = 1$, it follows that $\tau((\phi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \gamma) \Rightarrow (\phi \Rightarrow \gamma))) = 1$. By the same token, it is clear that $\tau((\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \gamma)) \not> \tau(\phi \Rightarrow \gamma)$. Therefore, the following theorem holds

Theorem 12. *Let $\phi, \psi, \gamma \in P(\mathbb{F}_4, \mathfrak{C})$. Then*

$$\begin{array}{r} \phi \Rightarrow \psi \\ \psi \Rightarrow \gamma \\ \hline \therefore \phi \Rightarrow \gamma \end{array}$$

is a sound rule of inference.

3.5 The Soundness and Completeness Theorems

We are now in a position to prove the Soundness and Completeness Theorems. Over $\mathbb{F}_4$, the full Completeness Theorem holds. However, we need to qualify the Soundness Theorem. Consider the following example. Suppose $\tau(\wedge\Gamma) = a$, $\tau(\phi) = a$, and $\Gamma \vdash \phi$. If we choose some ψ such that $\tau(\psi) = b$, there exists a proof of ϕ from the set $\Gamma \cup \{\psi\}$ (namely, the proof of ϕ from Γ alone). However $\Gamma \cup \{\phi\} \not\models_{bc} \phi$ since $\tau(\bigwedge(\Gamma \cup \{\phi\})) = 1$. Thus, we must restrict Γ to only those formulas that are required to prove ϕ; that is, we must restrict our consideration to any $\Sigma_i(\Gamma, \phi)$. Once we make this restriction, the Soundness theorem states:

Theorem 13. *Let Γ be a set of formulas and ϕ a formula. If $\Gamma \vdash \phi$ then there exists a $\Sigma_i(\Gamma, \phi) \subset \Gamma$ such that $\Sigma_i(\Gamma, \phi) \models_{bc} \phi$.*

Proof. Suppose $\Gamma \vdash \phi$. Then, there is a $\Sigma_i(\Gamma, \phi) \subset \Gamma$ such that the premises in $\Sigma_i(\Gamma, \phi) \vdash \phi$. Then there is a finite sequence of inferences $\{i_n\}$ that, when applied to the formulas in $\Sigma_i(\Gamma, \phi)$, yield ϕ. Since each i_n is sound, $i_{n+1} \circ i_n$ is sound. Thus, $\tau\left(\bigwedge \Sigma_i(\Gamma, \phi)\right) \not> \tau(\phi)$ for all τ. Therefore, $\Sigma_i(\Gamma, \phi) \models_{bc} \phi$.

□

We should note that, for Γ and ϕ such that $\Gamma \vdash \phi$, if we construct $\Gamma' = \Gamma \cup \Theta$ such that Θ consists of only 0 or 1-tautologies, then $\Gamma' \vdash \phi$ implies $\Gamma' \models \phi$ since adding a 1-tautology will not affect the value of the conjunction, thereby not affecting if $\Gamma' \models \phi$, and adding a 0-tautology will change the evaluation to 0, leaving $\Gamma' \models \phi$ as trivially true. Thus, the reduction of Γ to $\Sigma_i(\Gamma, \phi)$ is necessary only because $a \wedge b = 1$.

Theorem 14. *Let Γ be a set of formulas and ϕ be a formula. If $\Gamma \models_{bc} \phi$ then $\Gamma \vdash \phi$.*

Proof. Let $\Gamma \models_{bc} \phi$. Then $\tau\left(\bigwedge \Gamma\right) \not> \tau(\phi)$ for all truth functions ϕ. Observe that $\tau\left(\bigwedge \Gamma \Rightarrow \phi\right) = 1$. That is, $\bigwedge \Gamma \Rightarrow \phi$ is a 1-tautology. Since all 1-tautologies can be deduced from the empty set of premises, it follows that $\vdash \bigwedge \Gamma \Rightarrow \phi$. Thus, by the converse of the Deduction Theorem, $\bigwedge \Gamma \vdash \phi$ (this holds since the empty conjunction is 1). By Theorem 4, $\Gamma \vdash \bigwedge \Gamma$. Thus, by the transitivity of $\vdash$, $\Gamma \vdash \phi$, as needed.

