ABSTRACT

IMPROVED TSUNAMI MODELING VIA q-ADVANCED SPECIAL FUNCTIONS

by

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This thesis studies q-advanced functions that are used as forcing terms in the forced wave equation and the Korteweg-de Vries equation in modeling tsunamis. The model improves existing tsunami models and is compared to data collected from the 2011 Japanese tsunami. The main results show that one can modify existing q-advanced functions to obtain more accuracy in the forcing of a tsunami, and, in turn, gain a more accurate model of a tsunami propagation and its approach to land.

IMPROVED TSUNAMI MODELING VIA q-ADVANCED SPECIAL FUNCTIONS

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CHAPTER 1: Introduction

A tsunami is an incredible, unpredictable, and deadly mass of moving water that can cause an extreme amount of damage. Unlike the movies, a tsunami is barely noticeable as it emanates from its source and heads out through the ocean or sea. People can be sitting in a boat at sea and never notice a very large tsunami rolling under them as a tsunami may only raise the ocean a couple meters. However, the ocean may rise a couple meters for several minutes as the tsunami passes through. It is only when the tsunami nears land that its deadly form is truly noticeable, and by then it is typically too late for those who see it.

One way for a tsunami to form is from an earthquake [13], [14]. Earthquakes are the result of two or more tectonic plates moving against each other [10]. One specific type of earthquake is when one tectonic plate is moving under another. Due to friction, the plates hold onto each other until there is so much built up energy that they give way violently and release all that energy in a springboard type movement. If the site of the earthquake happens to be under water, it may push the water in an upward motion or drag it in a downward motion creating a resulting tsunami.

Because a shock wave travels through the earth faster than through the water, it can be felt hundreds of miles away long before a following tsunami. This earthquake ripples through the ocean floor disturbing the water above the disturbance, resulting in a precursor disturbance wave that can be measured by buoys floating on the water [9]. The data collected by sensors on these buoys can be used to accurately predict the size of the following tsunami and when the tsunami will hit land. This prediction can be used to give people an advanced warning of tsunamis, and possibly save many lives.

The data collected from the impact of the precursor waves on the buoys is an imitation of the forcing done by the earthquake. We will use new multiplicatively advanced functions in conjunction with the data from the precursor wave to model the forcing on the tsunami. There are then three parts that need to be considered [9]. The first is how to use this forcing to predict how a tsunami will form. For this, the wave equation can be used [9], [11]. Once the tsunami is formed, it will travel in the ocean for a period of time. For this second part, the Korteweg-de Vries equation (KdV equation) will be used as a propagation model [2], [9]. The third and final part of the tsunami is the approach to land [1], [9]. A run-up equation will be defined for this last part of the model.

In this thesis, we derive each equation associated with each stage of our model. We then describe, in detail, the numerical scheme associated with each equation. We will use a modified version of a forcing term previously used to model tsunamis [9]. The new forcing term is designed to match the 2011 Japanese tsunami precursor wave with greater accuracy This model, with the new forcing term, is then used to model the tsunami's course until it runs into land at Wake Island. The model will then be compared to actual data collected from at Wake Island.

CHAPTER 2: The Wave Equation

2.1 1-Dimensional Wave Equation

A forced wave equation is a wave equation used where there is a force acting upon a fluid, creating waves. We first derive the 1-dimensional wave equation which is a linear partial differential equation which we use to model tsunamis in their first stage of existence. Let $H(x,t)$ be the wave height at position x and time t.

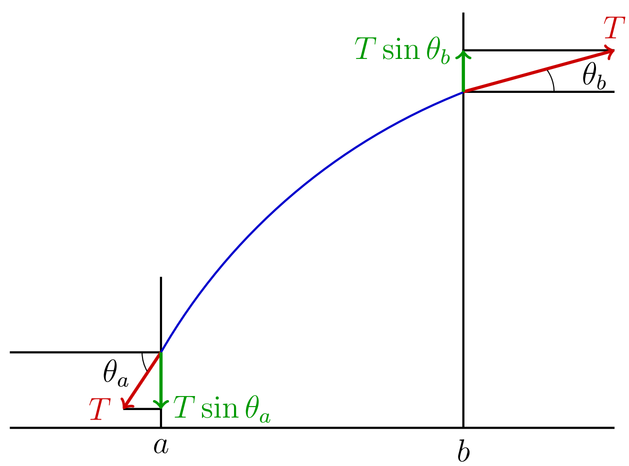

Figure 2.1: Diagram for tension T

Assume that the amplitude of the wave is not large, and furthermore assume the spacial gradient $\nabla H = \frac{\partial H}{\partial x} = H_x$ is small with $|\nabla H|^2$ negligible. Also assume there is no horizontal tension in the wave. Let $[a,b]$ be an arbitrary small interval as shown in Figure 2.1. Then the net vertical tension of the wave is

$$T\sin\theta_b - T\sin\theta_a \ , \tag{2.1}$$

where θ_b is the angle between the outer tangent vector $(1, \frac{\partial H}{\partial x})$ at $x = b$ and horizontal and θ_a is the angle between the outer tangent vector $(-1, -\frac{\partial H}{\partial x})$ at $x = a$ and horizontal. Now we can estimate $\sin\alpha \approx \tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ in (2.1) to get the net vertical tension

$$T\tan\theta_b - T\tan\theta_a = TH_x(b,t) - TH_x(a,t) = \int_a^b [TH_x(x)]_x dx \ ,$$

to order $O(|\nabla H|^2)$.

Note that the approximate $\sin\theta = \frac{\frac{\partial H}{\partial x}}{\sqrt{1+(\frac{\partial H}{\partial x})^2}} \approx \tan\theta = \frac{\frac{\partial H}{\partial x}}{1} = \frac{\partial H}{\partial x}$ is consistent with $|\frac{\partial H}{\partial x}|^2 = |\nabla H|^2 \approx 0$. Looking now at the force equation $F = ma$ we can use the density function $\rho(x) = \rho$ to get

$$F = \rho(x)dx \ H_{tt}(x,t) \ , \tag{2.2}$$

where $\rho(x)dx$ is mass of an element at x and $H_{tt}(x,t)$ its acceleration. The integral of (2.2) over $[a,b]$ is the total force, and thus we obtain the homogeneous wave equation

$$\int_a^b \rho(x)H_{tt}(x,t)dx = \int_a^b (TH_x(x))_x dx \ ,$$

where the total force = the net vertical force. Adding in an acceleration term $f(x,t)$ due to an external forcing gives

$$\int_a^b \rho(x)H_{tt}(x,t)dt = \int_a^b (TH_x(x))_x dx + \int_a^b \rho f(x,t)dx \ . \tag{2.3}$$

We can assume that the density function ρ and the tension T are both constant. Note that the integrands of (2.3) must be equal since $[a,b]$ is arbitrarily small, thus

$$\rho H_{tt} = T(H_{xx}) + \rho f$$

$$H_{tt} = \frac{T}{\rho} H_{xx} + f \ ,$$

and we have derived the wave equation in one dimension. Namely, for $c^2 = \frac{T}{\rho}$, where c is the speed of the wave.

$$H_{tt} - c^2 H_{xx} = f \ . \tag{2.4}$$

2.2 The Wave Equation in Higher Dimensions.

We can derive the wave equation in higher dimensions by using an analogue of the one dimensional case. Let R be an arbitrarily small region in $\mathbb{R}^n$ with $\vec{n}$ a unit outer normal to the boundary ∂R. Let $H(x_1, x_2, ..., x_n, t)$ be the height of the wave at time t in the $\vec{e}_{n+1}$ direction above $\vec{x} = (x_1, x_2, ..., x_n)$, where e_{n+1} is a unit vector perpendicular to the $x_1, x_2, ..., x_n$ hyperplane. Then $\nabla H \cdot \vec{n}$ is the directional derivative of H in the direction $\vec{n}$.

We next compute the net vertical tension on the surface $H(R)$ above R along with its boundary $H(\partial R)$ above the boundary ∂R. Let $\vec{N}$ be an outer unit normal to $H(\partial R)$ in the tangent space to $H(R)$. Let T be a scalar measuring tension per unit volume in $H(\partial R)$. The tension acting on a boundary volume element is then $T\vec{N}$. The net vertical tension is obtained by first computing the vertical component of the tension vector $T(\vec{N})$ in the direction $\vec{e}_{n+1}$, namely $T(\vec{N} \cdot \vec{e}_{n+1})$, and second, integrating $T(\vec{N} \cdot \vec{e}_{n+1})$ on $H(\partial R)$

$$\int_{H(\partial R)} \left(T \left(\vec{N} \cdot \vec{e}_{n+1} \right) \right) dV_{n-1} \ , \tag{2.5}$$

where T is the tension per unit boundary volume. We estimate (2.5) up to order $|\nabla H|^2$ terms by noting that if $\vec{n}$ is an outer unit normal to ∂R then $\vec{N} \approx (\vec{n}, \nabla H \cdot \vec{n})$.

This follows because if $\vec{x}$ is any tangent vector to ∂R then $(\vec{x}, \nabla H \cdot \vec{x})$ is tangent to $H(\partial R)$, resulting in

$$\begin{aligned} (\vec{n}, \nabla H \cdot \vec{n}) \cdot (\vec{x}, \nabla H \cdot \vec{x}) &= \vec{n} \cdot \vec{x} + (\nabla H \cdot \vec{n})(\nabla H \cdot \vec{x}) \\ &= 0 + (\nabla H \cdot \vec{n})(\nabla H \cdot \vec{x}) \\ &\approx 0 \qquad \text{up to order } \; O(|\nabla H|^2) \; . \end{aligned} \tag{2.6}$$

Furthermore $(\vec{n}, \nabla H \cdot \vec{n})$ is perpendicular to the normal vector to the graph of H, $(\nabla H, -1)$, as

$$(\vec{n}, \nabla H \cdot \vec{n}) \cdot (\nabla H, -1) = \nabla H \cdot \vec{n} - \nabla H \cdot \vec{n} = 0 \; .$$

Thus $(\vec{n}, \nabla H \cdot \vec{n})$ lies in the tangent surface to $H(R)$, perpendicular to $H(\partial R)$ (to order $O(|\nabla H|^2)$). Now,

$$|\,(\vec{n}, \nabla H \cdot \vec{n})\,| = \sqrt{\vec{n} \cdot \vec{n} + (\nabla H \cdot \vec{n})^2} = \sqrt{1 + (\nabla H \cdot \vec{n})^2}$$

Thus the unit vector

$$\frac{(\vec{n}, \nabla H \cdot \vec{n})}{\sqrt{1 + (\nabla H \cdot \vec{n})^2}} \approx \frac{(\vec{n}, \nabla H \cdot \vec{n})}{\sqrt{1}} = (\vec{n}, \nabla H \cdot \vec{n}) \quad \text{to order } \; O(|\nabla H|^2)$$

hence we use $(\vec{n}, \nabla H \cdot \vec{n})$ in place of $\vec{N}$ in (2.5) to obtain

$$\begin{aligned} \int_{H(\partial R)} T \left(\vec{N} \cdot \vec{e}_{n+1} \right) \, dV_{n-1} &\approx \int_{H(\partial R)} T \, (\vec{n}, \nabla H \cdot \vec{n}) \cdot \vec{e}_{n+1} \, dV_{n-1} \\ &= \int_{H(\partial R)} T \, (\nabla H \cdot \vec{n}) \; dV_{n-1} \; . \end{aligned} \tag{2.7}$$

Again, under the assumption that $|\nabla H|^2$ is small

$$dV_{n-1} \;\text{ on }\; H(\partial R) \;\; \approx \;\; dV_{n-1} \;\text{ on }\; \partial R$$

and (2.7) becomes

$$\int_{\partial R} T \nabla H \cdot \vec{n} \; dV_{n-1}$$

which by the divergence theorem gives

$$T\int_{\partial R} div(\nabla H)\ dV_{n-1}$$

as the net vertical tension on the surface $H(R)$ above R. We assume any horizontal tension is negligible. Then the total force on the element $H(R)$, $\int_R \rho H_{tt}(\vec{x},t)dV_n$, should be the sum of the net vertical tension and any external forcing due to f. Thus

$$\int_R \rho H_{tt}(\vec{x},t)dV_n = T\int_R div(\nabla H)dV_n + \int_R \rho f(\vec{x},t)dV_n\ . \tag{2.8}$$

Since (2.8) holds, $\int_R \rho H_{tt}(\vec{x},t)dV_n = \int_R \left[T div(\nabla H) + \rho f(\vec{x},t)\right] dV_{n-1}$ for all R we have

$$\rho H_{tt}(\vec{x}) = T div(\nabla H) + \rho f(\vec{x})\ ,$$

where

$$\begin{aligned} H_{tt}(\vec{x}) &= \frac{T}{\rho} div(\nabla H) + f(\vec{x}) \\ &= \frac{T}{\rho}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\right)\cdot\left(\frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2}, ..., \frac{\partial H}{\partial x_n}\right) + f(\vec{x}) \\ &= \frac{T}{\rho}(\nabla\cdot\nabla H) + \frac{f}{\rho} = \frac{T}{\rho}\left(\frac{\partial^2 H}{\partial x_1^2} + \frac{\partial^2 H}{\partial x_2^2} + ... + \frac{\partial^2 H}{\partial x_n^2}\right) + f(\vec{x})\ . \end{aligned}$$

Thus we have obtained the wave equation,

$$H_{tt} - c^2(H_{x_1x_1} + H_{x_2x_2} + ... + H_{x_nx_n}) = f(\vec{x})\ , \tag{2.9}$$

in dimension n, where $c^2 = \frac{T}{\rho}$ and c is the speed of the wave.