□

A useful consequence of the Completeness Theorem is that we can rewrite any 1-tautology with $\Rightarrow$ as the major connective as a sound rule of inference. Let $\phi, \psi \in P(\mathbb{F}_4, \mathfrak{C})$ such that $\phi \Rightarrow \psi$ be a 1-tautology. By the definition of $\Rightarrow$, we know $\phi \models_{bc} \psi$. Therefore, $\phi \vdash \psi$. It follows immediately that

$$\frac{\phi}{\therefore \psi}$$

is a sound rule of inference.

4 Some Considerations for Further Development

Having proved the Soundness and Completeness of $\mathbb{F}_4$, we conclude by discussing how the study of $\mathbb{F}_4$ might progress. We will begin with a discussion of some of the classic problems posed for two-valued propositional logics. One such problem asks whether there is an effective procedure for generating all tautologies over $\mathbb{F}_4$. There are two possible forms this procedure could take. The first is a strictly logic form. By Corollay 2, this is equivalent to generating all polynomials $p(x)$ over $\mathbb{F}_4$ that satisfy $p(x) + 1 = 0$.

Another such problem is the formal introduction of predicates and quantifiers. With the introduction of predicates and quantifiers to $\mathbb{F}_4$, we can define models and cary out model theoretic operations over $\mathbb{F}_4$, effectively allowing us to talk about mathematics over $\mathbb{F}_4$. With the introduction of quantifiers, we can also investigate whether predicate calculus over $\mathbb{F}_4$ is sound and complete.

We may also consider questions unique to $\mathbb{F}_4$. One benefit of inferences over $\mathbb{F}_2$ is that, if there exists a proof of ϕ from Γ and the formulas in Γ are all true, ϕ is also true. Since $\mathbb{F}_4$ has more than two truth values, there can be no such procedure this simple over $\mathbb{F}_4$. One method for determining the truth value of the conclusion is to determine the truth values of the premises, determine which truth functions allow that set of valuations, and apply those truth functions to the conclusion. However, this is a labor-intensive process. Thus, it would be much more efficient to develop an algebra of truth values, in which one needs merely to state the truth values of the premises and follow the rules of inference to find the truth value of the conclusion.

Finally, we can consider further extensions of $\mathbb{F}_2$. Consider the procedure used to construct $\mathbb{F}_4$. We constructed $\mathbb{F}_4$ as the quotient field of $\mathbb{F}_4[x]$ by "dividing out" by the ideal $\langle x^2 + x + 1 \rangle$, where 1 is the generator of $\mathbb{F}_2$. Suppose we repeat this construction. We seek to construct a chain of fields, each embedded in all of its successors, of the general form $\mathbb{F}_{q_n} = \mathbb{F}_{q_{n-1}}[x]/(x^2 + x + \gamma_{n-1})$, where q_n is the size of the field and γ_{n-1} is a generator of $\mathbb{F}_{q_{n-1}}$.

Theorem 15. *The general element* $\mathbb{F}_{q_{n-1}} = \mathbb{F}_{2^{2^{n-2}}}[x]/(x^2 + x + \gamma_{n-1}) = \mathbb{F}_{2^{2^{n-1}}}$, *where* i *is the place of the field in the chain and* γ_{n-1} *is the generator of* $\mathbb{F}_{2^{2^{n-2}}}$.

Proof. We proceed by induction. We can see that $[\mathbb{F}_4 : \mathbb{F}_2] = 2^{2-1} = 2$ since $\mathbb{F}_4 = \{ u + vx \mid u, v \in \mathbb{F}_2 \}$. Let $\mathbb{F}_{q_n}$ be a member of the chain. Suppose $[\mathbb{F}_{q_n} : \mathbb{F}_2] = 2^{n-1}$. We can see that

$$\begin{aligned} [\mathbb{F}_{q_{n+1}} : \mathbb{F}_2] &= [\mathbb{F}_{q_{n+1}} : \mathbb{F}_{q_n}] \cdot [\mathbb{F}_{q_n} : \mathbb{F}_2] \\ &= [\mathbb{F}_{q_{n+1}} : \mathbb{F}_{q_n}] \cdot 2^{n-1}. \end{aligned}$$