CHAPTER 3: Korteweg-de Vries Equation

3.1 Introduction

The Korteweg-de Vries, or KdV, equation gets its name from two men named Diederik Korteweg and Gustav de Vries who first published a paper about the equation in 1895 [4]. Although the equation is named after them, there were experiments done by Scott Russel that relate to the KdV equation as early as 1834, long before Korteweg and de Vries published their paper.

The KdV equation is a wave equation first used to model solitary waves flowing in a channel. The KdV equation is a variant of the wave equation that incorporates an extra non-linear term as well as a dispersion term. These non-linear and dispersion terms are negligible in the first stage of a tsunami, hence, the wave equation is used in the early stage. These non-linear and dispersion terms become more significant as the tsunami flows for some time t over the relatively constant depth of the ocean. Hence the KdV equation is used at this stage.

3.2 1-Dimensional Korteweg-de Vries Equation

We now derive the KdV, equation. We expand on the work of [3]. We can first set up our system where z is our vertical direction, and x is our horizontal direction. The velocity of the fluid is $\vec{v} = (\frac{dx}{dt}, \frac{dz}{dt})$. We must first assume that the fluid is irrotational, that is,

$$\nabla \times \vec{v} = 0 \ . \tag{3.1}$$

This implies the existence of a potential function ϕ, with $\vec{v} = (\phi_x, \phi_z)$ since our region is assumed to be simply connected. Thus $\vec{v} = \nabla\phi$. We next assume that the fluid is incompressible, namely,

$$\nabla \cdot \vec{v} = div\ \vec{v} = 0\ . \tag{3.2}$$

And thus

$$0 = \nabla \cdot \nabla \phi = \phi_{xx} + \phi_{zz}\ . \tag{3.3}$$

So the potential function is harmonic.

We next assume the density function ρ of the fluid is constant. Thus

$$\nabla \rho = 0 \qquad \rho_t = 0\ . \tag{3.4}$$

Now (3.4) together with (3.2) imply the conservation of mass, namely,

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0\ . \tag{3.5}$$

We remark that (3.5) holds more generally (without the assumption of (3.4) and (3.2)) via the statement

$$\frac{\partial}{\partial t} \int_R \rho\ dV = - \int_{\partial R} \rho \vec{v} \cdot \vec{n}\ dS \tag{3.6}$$

where the rate of change of mass in a region R equals the rate of flow of mass into the region R across the boundary ∂R. The divergence theorem applied to the flux integral in (3.6) then implies (3.5) in general.

In addition to conservation of mass in (3.5), one has Euler's equation [5],

$$\frac{\partial}{\partial t}(\rho \vec{v}) + \vec{v} \cdot \nabla(\rho \vec{v}) = -\nabla P + \rho \vec{f}\ , \tag{3.7}$$

which under the assumptions (3.2) and (3.4) simplifies to

$$\rho\frac{\partial}{\partial t}(\vec{v}) + \rho\vec{v}\cdot\nabla\vec{v} = -\nabla P + \rho\vec{f}\ , \tag{3.8}$$

where P is the internal pressure and ρf is any external forcing effect. Note that the basic principal behind Euler's Equation is the force law. In our case f will be acceleration due to gravity $\vec{f} = -g(0,1)$ yielding

$$\vec{f} = -g\nabla z\ . \tag{3.9}$$

Thus,

$$\frac{\partial}{\partial t}(\vec{v}) + \vec{v}\cdot\nabla\vec{v} = -\frac{\nabla P}{\rho} - g\nabla z\ , \tag{3.10}$$

where

$$\vec{v}\cdot\nabla\vec{v} = (\vec{v}\cdot\nabla v_1, \vec{v}\cdot\nabla v_2)\ .$$

Now, by Lemma 4.1 on page 47 below,

$$\vec{v}\cdot\nabla\vec{v} = \frac{1}{2}\nabla\left(|\vec{v}|^2\right) - \vec{v}\times\nabla\times\vec{v} \tag{3.11}$$

which becomes under (3.1)

$$\vec{v}\cdot\nabla\vec{v} = \frac{1}{2}\nabla\left(|\vec{v}|^2\right)\ .$$

And so (3.10) reduces to

$$\frac{\partial}{\partial t}(\vec{v}) + \frac{1}{2}\nabla\left(|\vec{v}|^2\right) = -\frac{\nabla P}{\rho} - g\nabla z\ ,$$

or equivalently, using the fact that $\vec{v} = \nabla\phi$,

$$\nabla\left[\phi_t + \frac{1}{2}|\nabla\phi|^2 + \frac{P}{\rho} + gz\right] = 0\ .$$

Thus

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + \frac{P}{\rho} + gz = B(t)\ . \tag{3.12}$$

At the surface $z = h + aH(x,t)$, where a is the amplitude of the wave and h is the undisturbed water level, the pressure P vanishes (since no water lies above the surface to create pressure). Thus (3.12) reduces to

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g(h + aH) = B(t)$$

or

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + gaH = B(t) - gh\ . \tag{3.13}$$

On the bottom, there should be no vertical component to the velocity, so

$$\phi_z = \frac{dz}{dt} = 0 \qquad \text{at } z = 0\ . \tag{3.14}$$

Finally, at the surface $z = h + aH$, by differentiating with respect to t and using the chain rule, one has

$$\frac{dz}{dt} = \phi_z = a\nabla H \cdot \nabla\phi + aH_t\ . \tag{3.15}$$

Thus the equations governing our fluid are (3.3) , (3.13), (3.14), and (3.15), which can be written as

$$\phi_{xx} + \phi_{zz} = 0 \qquad \forall x, z, t \qquad 0 \le z \le h + aH(x,t)\ \ , \tag{3.16}$$

$$\phi_z = aH_x\phi_x + aH_t \quad \text{at} \quad z = h + aH \ , \tag{3.17}$$

$$\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + gaH = B(t) - gh \quad \text{at} \quad z = h + aH, \tag{3.18}$$

and

$$\phi_z = 0 \qquad at \quad z = 0 \quad . \tag{3.19}$$

We deploy these to obtain the KdV equation.

We see by Remark 3.1 below that the solution to the linearized versions of (3.16)-(3.19) is of form (3.73) which is expressed as $c(k, z) \cdot \cos(k[x - \sqrt{gh}t])$. This exhibits this solution as a wave moving with velocity $\sqrt{gh}$. Furthermore, in a tsunami setting we should expect a large wavelength λ, and that for amplitude a above the depth of the ocean h, both $\left(\frac{a}{h}\right)$ and $\left(\frac{h}{\lambda}\right)$ should be reasonably small. We retain these assumptions when looking at the non-linear equations from (3.16)-(3.19).

We now record the effects on (3.16), (3.17), (3.18), and (3.19) of a series of successive change in variables to obtain a dimensionless system. First, we can use scaled variables. Let

$$\bar{x} = \frac{x}{\lambda} \ , \quad \bar{z} = \frac{z}{h} \ , \quad \bar{\phi} = \frac{h \ \phi}{\lambda a\sqrt{gh}} \quad \text{and} \quad \bar{t} = \frac{t\sqrt{gh}}{\lambda} \ . \tag{3.20}$$

By Proposition 3.2 , equations (3.16), (3.17), (3.18), and (3.19) transform under (3.20) to:

$$0 = \epsilon^2\bar{\phi}_{\bar{x}\bar{x}} + \bar{\phi}_{\bar{z}\bar{z}} \tag{3.21}$$

$$\text{at} \quad \bar{z} = 1 + \alpha H \quad \longrightarrow \quad \bar{\phi}_{\bar{z}} = \epsilon^2\left\{\alpha\bar{\phi}_{\bar{x}}H_{\bar{x}} + H_{\bar{t}}\right\} \tag{3.22}$$

$$\text{at } \bar{z} = 1 + \alpha H \quad \longrightarrow \quad \bar{\phi}_{\bar{t}} + \frac{1}{2}\alpha\left\{\left(\bar{\phi}_{\bar{x}}\right)^2 + \frac{1}{\epsilon^2}\left(\bar{\phi}_{\bar{z}}\right)^2\right\} + H = \frac{\tilde{B}(\bar{t}) - gh}{ga} \tag{3.23}$$

$$\text{at } \bar{z} = 0 \quad \longrightarrow \quad \bar{\phi}_{\bar{z}} = 0 \tag{3.24}$$

for $\epsilon = \frac{h}{\lambda}$, $\alpha = \frac{a}{h}$, and $\tilde{B}(\bar{t}) = B\left(\frac{\lambda}{\sqrt{gh}} \cdot \bar{t}\right) = B(t)$.

Next we can incorporate $\frac{\tilde{B}(\bar{t}) - gh}{ag}$ into the potential $\bar{\phi}$ in (3.23) by taking

$$\hat{B}(\bar{t}) = \int_0^{\bar{t}} \frac{\tilde{B}(s) - gh}{ga} ds$$

and letting $(\bar{\phi}_{new}) = \bar{\phi} - \hat{B}(\bar{t})$. Then all spacial derivatives of $(\bar{\phi}_{new})_{\bar{s}} = \bar{\phi}_{\bar{s}}$ (for $\bar{s} = \bar{x}, \bar{z}$) and one has $(\bar{\phi}_{new})_{\bar{t}} = \bar{\phi}_{\bar{t}} - \hat{B}_{\bar{t}} = \bar{\phi}_{\bar{t}} - \left[\frac{\tilde{B}(\bar{t}) - gh}{ah}\right]$. Thus from (3.21) we have

$$0 = \epsilon^2 (\bar{\phi}_{new})_{\bar{x}\bar{x}} + (\bar{\phi}_{new})_{\bar{z}\bar{z}} \ , \tag{3.25}$$

and, at the surface, from (3.22) we have

$$(\bar{\phi}_{new})_{\bar{z}} = \epsilon^2 \left\{\alpha(\bar{\phi}_{new})_{\bar{x}} H_{\bar{x}} + H_{\bar{t}}\right\} \ . \tag{3.26}$$

Similarly, using (3.23) at the surface, we have

$$(\bar{\phi}_{new})_{\bar{t}} + \frac{1}{2}\alpha\left\{\left[(\bar{\phi}_{new})_{\bar{x}}\right]^2 + \frac{1}{\epsilon^2}\left[(\bar{\phi}_{new})_{\bar{z}}\right]^2\right\} + H = 0 \ , \tag{3.27}$$

where the net effect is that the new potential $(\bar{\phi}_{new})_{\bar{t}}$ makes (3.27) a homogeneous version of (3.23). At the bottom using (3.24) we get

$$(\bar{\phi}_{new})_{\bar{z}} = 0 \qquad \text{at } \bar{z} = 0 \ . \tag{3.28}$$

So we can drop all subscripts *new* from here on, and we assume $\bar{\phi}_{new} = \bar{\phi}$.

We proceed with the next change of variables to obtain dimensionless equations: let

$$X = \frac{\alpha^{1/2}}{\epsilon}\left(\bar{x} + (\alpha - 1)\bar{t}\right) \quad , \quad \tau = \frac{\alpha^{3/2}}{\epsilon}\bar{t} \quad , \quad \psi = \frac{\alpha^{1/2}}{\epsilon}\bar{\phi} \ \text{ and } \ Z = \bar{z} \ . \tag{3.29}$$

By Proposition 3.3, equations (3.25), (3.26), (3.27), and (3.28) transform under (3.29) to

$$0 = \alpha\psi_{XX} + \psi_{ZZ} \tag{3.30}$$

$$\psi_Z = \alpha^2\psi_X H_X + (\alpha^2 - \alpha)H_X + \alpha^2 H_\tau \qquad \text{at} \qquad Z = 1 + \alpha H \tag{3.31}$$

$$\alpha\psi_X - \psi_X + \alpha\psi_\tau + \frac{1}{2}\{\alpha\psi_X^2 + \psi_Z^2\} + H = 0 \qquad \text{at} \qquad Z = 1 + \alpha H \tag{3.32}$$

$$\psi_Z = 0 \qquad \text{at} \qquad Z = 0 \tag{3.33}$$

Since both ψ and H are expressed in terms of X and τ, they both depend on α, thus we can expand each in terms of α to obtain

$$\psi = \psi_0 + \alpha\psi_1 + \alpha^2\psi_2 + O(\alpha^3) \tag{3.34}$$

and

$$H = H_0 + \alpha H_1 + O(\alpha^2) \ . \tag{3.35}$$

We can now substitute (3.34) and (3.35) into (3.30), (3.31), (3.32), and (3.33). For (3.30), one obtains

$$0 = \alpha\left(\psi_{0XX} + \alpha\psi_{1XX} + \alpha^2\psi_{2XX} + O(\alpha^3)\right) + \left(\psi_{0ZZ} + \alpha\psi_{1ZZ} + \alpha^2\psi_{2ZZ} + O(\alpha^3)\right) \ ,$$

and rearranging we get

$$0 = \psi_{0ZZ} + \alpha\left(\psi_{0XX} + \psi_{1ZZ}\right) + \alpha^2\left(\psi_{1XX} + \psi_{2ZZ}\right) + O(\alpha^3)\ ,$$

which gives for various orders of α:

$$O(\alpha^0) \qquad \psi_{0ZZ} = 0 \tag{3.36}$$
$$O(\alpha^1) \qquad \psi_{1ZZ} = -\psi_{0XX} \tag{3.37}$$
$$O(\alpha^2) \qquad \psi_{2ZZ} = -\psi_{1XX}\ . \tag{3.38}$$

By Proposition 3.4, equations (3.36), (3.37), and (3.38) in conjunction with the bottom boundary condition (3.33) imply

$$\text{(3.36) and (3.33)} \quad \Rightarrow \quad \psi_0 = B_0(X,\tau) \tag{3.39}$$
$$\text{(3.37) and (3.33)} \quad \Rightarrow \quad \psi_1 = -\frac{Z^2}{2}B_{0XX} + B_1(X,\tau) \tag{3.40}$$
$$\text{(3.38) and (3.33)} \quad \Rightarrow \quad \psi_2 = \frac{Z^4}{4!}B_{0XXXX} - \frac{Z^2}{2}B_{1XX} + B_2(X,\tau)\ . \tag{3.41}$$

By Proposition 3.5 below, to leading orders $O(\alpha^0)$ and $O(\alpha^1)$, equation (3.32) at the surface gives

$$O(\alpha^0): \qquad H_0 = \psi_{0X} = B_{0X} \tag{3.42}$$
$$O(\alpha^1): \qquad 0 = H_1 + B_{0X} + \frac{1}{2}B_{0XXX} - B_{1X} + B_{0\tau} + \frac{1}{2}B_{0X}^2\ . \tag{3.43}$$

By Proposition 3.6 below, to leading orders $O(\alpha^1)$ and $O(\alpha^2)$, equation (3.31) at the surface gives

$$O(\alpha^1): H_{0X} = B_{0XX} \tag{3.44}$$

$$\begin{aligned} O(\alpha^2): -H_0B_{0XX} + \frac{1}{6}B_{0XXXX} - B_{1XX} \\ = -H_{1X} + H_{0X} + H_{0\tau} + B_{0X}H_{0X}\ . \end{aligned} \tag{3.45}$$

From (3.45) one obtains, by moving the H_{1X} term over,

$$-H_0 B_{0XX} + \frac{1}{6} B_{0XXXX} + H_{1X} - B_{1XX} = H_{0X} + H_{0\tau} + B_{0X} H_{0X} \ . \tag{3.46}$$

By differentiating (3.43) we get

$$\begin{aligned} 0 &= H_{1X} + B_{0XX} + \frac{1}{2} B_{0XXXX} - B_{1XX} + B_{0\tau X} + B_{0X} B_{0XX} \\ H_{1X} - B_{1XX} &= -\frac{1}{2} B_{0XXXX} - B_{0XX} - B_{0\tau X} - B_{0X} B_{0XX} \ . \end{aligned} \tag{3.47}$$

Using (3.47) we can replace $H_{1X} - B_{1XX}$ in (3.46) to get

$$\begin{aligned} -H_0 B_{0XX} + \frac{1}{6} B_{0XXXX} - \frac{1}{2} B_{0XXXX} - B_{0\tau X} - B_{0X} B_{0XX} - B_{0XX} \\ = H_{0X} + H_{0\tau} + B_{0X} H_{0X} \ . \end{aligned} \tag{3.48}$$

We can rewrite (3.48) as

$$\begin{aligned} -H_0 B_{0XX} - \frac{1}{3} B_{0XXXX} - B_{0\tau X} - B_{0X} B_{0XX} - B_{0XX} \\ = H_{0X} + H_{0\tau} + B_{0X} H_{0X} \ . \end{aligned} \tag{3.49}$$

Now from (3.42), we can substitute $B_{0X} = H_0$ into (3.49) to get

$$-H_0 H_{0X} - \frac{1}{3} H_{0XXX} - H_{0\tau} - H_0 H_{0X} - H_{0X} = H_{0X} + H_{0\tau} + H_0 H_{0X} \ ,$$

which reduces to

$$0 = 2H_{0\tau} + 3H_0H_{0X} + \frac{1}{3}H_{0XXX} + 2H_{0X} \ .$$

Dividing through by 2 gives

$$0 = H_{0X} + H_{0\tau} + \frac{3}{2}H_0H_{0X} + \frac{1}{6}H_{0XXX} \ .$$

Thus

$$\begin{aligned}
H_X + & H_\tau + \frac{3}{2}HH_X + \frac{1}{6}H_{XXX} \\
= & H_{0X} + H_{0\tau} + \frac{3}{2}H_0H_{0X} + \frac{1}{6}H_{0XXX} \\
& + \alpha\left[H_{1X} + H_{1\tau} + \frac{3}{2}[H_0H_{1X} + H_1H_{0X}] + \frac{1}{6}H_{1XXX}\right] \\
& + O(\alpha^2) \\
= & 0 + O(\alpha) \ .
\end{aligned} \tag{3.50}$$

So the scaled wave $H(X,\tau) = H_0 + \alpha H_1 + O(\alpha^2)$ satisfies the KdV equation at the 0^{th} order in α. Thus

$$0 = H_X + H_\tau + \frac{3}{2}HH_X + \frac{1}{6}H_{XXX} + O(\alpha) \ .$$

So, for α small it is reasonable to assume

$$0 = H_X + H_\tau + \frac{3}{2}HH_X + \frac{1}{6}H_{XXX} \ , \tag{3.51}$$

which is the KdV equation in the form that is very commonly used.

We are interested in a transform of the KdV equation in (3.51). By Proposition 3.7 we see that (3.51) transforms under the appropriate change of variables, (3.99), into

$$0 = \hat{H}_{\hat{\tau}} + \hat{A}\hat{H}_{\hat{X}} + \frac{3}{2}\hat{B}\hat{H}\hat{H}_{\hat{X}} + \frac{1}{6}\hat{C}\hat{H}_{\hat{X}\hat{X}\hat{X}} \ , \tag{3.52}$$

where $\hat{A}$, $\hat{B}$, and $\hat{C}$ are arbitrary constants. This is a more general form of the KdV equation.

We can now choose the arbitrary constants to be as follows:

$$\hat{A} = \sqrt{gh} \ , \quad \hat{B} = \frac{a}{h} = \alpha \text{ and } \ \hat{C} = \left(\frac{h}{\lambda}\right)^2 = \epsilon^2$$

and obtain the KdV equation in the form we are interested in, namely,

$$0 = \hat{H}_{\hat{\tau}} + \sqrt{gh}\hat{H}_{\hat{X}} + \frac{3}{2}\alpha\hat{H}\hat{H}_{\hat{X}} + \frac{1}{6}\epsilon^2\hat{H}_{\hat{X}\hat{X}\hat{X}} \quad . \tag{3.53}$$

This form of the KdV equation is chosen because it is consistent with that used in [9].

Remark 3.1. Given the equations from (3.16)-(3.19) namely,

$$\phi_{xx} + \phi_{zz} = 0 \qquad \forall x, z, t \qquad 0 \leq z \leq h + aH(x,t) \quad , \tag{3.54}$$

$$\phi_z = aH_x\phi_x + aH_t \ , \quad \text{at} \quad z = h + aH \tag{3.55}$$

$$\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + gaH = B(t) - gh \quad , \quad \text{at} \quad z = h + aH \tag{3.56}$$

and

$$\phi_z = 0 \qquad at \;\; z = 0 \;\; , \tag{3.57}$$

one can linearize these equations (3.54)-(3.57) and obtain

$$\phi_{xx} + \phi_{zz} = 0 \qquad \forall x, z, t \qquad 0 \leq z \leq h + aH(x,t) \;\; , \tag{3.58}$$

$$\phi_z = aH_t \;, \quad \text{at} \;\; z = h + aH \tag{3.59}$$

$$\phi_t + gaH + gh = 0 \;\; , \quad \text{at} \;\; z = h + aH \tag{3.60}$$

and

$$\phi_z = 0 \qquad at \;\; z = 0 \;\; , \tag{3.61}$$

where:

(i) $B(t)$ has been absorbed into the potential function ϕ, that is, a new term ϕ is $\phi - \int_{t_0}^{t} B(s)ds$,

(ii) the non-linear term $aH_x\phi_x$ is dropped from (3.55) , and

(iii) the non-linear term $\frac{1}{2}(\phi_x^2 + \phi_z^2)$ is dropped from (3.56) .