It is sufficient to show that $[\mathbb{F}_{q_{n+1}} : \mathbb{F}_{q_n}] = 2$. Since $\mathbb{F}_{q_{n+1}} = \mathbb{F}_{q_n}[x]/(x^2 + x + \gamma_n)$, by Euclidean Division, the polynomails are equivalent to linear terms with coefficients in $\mathbb{F}_{q_n}$; that is,

$$\mathbb{F}_{q_{n+1}} = \{ u + v\gamma_{n+1} \mid u, v \in \mathbb{F}_{q_n} \}.$$

Thus, $[\mathbb{F}_{q_{n+1}} : \mathbb{F}_{q_n}] = 2$, as needed. By a theorem from (Serre 1973, p. 3), $|\mathbb{F}_{q_n}| = 2^{2^{n-1}}$. Therefore, $\mathbb{F}_{q_n} = \mathbb{F}_{2^{2^{n-1}}}$.

□

From this theorem, we can see that the chain proceeds as

$$\mathbb{F}_2 \hookrightarrow \mathbb{F}_4 \hookrightarrow \mathbb{F}_{16} \hookrightarrow \mathbb{F}_{256} \hookrightarrow \mathbb{F}_{65,536} \hookrightarrow \dots$$

We may investigate several aspects of the elements of the chain. Suppose we define $\wedge$, $\vee$, and $\neg$ over $\mathbb{F}_{q_n}$ as we have for $\mathbb{F}_2$ and $\mathbb{F}_4$. To show that an inferential system over $\mathbb{F}_{q_n}$ with connectives $\wedge$, $\vee$, and $\neg$ is sound and complete, we must show that $\mathbb{F}_{q_n}$ admits functional completeness, a partial order, and a definition of semantic implication. Supposing that a single variable formula $\chi_\alpha(p)$ can be constructed such that $\chi_\alpha(p) = 1$ when $\tau(p) = \alpha$

and 0 otherwise for every $\alpha \in F_{q_n}$, the proof given for functional completeness will hold for $\mathbb{F}_{q_n}$. It remains to find each $\chi_\alpha(p)$.

Since $q_n = 2^{2^n}$, there exists a Boolean algebra $\mathbb{B}_{q_n}$ such that $|\mathbb{B}_{q_n}| = |\mathbb{F}_{q_n}|$. Thus, to induce a partial order over F_{q_n}, we must show the diagram

$$\begin{array}{ccc} \mathbb{B}_2 & \xrightarrow{f} & \mathbb{B}_4 \\ \downarrow h & & \downarrow \sigma \\ \mathbb{F}_2 & \xrightarrow{g} & \mathbb{F}_4 \end{array}$$

commute with f, g, h, and σ defined as they in Section 3.1.

Once $\mathbb{F}_{q_n}$ has been given a partial order, we need to define semantic implication. As was shown in Section 3.1, the relations $\models_{card}$ and $\models_{bc}$ are equivalent over $\mathbb{F}_4$. However, over $\mathbb{F}_{16}$, they are not. As we can see in Fig. 1d, the element $\{1,2\} \models_{bc} \{3\}$, since $\{1,2\} \not> \{3\}$, but $\{1,2\} \not\models_{card} \{3\}$, since $|\{3\}| < |\{1,2\}|$. By similar reasoning, $\models_{card}$ and $\models_{bc}$ are not equivalent in any $\mathbb{F}_{q_n}$ beyond $\mathbb{F}_4$. Thus, to determine how to define semantic implication over $\mathbb{F}_{q_n}$, a comparative study of $\models_{card}$ and $\models_{bc}$ must be conducted.

Supposing each $\mathbb{F}_{q_n}$ admits functional completeness, can be imbued with a partial order, and has a clear notion of semantic implication, many of the proofs presented in this paper will hold in $\mathbb{F}_{q_n}$. By functional completeness, we can define $\Rightarrow$ such that it models the axiom scheme of the material implication, from which we can prove the Deduction Theorem. Furthermore, if Corollary 5 can be proved over $\mathbb{F}_{q_n}$, then the Soundness and Completeness Theorems follow as above. That is, each $\mathbb{F}_{q_n}$ is imbued with a well-behaving inferential system if we can prove the set $\mathbb{C} \cup \mathbb{F}_{q_n}$ is functionally complete, imbue $\mathbb{F}_{q_n}$ with a Boolean partial order, and find a suitable notion of semantic implication.

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