Now, (3.59) gives $\phi_z = aH_t$, and antidifferentiating (3.60) with respect to t gives

$$\phi_{tt} + agH_t = 0 \; . \tag{3.62}$$

Substituting (3.59) into (3.62) yields

$$\phi_{tt} + g\phi_z = 0 \ . \tag{3.63}$$

Next assume that the potential ϕ has the form

$$\phi = Y(z)\sin(kx - \omega t) \ .$$

Then,

$$\phi_{xx} = -k^2 Y(z)\sin(kx - \omega t)$$

and

$$\phi_{zz} = Y_{zz}(z)\sin(kx - \omega t) \ . \tag{3.64}$$

Then (3.58) becomes

$$0 = \phi_{xx} + \phi_{zz} = \left[-k^2 Y(z) + Y_{zz}(z)\right]\sin(kx - \omega t)$$

or

$$Y_{zz}(z) - k^2 Y(z) = 0 \ . \tag{3.65}$$

Since e^{kz} and e^{-kz} both satisfy (3.65), we have

$$Y(z) = Ae^{kz} + Be^{-kz} \ .$$

Thus

$$\phi = (Ae^{kz} + Be^{-kz})\sin(kx - \omega t) \ .$$

(3.61) then gives

$$0 = \phi_z|_{z=0} = ((kAe^{kz} - kBe^{-kz})|_{z=0} \sin(kx - \omega t)$$

so

$$0 = (kA - kB) \sin(kx - \omega t) \ .$$

Thus

$$kA - kB = 0$$

and so

$$A = B \ .$$

Thus

$$\phi = (Ae^{kz} + Ae^{-kz}) \sin(kx - \omega t)$$

or

$$\phi = 2A \cosh(kz) \sin(kx - \omega t) \ . \tag{3.66}$$

Next, at $z = h + aH$, (3.63) gives

$$0 = \phi_{tt} + g\phi_z = -\omega^2 2A \cosh(kz) \sin(kx - \omega t) + gk2A \sinh(kz) \sin(kx - \omega t) \ .$$

Thus

$$\omega^2 2A \cosh(kz) \sin(kx - \omega t) = gk2A \sinh(kz) \sin(kx - \omega t) \ ,$$

so

$$\omega^2 = gk \tanh(kz) \qquad \text{at } z = h + aH \ ,$$

and therefore

$$\omega^2 = gk\tanh(k(h + aH)) \ .$$

Thus,

$$\omega^2 = gk\tanh\left(kh\left(1 + \frac{a}{h}H\right)\right) \ . \tag{3.67}$$

To recover (aH) from the linearized problem we rely on (3.60), that is,

$$aH = \frac{-\phi_t - gh}{g} = -\frac{\phi_t}{g} - h \ . \tag{3.68}$$

Differentiating (3.66) with respect to t and substituting into (3.68) yields

$$aH = -\frac{1}{g}\left\{2A\cosh(kz)(-\omega)\cos(kx - \omega t)\right\} - h \tag{3.69}$$

which is a wave with wavelength $\lambda = \frac{2\pi}{k}$ and ω the angular frequency relative to $\cos(t)$. Thus a longer wavelength λ corresponds to a smaller wavenumber $k = \frac{2\pi}{\lambda}$, so substituting for k in (3.67) yields

$$\omega^2 = gk\tanh\left(2\pi\frac{h}{\lambda}\left[1 + \left(\frac{a}{h}\right)H\right]\right) \ . \tag{3.70}$$

It is reasonable to assume for a tsunami wave in the ocean that the amplitude term a is small when compared to the ocean depth h. Thus we assume that $\frac{a}{h}$ is small and obtain from (3.67) and (3.70)

$$\omega^2 \approx gk\tanh(kh) = gk\tanh\left(2\pi\left(\frac{h}{\lambda}\right)\right) \ . \tag{3.71}$$

Furthermore, since tsunamis generate a longer wavelength profile, it is reasonable to assume that λ is large when compared to h. Then we assume k and kh are small, also $\frac{h}{\lambda}$ is small, to use $\tanh(x) \approx x$ for small x. We then obtain from (3.71) that

$$\omega^2 \approx gkkh \qquad \text{or} \qquad \omega^2 \approx gk2\pi\left(\frac{h}{\lambda}\right)$$

giving

$$\omega \approx k\sqrt{gh} \qquad \text{or} \qquad \omega \approx \sqrt{gk2\pi\left(\frac{h}{\lambda}\right)} \tag{3.72}$$

which exhibits ω to be low frequency in addition to the longer wavelength λ profile. Substituting (3.72) into (3.69) shows

$$aH = -\frac{1}{g}\left\{2A\cos(kz)(-\omega)\cos(kx - k\sqrt{gh}t)\right\} - h$$

or

$$aH = \frac{\omega}{g}\left\{2A\cos(kz)\cos\left(k\left[x - \sqrt{gh}t\right]\right)\right\} - h \tag{3.73}$$

which is a long length wave traveling at velocity $c = \sqrt{gh}$.

Proposition 3.2. *Equations* (3.16), (3.17), (3.18), *and* (3.19) *transform under* (3.20) *to:*

$$\begin{aligned} & 0 = \epsilon^2\bar{\phi}_{\bar{x}\bar{x}} + \bar{\phi}_{\bar{z}\bar{z}} \\ at \;\; \bar{z} = 1 + \alpha H \;\; \longrightarrow \;\; & \bar{\phi}_{\bar{z}} = \epsilon^2\left\{\alpha\bar{\phi}_{\bar{x}}H_{\bar{x}} + H_{\bar{t}}\right\} \\ at \;\; \bar{z} = 1 + \alpha H \;\; \longrightarrow \;\; & \bar{\phi}_{\bar{t}} + \frac{1}{2}\alpha\left\{\left(\bar{\phi}_{\bar{x}}\right)^2 + \frac{1}{\epsilon^2}\left(\bar{\phi}_{\bar{z}}\right)^2\right\} + H = \frac{\tilde{B}(\bar{t}) - gh}{ga} \\ at \;\; \bar{z} = 0 \;\; \longrightarrow \;\; & \bar{\phi}_{\bar{z}} = 0 \end{aligned}$$

for $\epsilon = \frac{x}{\lambda}$, $\alpha = \frac{a}{h}$, *and* $\tilde{B}(\bar{t}) = B\left(\frac{\lambda}{\sqrt{gh}} \cdot \bar{t}\right) = B(t)$.

Proof. Recall from (3.20) that

$$\bar{x} = \frac{x}{\lambda}\ , \quad \bar{z} = \frac{z}{h}\ , \quad \bar{\phi} = \frac{h\ \phi}{\lambda a\sqrt{gh}}\ , \quad \bar{t} = \frac{t\sqrt{gh}}{\lambda}\ .$$

By the chain rule

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \bar{x}} \cdot \frac{\partial \bar{x}}{\partial x} = \frac{\partial}{\partial \bar{x}} \cdot \frac{1}{\lambda} \quad , \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial \bar{z}} \cdot \frac{\partial \bar{z}}{\partial z} = \frac{\partial}{\partial \bar{z}} \cdot \frac{1}{h}$$
$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \bar{t}} \cdot \frac{\partial \bar{t}}{\partial t} = \frac{\partial}{\partial \bar{t}} \cdot \frac{\sqrt{gh}}{\lambda} \ , \tag{3.74}$$

Then from (3.74) we have

$$\phi_t = \frac{\sqrt{gh}}{\lambda}\phi_{\bar{t}} \ , \ \phi_x = \frac{1}{\lambda}\phi_{\bar{x}} \ \text{ and } \ \phi_z = \frac{1}{h}\phi_{\bar{z}} \ . \tag{3.75}$$

Since $\phi = \left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1} \bar{\phi}$ in (3.20) we have

$$\phi_t = \frac{\sqrt{gh}}{\lambda}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1} \bar{\phi}_{\bar{t}} = ag\bar{\phi}_{\bar{t}} \quad , \quad \phi_x = \frac{1}{\lambda}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1} \bar{\phi}_{\bar{x}}$$
$$\text{and } \ \phi_z = \frac{1}{h}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1} \bar{\phi}_{\bar{z}} \ . \tag{3.76}$$

Furthermore, by repeated application of (3.74),

$$\phi_{xx} = \frac{1}{\lambda^2}\phi_{\bar{x}\bar{x}} \ \text{ and } \ \phi_{zz} = \frac{1}{h^2}\phi_{\bar{z}\bar{z}}$$

and

$$\phi_{xx} = \frac{1}{\lambda^2}\phi_{\bar{x}\bar{x}} = \frac{1}{\lambda^2}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1} \bar{\phi}_{\bar{x}\bar{x}} \ \text{ and } \ \phi_{zz} = \frac{1}{h^2}\phi_{\bar{z}\bar{z}} = \frac{1}{h^2}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1} \bar{\phi}_{\bar{z}\bar{z}} \ .$$

Therefore, (3.16) gives

$$\begin{aligned} 0 &= \phi_{xx} + \phi_{zz} = \left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1} \cdot \frac{1}{\lambda^2} \ \bar{\phi}_{\bar{x}\bar{x}} + \left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1} \cdot \frac{1}{h^2} \ \bar{\phi}_{\bar{z}\bar{z}} \\ \Rightarrow \quad 0 &= \frac{1}{\lambda^2}\bar{\phi}_{\bar{x}\bar{x}} + \frac{1}{h^2}\bar{\phi}_{\bar{z}\bar{z}} \end{aligned}$$

$$\Rightarrow \qquad 0 \quad = \quad \frac{h^2}{\lambda^2}\bar{\phi}_{\bar{x}\bar{x}} + \bar{\phi}_{\bar{z}\bar{z}} \ .$$

Finally (3.16) transforms to

$$0 = \epsilon^2 \bar{\phi}_{\bar{x}\bar{x}} + \bar{\phi}_{\bar{z}\bar{z}} \quad \text{where} \ \ \epsilon = \frac{h}{\lambda} \ ,$$

giving (3.21). As a consequence of (3.74), we have

$$H_x = H_{\bar{x}}\frac{\partial \bar{x}}{\partial x} = H_{\bar{x}} \cdot \frac{1}{\lambda} \quad \text{and} \quad H_t = H_{\bar{t}}\frac{\partial \bar{t}}{\partial t} = H_{\bar{t}} \cdot \frac{\sqrt{gh}}{\lambda} \ . \tag{3.77}$$

We can substitute (3.77) into (3.17) with (3.76) to get

$$\begin{aligned}
\phi_z &= aH_x\phi_x + aH_t \\
\Rightarrow \bar{\phi}_{\bar{z}}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\frac{1}{h} &= aH_{\bar{x}} \cdot \frac{1}{\lambda}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\frac{1}{\lambda}\bar{\phi}_{\bar{x}} + aH_{\bar{t}}\frac{\sqrt{gh}}{\lambda} \\
\Rightarrow \bar{\phi}_{\bar{z}}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\frac{1}{h} &= a\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\frac{1}{\lambda^2}\bar{\phi}_{\bar{x}}H_{\bar{x}} + \frac{a}{\lambda}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\sqrt{gh}\left(\frac{h}{\lambda a\sqrt{gh}}\right)H_{\bar{t}} \\
\Rightarrow \qquad \bar{\phi}_{\bar{z}}\frac{1}{h} &= \frac{a}{\lambda^2}\bar{\phi}_{\bar{x}}H_{\bar{x}} + \sqrt{gh}\frac{h}{\lambda^2\sqrt{gh}}H_{\bar{t}} \\
\Rightarrow \qquad \bar{\phi}_{\bar{z}} &= \frac{ha}{\lambda^2}\bar{\phi}_{\bar{x}}H_{\bar{x}} + \frac{h^2}{\lambda^2}H_{\bar{t}} \\
\Rightarrow \qquad \bar{\phi}_{\bar{z}} &= \frac{h^2}{\lambda^2}\left\{\frac{a}{h}\bar{\phi}_{\bar{x}}H_{\bar{x}} + H_{\bar{t}}\right\} \\
\Rightarrow \qquad \bar{\phi}_{\bar{z}} &= \epsilon^2\left\{\alpha\bar{\phi}_{\bar{x}}H_{\bar{x}} + H_{\bar{t}}\right\} \ ,
\end{aligned}$$

where $\alpha = \frac{a}{h}$ and $\epsilon = \frac{h}{\lambda}$. Thus (3.17) has transformed to (3.22). Next, we can substitute (3.75) and (3.76) into (3.18) to get

$$\begin{aligned}
&\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + gaH = B(t) - gh \\
\Rightarrow \qquad &ag\bar{\phi}_{\bar{t}} + \frac{1}{2}\left(\left[\frac{a\sqrt{gh}}{h}\bar{\phi}_{\bar{x}}\right]^2 + \left[\frac{\lambda a\sqrt{gh}}{h^2}\bar{\phi}_{\bar{z}}\right]^2\right) + gaH = B(t) - gh \ .
\end{aligned}$$

Now $\tilde{B}\left(\,\bar{t}\,\right) \equiv B\left(\frac{\lambda}{\sqrt{gh}}\left[t\frac{\sqrt{gh}}{\lambda}\right]\right) = B(t)$

$$\Rightarrow \qquad ag\bar{\phi}_{\bar{t}} + \frac{1}{2}\left\{\frac{a^2 gh}{h^2}\left[\bar{\phi}_{\bar{x}}\right]^2 + \frac{\lambda^2 a^2 gh}{h^4}\left[\bar{\phi}_{\bar{z}}\right]^2\right\} + gaH = \tilde{B}(\bar{t}) - gh$$
$$\Rightarrow \qquad \bar{\phi}_{\bar{t}} + \frac{1}{2}\left\{\frac{a}{h}\left(\bar{\phi}_{\bar{x}}\right)^2 + \frac{a}{h}\frac{\lambda^2}{h^2}\left(\bar{\phi}_{\bar{z}}\right)^2\right\} + H = \frac{\tilde{B}(\bar{t}) - gh}{ga}$$
$$\Rightarrow \qquad \bar{\phi}_{\bar{t}} + \frac{1}{2}\alpha\left\{\left(\bar{\phi}_{\bar{x}}\right)^2 + \frac{1}{\epsilon^2}\left(\bar{\phi}_{\bar{z}}\right)^2\right\} + H = \frac{\tilde{B}(\bar{t}) - gh}{ga} \ .$$

Thus (3.18) has transformed to (3.23). Next, from (3.19) using (3.76) we get

$$\begin{aligned} \phi_z &= 0 && \text{at } z = 0 \\ \Rightarrow \qquad \frac{\lambda a\sqrt{gh}}{h^2}\bar{\phi}_{\bar{z}} &= 0 && \text{at } \bar{z} = 0 \\ \Rightarrow \qquad \bar{\phi}_{\bar{z}} &= 0 && \text{at } \bar{z} = 0 \ . \end{aligned}$$

Thus (3.19) has transformed to (3.24). □

Proposition 3.3. *Equations* (3.25), (3.26), (3.27), *and* (3.28) *transform under* (3.29) *to*

$$\begin{aligned} 0 &= \alpha\psi_{XX} + \psi_{ZZ} \\ \psi_Z = \alpha^2\psi_X H_X + (\alpha^2 - \alpha)H_X + \alpha^2 H_\tau & \qquad \text{at} \qquad Z = 1 + \alpha H \\ \alpha\psi_X - \psi_X + \alpha\psi_Z + \frac{1}{2}\{\alpha\psi_X^2 + \psi_Z^2\} + H = 0 & \qquad \text{at} \qquad Z = 1 + \alpha H \\ \psi_Z = 0 & \qquad \text{at} \qquad Z = 0 \end{aligned}$$

Proof. Recall from (3.29) that

$$X = \frac{\alpha^{1/2}}{\epsilon}\left(\bar{x} + (\alpha - 1)\bar{t}\right), \ Z = \bar{z}, \ \tau = \frac{\alpha^{3/2}}{\epsilon}\bar{t}, \ \text{ and } \ \psi = \frac{\alpha^{1/2}}{\epsilon}\bar{\phi} \ .$$

Assume $\alpha^{1/2} = k\epsilon$ for some $k \in \mathbb{R}$. Then

$$\frac{\partial}{\partial\bar{x}} = \frac{\partial}{\partial X}\frac{\partial X}{\partial\bar{x}} = \frac{\partial}{\partial X}\frac{\alpha^{1/2}}{\epsilon} \ , \text{ and } \ \frac{\partial}{\partial\bar{z}} = \frac{\partial}{\partial Z}\frac{\partial Z}{\partial\bar{z}} = \frac{\partial}{\partial Z} \qquad (3.78)$$

$$\frac{\partial}{\partial \bar{t}} = \frac{\partial}{\partial X}\frac{\partial X}{\partial \bar{t}} + \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial \bar{t}} = \frac{\partial}{\partial X}\frac{\alpha^{1/2}}{\epsilon}(\alpha - 1) + \frac{\partial}{\partial \tau}\frac{\alpha^{3/2}}{\epsilon} \quad . \tag{3.79}$$

Now from (3.25), (3.78) and (3.79)

$$\begin{aligned}
0 &= \epsilon^2 \bar{\phi}_{\ \bar{x}\bar{x}} + \bar{\phi}_{\ \bar{z}\bar{z}} \\
0 &= \epsilon^2 \frac{\partial}{\partial \bar{x}}\frac{\partial}{\partial \bar{x}}\bar{\phi} + \frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial \bar{z}}\bar{\phi} \\
0 &= \epsilon^2 \frac{\alpha}{\epsilon^2}\frac{\partial}{\partial X}\frac{\partial}{\partial X}\bar{\phi} + \frac{\partial}{\partial Z}\frac{\partial}{\partial Z}\bar{\phi} \\
0 &= \epsilon^2 \frac{\alpha}{\epsilon^2}\bar{\phi}_{XX} + \bar{\phi}_{ZZ} \\
0 &= \alpha \frac{\epsilon}{\alpha^{1/2}}\psi_{XX} + \frac{\epsilon}{\alpha^{1/2}}\psi_{ZZ} \\
0 &= \alpha \psi_{XX} + \psi_{ZZ} \ .
\end{aligned}$$

Thus (3.25) has transformed to (3.30). Next, apply (3.78), (3.79) and (3.29) to (3.26) to get

$$\begin{aligned}
\bar{\phi}_{\ \bar{z}} &= \epsilon^2 \left\{ \alpha \bar{\phi}_{\ \bar{x}} H_{\bar{x}} + H_{\bar{t}} \right\} \\
\bar{\phi}_{\ Z} &= \epsilon^2 \left\{ \alpha \left(\frac{\alpha^{1/2}}{\epsilon}\bar{\phi}_X \frac{\alpha^{1/2}}{\epsilon} H_X \right) + \frac{\alpha^{1/2}}{\epsilon}(\alpha - 1) H_X + \frac{\alpha^{3/2}}{\epsilon} H_\tau \right\} \\
\bar{\phi}_{\ Z} &= \left\{ \alpha^2 \bar{\phi}_X H_X + \left(\alpha^{3/2} - \alpha^{1/2}\right) \epsilon H_X + \alpha^{3/2} \epsilon H_\tau \right\} \\
\frac{\epsilon}{\alpha^{1/2}} \psi_{\ Z} &= \left\{ \alpha^2 \frac{\epsilon}{\alpha^{1/2}} \psi_X H_X + \left(\alpha^{3/2} - \alpha^{1/2}\right) \epsilon H_X + \alpha^{3/2} \epsilon H_\tau \right\} \\
\psi_{\ Z} &= \alpha^2 \psi_X H_X + \left(\alpha^2 - \alpha\right) H_X + \alpha^2 H_\tau \qquad \text{when } Z = 1 + \alpha H \ .
\end{aligned}$$

Thus (3.26) has transformed to (3.31). Now from (3.27) we can obtain the Bernoulli equation as follows:

$$\begin{aligned}
& \bar{\phi}_{\ \bar{t}} + \frac{1}{2}\alpha \left\{ \left(\bar{\phi}_{\ \bar{x}}\right)^2 + \frac{1}{\epsilon^2}\left(\bar{\phi}_{\ \bar{z}}\right)^2 \right\} + H = 0 \\
\Rightarrow \quad & \frac{\alpha^{1/2}}{\epsilon}(\alpha - 1)\bar{\phi}_X + \frac{\alpha^{3/2}}{\epsilon}\bar{\phi}_\tau + \frac{1}{2}\alpha \left\{ \left(\frac{\alpha^{1/2}}{\epsilon}\bar{\phi}_{\bar{X}} \right)^2 + \frac{1}{\epsilon^2}\left(\bar{\phi}_Z\right)^2 \right\} + H = 0 \\
\Rightarrow \quad & \frac{\epsilon}{\alpha^{1/2}} \left(\frac{\alpha^{1/2}}{\epsilon}(\alpha - 1)\psi_X \right) + \frac{\alpha^{3/2}}{\epsilon}\frac{\epsilon}{\alpha^{1/2}}\psi_\tau \\
& \qquad + \frac{1}{2}\alpha \left\{ \left(\frac{\alpha^{1/2}}{\epsilon}\frac{\epsilon}{\alpha^{1/2}}\psi_X \right)^2 + \frac{1}{\epsilon^2}\left(\frac{\epsilon}{\alpha^{1/2}}\psi_Z \right)^2 \right\} + H = 0
\end{aligned}$$

$$\Rightarrow \qquad (\alpha-1)\psi_X + \alpha\psi_\tau + \frac{1}{2}\alpha\left\{(\psi_X)^2 + \frac{1}{\alpha}(\psi_Z)^2\right\} + H = 0$$

$$\Rightarrow \qquad \alpha\psi_X - \psi_X + \alpha\psi_\tau + \frac{1}{2}\left\{\alpha\psi_X^2 + \psi_Z^2\right\} + H = 0 \quad \text{at} \quad Z = 1 + \alpha H \ .$$

This gives (3.32).

We can examine (3.28) and obtain the following:

$$\begin{aligned}
& \bar{\phi}_{\ \bar{z}} = 0 && \text{at } \bar{z} = 0 \\
\Rightarrow \quad & \bar{\phi}_{\ Z} = 0 && \text{at } Z = 0 \\
\Rightarrow \quad & \frac{\epsilon}{\alpha^{1/2}}\psi_{\ Z} = 0 && \text{at } Z = 0 \\
\Rightarrow \quad & \psi_{\ Z} = 0 && \text{at } Z = 0
\end{aligned}$$

This gives (3.33). □

Proposition 3.4. *Equations* (3.36), (3.37), *and* (3.38) *in conjunction with the bottom boundary condition* (3.33) *imply the following:*

$$\begin{aligned}
&(3.36) \ \textit{and} \ (3.33) \quad \Rightarrow \quad \psi_0 = B_0(X,\tau) \\
&(3.37) \ \textit{and} \ (3.33) \quad \Rightarrow \quad \psi_1 = -\frac{Z^2}{2}B_{0XX} + B_1(X,\tau) \\
&(3.38) \ \textit{and} \ (3.33) \quad \Rightarrow \quad \psi_2 = \frac{Z^4}{4!}B_{0XXXX} - \frac{Z^2}{2}B_{1XX} + B_2(X,\tau) \ .
\end{aligned}$$

Proof. Antidifferentiate (3.36) once with respect to Z to get $\psi_{0Z} = C_0(X,\tau)$ and again to get

$$\psi_0 = B_0(X,\tau) + ZC_0(X,\tau) \ . \tag{3.80}$$

Now (3.33) gives

$$0 = \psi_Z = \psi_{0Z} + \alpha\psi_{1Z} + \alpha^2\psi_{2Z} + O(\alpha^3) \qquad \text{for} \ \ Z = 0 \ ,$$

so for $Z = 0$

$$0 = \psi_{0Z} = \psi_{1Z} = \psi_{2Z} \ . \tag{3.81}$$

Thus by (3.80) $0=\psi_{0Z}=C_0(X,\tau)$ for all Z. Thus, (3.36), (3.80), and (3.81) imply that

$$\psi_0 = B_0(X,\tau) \ .$$

Note that ψ_0 does not depend on Z, nor does $\psi_{0XX}=B_{0XX}$. Next, anti-differentiating (3.37) once with respect to Z gives

$$\psi_{1Z} = (-\psi_{0XX})\, Z + D(X,\tau) \ . \tag{3.82}$$

Then, from (3.81) we get

$$\psi_{1Z} = 0 = 0 + D(X,\tau) \qquad \text{at} \quad Z=0 \ ,$$

so $D(X,\tau)=0$ for all Z. Thus,

$$\psi_{1Z} = -Z\psi_{0XX} \ . \tag{3.83}$$

Anti-differentiating again with respect to Z gives

$$\psi_1 = -\frac{Z^2}{2}\psi_{0XX} + B_1(X,\tau)$$

and (3.39) implies $\psi_{0XX}=B_{0XX}$, so

$$\psi_1 = -\frac{Z^2}{2}B_{0XX} + B_1 \ . \tag{3.84}$$

Next (3.38) gives $\psi_{2ZZ}=-\psi_{1XX}$ so, by (3.84) one obtains

$$\psi_{2ZZ} = \frac{Z^2}{2}B_{0XXXX} - B_{1XX} \ . \tag{3.85}$$

Anti-differentiating with respect to Z gives

$$\psi_{2Z} = \frac{Z^3}{3!} B_{0XXXX} - Z B_{1XX} + E(X, \tau) \ . \tag{3.86}$$

Now from (3.81) at $Z = 0$, we get

$$\psi_{2Z}|_{Z=0} = 0 = 0 - 0 + E$$

and so we get that $E = 0$ for all Z. Thus,

$$\psi_{2Z} = \frac{Z^3}{3!} B_{0XXXX} - Z B_{1XX} \ .$$

Anti-differentiating once more with respect to Z, we get

$$\psi_2 = \frac{Z^4}{4!} B_{0XXXX} - \frac{Z^2}{2} B_{1XX} + B_2(X, \tau) \ .$$

□

Proposition 3.5. *The leading orders* $O(\alpha^0)$ *and* $O(\alpha^1)$ *of equation* (3.32) *at the surface give:*

$$\begin{aligned} O(\alpha^0): &\qquad H_0 = \psi_{0X} = B_{0X} \\ O(\alpha^1): &\qquad 0 = H_1 + B_{0X} + \frac{1}{2} B_{0XXX} - B_{1X} + B_{0\tau} + \frac{1}{2} B_{0X}^2 \end{aligned}$$

Proof. At $Z = 1 + \alpha H$, (3.32) gives

$$H + \alpha \psi_X - \psi_X + \alpha \psi_\tau + \frac{1}{2} \left\{ \alpha \psi_X^2 + \psi_Z^2 \right\} = 0 \ ,$$

and then (3.34) and (3.35) give

$$\begin{aligned}0 =& H_0 + \alpha H_1 + O(\alpha^2) + \alpha\psi_{0X} + \alpha^2\psi_{1X} + \alpha^3\psi_{2X} + O(\alpha^4)\\ &- \psi_{0X} - \alpha\psi_{1X} - \alpha^2\psi_{2X} - O(\alpha^3)\\ &+ \alpha\psi_{0\tau} + \alpha^2\psi_{1\tau} + \alpha^3\psi_{2\tau} + O(\alpha^4)\\ &+ \alpha\frac{1}{2}\left[\psi_{0X} + \alpha\psi_{1X} + \alpha^2\psi_{2X} + O(\alpha^3)\right]^2\\ &+ \frac{1}{2}\left[\psi_{0Z} + \alpha\psi_{1Z} + \alpha^2\psi_{2Z} + O(\alpha^3)\right]^2 \ ,\end{aligned}$$

and we get

$$\begin{aligned}0 = & H_0 + \alpha H_1 + O(\alpha^2)\\ & + \alpha\psi_{0X} + \alpha^2\psi_{1X} + \alpha^3\psi_{2X} + O(\alpha^4)\\ & -\psi_{0X} - \alpha\psi_{1X} - \alpha^2\psi_{2X} - O(\alpha^3)\\ & + \alpha\psi_{0\tau} + \alpha^2\psi_{1\tau} + \alpha^3\psi_{2\tau} + O(\alpha^4)\\ & + \alpha\frac{1}{2}\psi_{0X}^2 + \alpha^2\psi_{0X}\psi_{1X} + O(\alpha^3)\\ & +\frac{1}{2}\psi_{0Z}^2 + \alpha\psi_{0Z}\psi_{1Z} + O(\alpha^2) \ .\end{aligned}$$

Therefore,

$$\begin{aligned}0 =& \left[H_0 - \psi_{0X} + \frac{1}{2}\psi_{0Z}^2\right]\\ &+ \alpha\left[H_1 + \psi_{0X} - \psi_{1X} + \psi_{0\tau} + \frac{1}{2}\psi_{0X}^2 + \psi_{0Z}\psi_{1Z}\right]\\ &+ O(\alpha^2) \ .\end{aligned} \tag{3.87}$$

Thus, from the α^0 term of (3.87), we get

$$0 = H_0 - \psi_{0X} + \frac{1}{2}\psi_{0Z}^2 \tag{3.88}$$

and from the α^1 term of (3.87),

$$0 = H_1 + \psi_{0X} - \psi_{1X} + \psi_{0\tau} + \frac{1}{2}\psi_{0X}^2 + \psi_{0Z}\psi_{1Z} \tag{3.89}$$

at $Z = 1 + \alpha H$.

Thus from (3.88), (3.81), and (3.39) we get

$$\begin{aligned} 0 &= H_0 - B_{0X} + \frac{1}{2}0^2 \\ &= H_0 - B_{0X} \ . \end{aligned} \tag{3.90}$$

Thus

$$H_0 = \psi_{0X} = B_{0X} \ . \tag{3.91}$$

Furthermore,

$$\psi_{0\tau} = B_{0\tau} \ . \tag{3.92}$$

Also, (3.40) gives

$$\psi_1 = -\frac{Z^2}{2}B_{0XX} + B_1 \ ,$$

and differentiating with respect to X we have

$$\psi_{1X} = -\frac{Z^2}{2}B_{0XXX} + B_{1X} \ .$$

Thus,

$$\psi_{1X}|_{Z=1+\alpha H} = -\frac{1}{2}(1+2\alpha H+\alpha^2H^2)B_{0XXX} + B_{1X} \ . \tag{3.93}$$

Now the α and α^2 terms in (3.93) create $O(\alpha^2)$ and $O(\alpha^3)$ terms in (3.87), or higher order terms in (3.89) so we can drop both terms in these settings to obtain

$$\psi_{1X}|_{Z=1+\alpha H} = -\frac{1}{2}B_{0XXX} + B_{1X} \ . \tag{3.94}$$

Next, from (3.89), (3.91), (3.81), (3.92), and (3.94), we get

$$0 = H_1 + B_{0X} - \left(-\frac{1}{2}B_{0XXX} + B_{1X}\right) + B_{0\tau} + \frac{1}{2}B_{0X}^2 + 0 \ ,$$

so

$$0 = H_1 + B_{0X} + \frac{1}{2}B_{0XXX} - B_{1X} + B_{0\tau} + \frac{1}{2}B_{0X}^2 \ .$$

□

Proposition 3.6. *The leading orders* $O(\alpha^1)$ *and* $O(\alpha^2)$ *of equation* (3.31) *at the surface give thw following:*

$$O(\alpha^1) : H_{0X} = B_{0XX}$$

$$O(\alpha^2) : -H_0B_{0XX} + \frac{1}{6}B_{0XXXX} - B_{1XX} = -H_{1X} + H_{0X} + H_{02} + B_{0X}H_{0X}$$

Proof. At $Z = 1+\alpha H = 1+\alpha H_0 + \alpha^2 H_1 + O(\alpha^3)$, the kinematic equation (3.31) gives

$$\psi_Z = \alpha^2\psi_X H_X + \alpha^2 H_X - \alpha H_X + \alpha^2 H_\tau \ .$$

Thus, substituting in (3.34) and (3.35) gives

$$
\begin{aligned}
\psi_{0Z} + \alpha\psi_{1Z} + &\alpha^2\psi_{2Z} + O(\alpha^3) \\
&= -\alpha[H_{0X} + \alpha H_{1X} + O(\alpha^2)] \\
&\quad + \alpha^2[H_{0X} + \alpha H_{1X} + O(\alpha^2)] \\
&\quad + \alpha^2[H_{0\tau} + \alpha H_{1\tau} + O(\alpha^2)] \\
&\quad + \alpha^2[\psi_{0X} + \alpha\psi_{1X} + \alpha^2\psi_{2X} + O(\alpha^3)][H_{0X} + \alpha H_{1X} + O(\alpha^2)] \ .
\end{aligned} \tag{3.95}
$$

Now, from (3.40) and (3.41) we have

$$
\begin{aligned}
\psi_{0Z} + \alpha\psi_{1Z} + &\alpha^2\psi_{2Z} + O(\alpha^3) \\
&= 0 + \alpha(-ZB_{0XX}) + \alpha^2\left(\frac{1}{6}Z^3B_{0XXXX} - ZB_{1XX}\right) + O(\alpha^3) \ .
\end{aligned}
$$

At $Z = 1 + \alpha H$ we expand using (3.35) and get

$$
\begin{aligned}
\psi_{0Z} + \alpha\psi_{1Z} + &\alpha^2\psi_{2Z} + O(\alpha^3) \\
&= -\alpha[1 + \alpha H_0 + \alpha^2 H_1 + O(\alpha^3)]B_{0XX} \\
&\quad + \alpha^2\frac{1}{6}[1 + \alpha H_0 + \alpha^2 H_1 + O(\alpha^3)]^3 B_{0XXXX} \\
&\quad - \alpha^2[1 + \alpha H_0 + \alpha^2 H_1 + O(\alpha^3)]B_{1XX} + 0 + O(\alpha^3) \ .
\end{aligned}
$$

Moving the α^3 terms into $O(\alpha^3)$, we have

$$
\begin{aligned}
\psi_{0Z} + \alpha\psi_{1Z} + \alpha^2\psi_{2Z} + O(\alpha^3) & \\
&= -\alpha B_{0XX} - \alpha^2 H_0 B_{0XX} + O(\alpha^3) \\
&\qquad + \frac{1}{6}\alpha^2 B_{0XXXX} + O(\alpha^3) \\
&\qquad - \alpha^2 B_{1XX} + O(\alpha^3)\ .
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\psi_{0Z} + \alpha\psi_{1Z} + &\alpha^2\psi_{2Z} + O(\alpha^3) \\
&= -\alpha B_{0XX} + \alpha^2\left[-H_0 B_{0XX} + \frac{1}{6}B_{0XXXX} - B_{1XX}\right] + O(\alpha^3)\ .
\end{aligned}
\tag{3.96}
$$

Now substituting (3.96) into (3.95) we get

$$
\begin{aligned}
-\alpha B_{0XX} &+ \alpha^2\left[-H_0 B_{0XX} + \frac{1}{6}B_{0XXXX} - B_{1XX}\right] + O(\alpha^3) \\
&= -\alpha H_{0X} - \alpha^2 H_{1X} + O(\alpha^3) \\
&\qquad + \alpha^2 H_{0X} + O(\alpha^3) \\
&\qquad + \alpha^2 H_{0\tau} + O(\alpha^3) \\
&\qquad + \alpha^2 \psi_{0X} H_{0X} + O(\alpha^3)\ .
\end{aligned}
$$

So matching α^1 terms implies $B_{0XX} = H_{0X}$ which is consistant with (3.90). Matching α^2 terms implies

$$
-H_0 B_{0XX} + \frac{1}{6}B_{0XXXX} - B_{1XX} = -H_{1X} + H_{0X} + H_{0\tau} + \psi_{0X} H_{0X}\ .
$$

Substituting $\psi_{0X} = B_{0X}$ from (3.92) gives

$$-H_0 B_{0XX} + \frac{1}{6} B_{0XXXX} - B_{1XX} = -H_{1X} + H_{0X} + H_{0\tau} + B_{0X} H_{0X} \ . \tag{3.97}$$

□

Proposition 3.7. (3.51) *can be transformed into*

$$0 = \hat{H}_{\hat{\tau}} + \hat{A}\hat{H}_{\hat{X}} + \frac{3}{2}\hat{B}\hat{H}\hat{H}_{\hat{X}} + \frac{1}{6}\hat{C}\hat{H}_{\hat{X}\hat{X}\hat{X}}$$

under the appropriate change of variables, where $\hat{A}$, $\hat{B}$, and $\hat{C}$ are arbitrary constants.

Proof. From (3.51) we have

$$0 = H_X + H_\tau + \frac{3}{2} H H_X + \frac{1}{6} H_{XXX} \ . \tag{3.98}$$

We can then perform a change of variables, namely

$$\begin{aligned} \hat{X} = BX \quad &\Rightarrow \quad \frac{\partial}{\partial X} = \frac{\partial}{\partial \hat{X}} \frac{\partial \hat{X}}{\partial X} = \frac{\partial}{\partial \hat{X}} \cdot B \\ \hat{\tau} = C\tau \quad &\Rightarrow \quad \frac{\partial}{\partial \tau} = \frac{\partial}{\partial \hat{\tau}} \frac{\partial \hat{\tau}}{\partial \tau} = \frac{\partial}{\partial \hat{\tau}} \cdot C \\ \hat{H} = L^{-1}H + K \quad &\Rightarrow \quad L\hat{H} - KL = H \ . \end{aligned} \tag{3.99}$$

Then (3.98) becomes

$$(L\hat{H} - KL)_X + (L\hat{H} - KL)_\tau + \frac{3}{2}(L\hat{H} - KL)(L\hat{H} - KL)_X + \frac{1}{6}(L\hat{H} - KL)_{XXX} = 0$$

or

$$BL\hat{H}_{\hat{X}} + LC\hat{H}_{\hat{\tau}} + \frac{3}{2}(L\hat{H} - KL)(LB\hat{H}_{\hat{X}}) + \frac{1}{6}LB^3\hat{H}_{\hat{X}\hat{X}\hat{X}} = 0$$

and therefore,

$$BL\left\{1-\frac{3}{2}KL\right\}\hat{H}_{\hat{X}}+LC\hat{H}_{\hat{\tau}}+\frac{3}{2}L^2B\hat{H}\hat{H}_{\hat{X}}+\frac{1}{6}LB^3\hat{H}_{\hat{X}\hat{X}\hat{X}}=0\ .$$

Thus dividing by LC gives:

$$\left\{\frac{B}{C}\right\}\left\{1-\frac{3}{2}KL\right\}\hat{H}_{\hat{X}}+\hat{H}_{\hat{\tau}}+\frac{3}{2}L\left\{\frac{B}{C}\right\}\hat{H}\hat{H}_{\hat{X}}+\frac{1}{6}\left\{\frac{B}{C}\right\}B^2\hat{H}_{\hat{X}\hat{X}\hat{X}}=0\ .$$

Now:

1. Choose C first
2. Choose B next, determining the coefficient of $\bar{H}_{\bar{X}\bar{X}\bar{X}}$ as $\hat{C}=\left\{\frac{B}{C}\right\}B^2$
3. Choose L next, determining the coefficient of $\bar{H}\bar{H}_{\bar{X}}$ as $\hat{B}=L\left\{\frac{B}{C}\right\}$
4. Choose K next, determining the coefficient of $\bar{H}_{\bar{X}}$ as $\left\{\frac{B}{C}\right\}\left\{1-\frac{3}{2}KL\right\}$.

Thus we obtain a more general form of the KdV equation, namely,

$$0=\hat{H}_{\hat{\tau}}+\hat{A}\hat{H}_{\hat{X}}+\frac{3}{2}\hat{B}\hat{H}\hat{H}_{\hat{X}}+\frac{1}{6}\hat{C}\hat{H}_{\hat{X}\hat{X}\hat{X}}$$

where $\hat{A}$, $\hat{B}$, and $\hat{C}$ are arbitrary constants.

$\square$

CHAPTER 4: 2-Dimensional Korteweg-de Vries Equation

We will derive the 2 dimensional KdV, equation. We expand on the work in [3]. We set up our system such that z is our vertical direction, and x and y are our horizontal directions. The velocity of the fluid is $\vec{v} = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})$. We must first assume that the fluid is irrotational, i.e.,

$$\nabla \times \vec{v} = 0 \ . \tag{4.1}$$

This implies the existence of a potential function ϕ, with $\vec{v} = (\phi_x, \phi_y, \phi_z)$. Thus $\vec{v} = \nabla\phi$. We next assume that the fluid is incompressible, namely,

$$\nabla \cdot \vec{v} = div\ \vec{v} = 0 \ . \tag{4.2}$$

And thus

$$0 = \nabla \cdot \nabla\phi = \phi_{xx} + \phi_{yy} + \phi_{zz} \ . \tag{4.3}$$

So the potential function is harmonic.

We next assume the density function ρ of the fluid is constant. Thus

$$\nabla\rho = 0 \qquad \rho_t = 0 \ . \tag{4.4}$$

Now (4.4) together with (4.2) imply the conservation of mass

$$\partial_t\rho + \nabla \cdot (\rho\vec{v}) = 0 \ . \tag{4.5}$$

We remark that (4.5) holds more generally (without the assumption of (4.4) and

(4.2)) via the statement

$$\frac{\partial}{\partial t}\int_R \rho \; dV = -\int_{\partial R} \rho\vec{v}\cdot\vec{n} \; dS \tag{4.6}$$

where the rate of change of mass in R equals the rate of flow of mass into R across the boundary ∂R. The divergence theorem applied to the flux integral in (4.6) then implies (4.5) in general.

In addition to conservation of mass in (4.5), one has Euler's equation [5], (due to the force law)

$$\frac{\partial}{\partial t}(\rho\vec{v}) + \vec{v}\cdot\nabla(\rho\vec{v}) = -\nabla P + \rho\vec{f}\ , \tag{4.7}$$

which under the assumptions (4.2) and (4.4) simplifies to

$$\rho\frac{\partial}{\partial t}(\vec{v}) + \rho\vec{v}\cdot\nabla\vec{v} = -\nabla P + \rho\vec{f} \tag{4.8}$$

where p is the internal pressure and ρf is any external forcing effect. In our case f will be acceleration due to gravity $\vec{f} = -g(0,0,1)$ yielding

$$\vec{f} = -g\nabla z\ . \tag{4.9}$$

Thus,

$$\frac{\partial}{\partial t}(\vec{v}) + \vec{v}\cdot\nabla\vec{v} = -\frac{\nabla P}{\rho} - g\nabla z \tag{4.10}$$

where

$$\vec{v}\cdot\nabla\vec{v} = (\vec{v}\cdot\nabla v_1, \vec{v}\cdot\nabla v_2, \vec{v}\cdot\nabla v_3)\ .$$

Now by Lemma 4.1

$$\vec{v} \cdot \nabla \vec{v} = \frac{1}{2} \nabla \left(|\vec{v}|^2 \right) - \vec{v} \times \nabla \times \vec{v} \tag{4.11}$$

which becomes under (4.1)

$$\vec{v} \cdot \nabla \vec{v} = \frac{1}{2} \nabla \left(|\vec{v}|^2 \right) \ .$$

So (4.10) reduces to

$$\frac{\partial}{\partial t}(\vec{v}) + \frac{1}{2} \nabla \left(|\vec{v}|^2 \right) = -\frac{\nabla P}{\rho} - g \nabla z \ ,$$

or equivalently using the fact that $\vec{v} = \nabla \phi$

$$\nabla \left[\phi_t + \frac{1}{2} |\nabla \phi|^2 + \frac{P}{\rho} + gz \right] = 0 \ .$$

Thus,

$$\phi_t + \frac{1}{2} |\nabla \phi|^2 + \frac{P}{\rho} + gz = B(t) \ . \tag{4.12}$$

At the surface $z = h + aH(x, y, t)$, where a is the amplitude of the wave and h is the undisturbed water level, the pressure P vanishes, and (4.12) reduces to

$$\phi_t + \frac{1}{2} |\nabla \phi|^2 + g(h + aH) = B(t)$$

or

$$\phi_t + \frac{1}{2} |\nabla \phi|^2 + gaH = B(t) - gh \ . \tag{4.13}$$

On the bottom, there should be no vertical component to the velocity, so

$$\phi_z = \frac{dz}{dt} = 0 \qquad \text{at } z = 0 \ . \tag{4.14}$$

Finally, at the surface $z = h + aH$ one has, by differentiating with respect to t and using on the chain rule, one has

$$\frac{dz}{dt} = \phi_z = a\nabla H \cdot \nabla\phi + aH_t \ . \tag{4.15}$$

Thus the equations governing our fluid are (4.3), (4.13), (4.14), and (4.15). We can rewrite these as:

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \qquad \forall x, y, z, t \qquad 0 \leq z \leq h + aH(x, y, t) \ \ , \tag{4.16}$$

$$\phi_z = aH_x\phi_x + aH_y\phi_y + aH_t \ , \tag{4.17}$$

$$\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) + gaH = B(t) - gh \ \ , \tag{4.18}$$

and

$$\phi_z = 0 \qquad at \ \ z = 0 \ \ . \tag{4.19}$$

We deploy these to obtain the KdV equation.

Analogous to the 1 dimensional case, we assume that $\frac{h}{\lambda}$ and $\frac{a}{h}$ are small with a wave speed of $\sqrt{gh}$. We now record the effects on (4.16), (4.17), (4.18), and (4.19) of a series of successive changes in variables to obtain a dimensionless system. First, we use scaled variables. Let

$$\bar{x} = \frac{x}{\lambda} \ , \quad \bar{y} = \frac{y}{\lambda} \ , \quad \bar{z} = \frac{z}{h} \ , \quad \bar{\phi} = \frac{h\ \phi}{\lambda a\sqrt{gh}} \ \text{ and } \ \bar{t} = \frac{t\sqrt{gh}}{\lambda} \ . \tag{4.20}$$

By Proposition 4.2 , equations (4.16), (4.17), (4.18), and (4.19) transform under (4.20) to the following:

$$0 = \epsilon^2 \bar{\phi}_{\bar{x}\bar{x}} + \epsilon^2 \bar{\phi}_{\bar{y}\bar{y}} + \bar{\phi}_{\bar{z}\bar{z}} \tag{4.21}$$

$$\bar{\phi}_{\bar{z}} = \epsilon^2 \left\{ \alpha \bar{\phi}_{\bar{x}} H_{\bar{x}} + \alpha \bar{\phi}_{\bar{y}} H_{\bar{y}} + H_{\bar{t}} \right\} \text{ at } \bar{z} = 1 + \alpha H \tag{4.22}$$

$$\bar{\phi}_{\bar{t}} + \frac{1}{2}\alpha \left\{ \left(\bar{\phi}_{\bar{x}}\right)^2 + \left(\bar{\phi}_{\bar{y}}\right)^2 + \frac{1}{\epsilon^2}\left(\bar{\phi}_{\bar{z}}\right)^2 \right\} + H = \frac{\tilde{B}(\bar{t}) - gh}{ga} \text{ at } \bar{z} = 1 + \alpha H \tag{4.23}$$

$$\bar{\phi}_{\bar{z}} = 0 \text{ at } \bar{z} = 0 \tag{4.24}$$

for $\epsilon = \frac{h}{\lambda}$, $\alpha = \frac{a}{h}$, and $\tilde{B}(\bar{t}) = B\left(\frac{\lambda}{\sqrt{gh}} \cdot \bar{t}\right) = B(t)$.

Next we can incorporate $\frac{\tilde{B}(\bar{t}) - gh}{ag}$ into the potential $\bar{\phi}$ in (4.23) by taking

$$\hat{B}(\bar{t}) = \int_0^{\bar{t}} \frac{\tilde{B}(s) - gh}{ga} ds$$

and letting $(\bar{\phi}_{new}) = \bar{\phi} - \hat{B}(\bar{t})$. Then all spacial derivatives of $(\bar{\phi}_{new})_{\bar{s}} = \bar{\phi}_{\bar{s}}$ (for $\bar{s} = \bar{x}, \bar{y}, \bar{z}$) and one has $(\bar{\phi}_{new})_{\bar{t}} = \bar{\phi}_{\bar{t}} - \hat{B}_{\bar{t}} = \bar{\phi}_{\bar{t}} - \left[\frac{\tilde{B}(\bar{t}) - gh}{ah}\right]$. Thus from (4.21) we have

$$0 = \epsilon^2 (\bar{\phi}_{new})_{\bar{x}\bar{x}} + \epsilon^2 (\bar{\phi}_{new})_{\bar{y}\bar{y}} + (\bar{\phi}_{new})_{\bar{z}\bar{z}} \ , \tag{4.25}$$

and, at the surface, from (4.22) we have

$$(\bar{\phi}_{new})_{\bar{z}} = \epsilon^2 \left\{ \alpha (\bar{\phi}_{new})_{\bar{x}} H_{\bar{x}} + \alpha (\bar{\phi}_{new})_{\bar{y}} H_{\bar{y}} + H_{\bar{t}} \right\} \ . \tag{4.26}$$

Similarly, using (4.23) at the surface, we have

$$(\bar{\phi}_{new})_{\bar{t}} + \frac{1}{2}\alpha \left\{ \left[(\bar{\phi}_{new})_{\bar{x}}\right]^2 + \left[(\bar{\phi}_{new})_{\bar{y}}\right]^2 + \frac{1}{\epsilon^2}\left[(\bar{\phi}_{new})_{\bar{z}}\right]^2 \right\} + H = 0 \ , \tag{4.27}$$

where (4.27) has become a homogeneous version of (4.18). At the bottom using (4.24) we get

$$(\bar{\phi}_{new})_{\bar{z}} = 0 \qquad \text{at } \bar{z} = 0\ . \tag{4.28}$$

So we can drop all subscripts *new* from here on and assume $\bar{\phi}_{new} = \bar{\phi}$.

We proceed with the next change of variables to obtain dimensionless equations: let

$$X = \frac{\alpha^{1/2}}{\epsilon}\left(\bar{x} + (\alpha - 1)\bar{t}\right) \ , \ Y = \frac{\alpha}{\epsilon}\bar{y} \ , \ \tau = \frac{\alpha^{3/2}}{\epsilon}\bar{t} \ , \ \psi = \frac{\alpha^{1/2}}{\epsilon}\bar{\phi} \ , \ Z = \bar{z}\ . \tag{4.29}$$

By Proposition 4.3, equations (4.25), (4.26), (4.27), and (4.28) transform under (4.29) to

$$0 = \alpha\psi_{XX} + \alpha^2\psi_{YY} + \psi_{ZZ} \tag{4.30}$$

$$\psi_Z = \alpha^2\psi_X H_X + \alpha^3\psi_Y H_Y + (\alpha^2 - \alpha)H_X + \alpha^2 H_\tau \qquad \text{at} \quad Z = 1 + \alpha H \tag{4.31}$$

$$\alpha\psi_X - \psi_X + \alpha\psi_\tau + \frac{1}{2}\{\alpha\psi_X^2 + \alpha^2\psi_Y^2 + \psi_Z^2\} + H = 0 \qquad \text{at} \quad Z = 1 + \alpha H \tag{4.32}$$

$$\psi_Z = 0 \qquad \text{at} \quad Z = 0\ . \tag{4.33}$$

Since both ψ and H are expressed in terms of X and τ, they both depend on α. Thus we assume that we can expand each in terms of α to obtain

$$\psi = \psi_0 + \alpha\psi_1 + \alpha^2\psi_2 + O(\alpha^3) \tag{4.34}$$

and

$$H = H_0 + \alpha H_1 + O(\alpha^2)\ . \tag{4.35}$$

We can now substitute (4.34) and (4.35) into (4.30), (4.31), (4.32), and (4.33). For (4.30), one obtains

$$0 = \alpha\left(\psi_{0XX} + \alpha\psi_{1XX} + \alpha^2\psi_{2XX} + O(\alpha^3)\right) + \alpha^2\left(\psi_{0YY} + \alpha\psi_{1YY} + \alpha^2\psi_{2YY} + O(\alpha^3)\right)$$
$$+ \left(\psi_{0ZZ} + \alpha\psi_{1ZZ} + \alpha^2\psi_{2ZZ} + O(\alpha^3)\right)$$

and rearranging we get

$$0 = \psi_{0ZZ} + \alpha\left(\psi_{0XX} + \psi_{1ZZ}\right) + \alpha^2\left(\psi_{1XX} + \psi_{0YY} + \psi_{2ZZ}\right) + O(\alpha^3)$$

which gives for various orders of α:

$$O(\alpha^0) \qquad \psi_{0ZZ} = 0 \tag{4.36}$$
$$O(\alpha^1) \qquad \psi_{1ZZ} = -\psi_{0XX} \tag{4.37}$$
$$O(\alpha^2) \qquad \psi_{2ZZ} = -\psi_{1XX} - \psi_{0YY}\ . \tag{4.38}$$

By Proposition 4.4, equations (4.36), (4.37), and (4.38) in conjunction with the bottom boundary condition (4.33) imply

$$(4.36),(4.33) \ \Rightarrow\ \psi_0 = B_0(X,Y,\tau) \tag{4.39}$$
$$(4.37),(4.33) \ \Rightarrow\ \psi_1 = -\frac{Z^2}{2}B_{0XX} + B_1(X,Y,\tau) \tag{4.40}$$
$$(4.38),(4.33) \ \Rightarrow\ \psi_2 = \frac{Z^4}{4!}B_{0XXXX} - \frac{Z^2}{2}B_{1XX} - \frac{Z^2}{2}\psi_{0YY} + B_2(X,Y,\tau) \tag{4.41}$$

By Proposition 4.5 below, to leading orders $O(\alpha^0)$ and $O(\alpha^1)$, equation (4.32) at the surface gives

$$O(\alpha^0): \qquad H_0 = \psi_{0X} = B_{0X} \tag{4.42}$$
$$O(\alpha^1): \qquad 0 = H_1 + B_{0X} + \frac{1}{2}B_{0XXX} - B_{1X} + B_{0\tau} + \frac{1}{2}B_{0X}^2\ . \tag{4.43}$$

By Proposition 4.6 below, to leading orders $O(\alpha^1)$ and $O(\alpha^2)$, equation (4.31) at the surface gives

$$O(\alpha^1) : H_{0X} = B_{0XX} \tag{4.44}$$

$$\begin{aligned} O(\alpha^2) : -H_0 B_{0XX} + \frac{1}{6} B_{0XXXX} - B_{1XX} - \psi_{0YY} \\ = -H_{1X} + H_{0X} + H_{0\tau} + B_{0X} H_{0X} \end{aligned} \tag{4.45}$$

From (4.45) one obtains, by moving the H_{1X} term over,

$$-H_0 B_{0XX} + \frac{1}{6} B_{0XXXX} - \psi_{0YY} + H_{1X} - B_{1XX} = H_{0X} + H_{0\tau} + B_{0X} H_{0X} \ . \tag{4.46}$$

By differentiating (4.43) with respect to X we get

$$\begin{aligned} 0 &= H_{1X} + B_{0XX} + \frac{1}{2} B_{0XXXX} - B_{1XX} + B_{0\tau X} + B_{0X} B_{0XX} \\ H_{1X} - B_{1XX} &= -\frac{1}{2} B_{0XXXX} - B_{0XX} - B_{0\tau X} - B_{0X} B_{0XX} \ . \end{aligned} \tag{4.47}$$

Using (4.47) we can replace $H_{1X} - B_{1XX}$ in (4.46) to get

$$\begin{aligned} -H_0 B_{0XX} + \frac{1}{6} B_{0XXXX} - \psi_{0YY} - \frac{1}{2} B_{0XXXX} - B_{0\tau X} - B_{0X} B_{0XX} - B_{0XX} \\ = H_{0X} + H_{0\tau} + B_{0X} H_{0X} \ . \end{aligned} \tag{4.48}$$

We can rewrite (4.48) as

$$\begin{aligned} -H_0 B_{0XX} - \frac{1}{3} B_{0XXXX} - \psi_{0YY} - B_{0\tau X} - B_{0X} B_{0XX} - B_{0XX} \\ = H_{0X} + H_{0\tau} + B_{0X} H_{0X} \ . \end{aligned} \tag{4.49}$$

Now from (4.72) we can substitute $B_{0X} = H_0$ into (4.49) to get

$$-H_0 H_{0X} - \frac{1}{3} H_{0XXX} - \psi_{0YY} - H_{0\tau} - H_0 H_{0X} - H_{0X} = H_{0X} + H_{0\tau} + H_0 H_{0X} \ ,$$

or

$$0 = 2H_{0\tau} + 3H_0 H_{0X} + \frac{1}{3} H_{0XXX} + 2H_{0X} + \psi_{0YY} \ .$$

So the scaled wave $H(X, Y, \tau) = H_0 + \alpha H_1 + O(\alpha^2)$ satisfies a variant of the KdV at the 0th order in α due to the ψ_{0YY} term. Thus, for α small, it is reasonable to assume that

$$0 = 2H_\tau + 3HH_X + \frac{1}{3} H_{XXX} + 2H_X + \psi_{YY} \ . \tag{4.50}$$

From (4.42) it is reasonable to assume when α is small that

$$H = \psi_X \ . \tag{4.51}$$

Differentiating (4.50) with respect to X gives

$$0 = \left[2H_\tau + 3HH_X + \frac{1}{3} H_{XXX} + 2H_X \right]_X + \psi_{YYX} \ ,$$

and by (4.51)

$$0 = \left[2H_\tau + 3HH_X + \frac{1}{3} H_{XXX} + 2H_X \right]_X + H_{YY} \ .$$

This can be rewritten as

$$0 = \left[H_X + H_\tau + \frac{3}{2} H H_X + \frac{1}{6} H_{XXX} \right]_X + \frac{1}{2} H_{YY} \ , \tag{4.52}$$

which is the KdV equation in 2 dimensions in a commonly used form.

Now, by Proposition (4.7), equation (4.52) transforms under the appropriate change of variables into

$$\left[\hat{H}_{\hat{\tau}} + \hat{A}\hat{H}_{\hat{X}} + \frac{3}{2}\hat{B}\hat{H}\hat{H}_{\hat{X}} + \frac{1}{6}\hat{C}\hat{H}_{\hat{X}\hat{X}\hat{X}} \right]_{\hat{X}} + \frac{1}{2}\hat{D}\hat{H}_{\hat{Y}\hat{Y}} = 0 \tag{4.53}$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ are arbitrary constants with the stipulation that the sign of $\hat{C}$ and the sign of $\hat{D}$ must be the same. This is a more general form of the KdV equation. We can then choose $\hat{A} = \sqrt{gh}$, $\hat{B} = \alpha$, $\hat{C} = \epsilon^2$, and $\hat{D} = 2\sqrt{gh}$ to get a specific form of the KdV that we are most interested in, namely

$$\left[\hat{H}_{\hat{\tau}} + \sqrt{gh}\hat{H}_{\hat{X}} + \frac{3}{2}\alpha\hat{H}\hat{H}_{\hat{X}} + \frac{1}{6}\epsilon^2\hat{H}_{\hat{X}\hat{X}\hat{X}} \right]_{\hat{X}} + \sqrt{gh}\hat{H}_{\hat{Y}\hat{Y}} = 0 \ , \tag{4.54}$$

which relates closely to (3.53).

Lemma 4.1.

$$\vec{v} \times \nabla \times \vec{v} + \vec{v} \cdot \nabla \vec{v} = \frac{1}{2} \nabla \left[v_1^2 + v_2^2 + v_3^2 \right]$$

Proof. We will prove this for the 2 dimensional case, and leave to the reader to show the 1 dimensional case. We begin with

$$\nabla \times \vec{v} = det \begin{pmatrix} i & j & k \\ D_x & D_y & D_z \\ v_1 & v_2 & v_3 \end{pmatrix} = (v_{3y} - v_{2z}, -v_{3x} + v_{1z}, v_{2x} - v_{1y})$$

so,

$$\vec{v}\times\nabla\times\vec{v} = det\begin{pmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ v_{3y}-v_{2z} & -v_{3x}+v_{1z} & v_{2x}-v_{1y}\end{pmatrix}$$
$$= \begin{pmatrix} v_2[v_{2x}-v_{1y}] - v_3[-v_{3x}+v_{1z}] \\ -v_1[v_{2x}-v_{1y}] + v_3[v_{3y}-v_{2z}] \\ v_1[-v_{3x}+v_{1z}] - v_2[v_{3y}-v_{2z}]\end{pmatrix} .$$

Next,

$$\vec{v}\cdot\nabla\vec{v} = \begin{pmatrix} \vec{v}\cdot\nabla v_1 \\ \vec{v}\cdot\nabla v_2 \\ \vec{v}\cdot\nabla v_3\end{pmatrix} = \begin{pmatrix} v_1v_{1x}+v_2v_{1y}+v_3v_{1z} \\ v_1v_{2x}+v_2v_{2y}+v_3v_{2z} \\ v_1v_{3x}+v_2v_{3y}+v_3v_{3z}\end{pmatrix} . \tag{4.55}$$

Thus,

$$\vec{v}\times\nabla\times\vec{v} + \vec{v}\cdot\nabla\vec{v}$$
$$= \begin{pmatrix} v_2[v_{2x}-v_{1y}] - v_3[-v_{3x}+v_{1z}] \\ -v_1[v_{2x}-v_{1y}] + v_3[v_{3y}-v_{2z}] \\ v_1[-v_{3x}+v_{1z}] - v_2[v_{3y}-v_{2z}]\end{pmatrix} + \begin{pmatrix} v_1v_{1x}+v_2v_{1y}+v_3v_{1z} \\ v_1v_{2x}+v_2v_{2y}+v_3v_{2z} \\ v_1v_{3x}+v_2v_{3y}+v_3v_{3z}\end{pmatrix}$$
$$= \begin{pmatrix} v_1v_{1x}+v_2v_{2x}+v_3v_{3x} \\ v_1v_{1y}+v_2v_{2y}+v_3v_{3y} \\ v_1v_{1z}+v_2v_{2z}+v_3v_{3z}\end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2}\left[v_1^2+v_2^2+v_3^2\right]_x \\ \frac{1}{2}\left[v_1^2+v_2^2+v_3^2\right]_y \\ \frac{1}{2}\left[v_1^2+v_2^2+v_3^2\right]_z\end{pmatrix}$$
$$= \frac{1}{2}\nabla\left[v_1^2+v_2^2+v_3^2\right] .$$

Therefore

$$\vec{v}\times\nabla\times\vec{v} + \vec{v}\cdot\nabla\vec{v} = \frac{1}{2}\nabla\left[v_1^2+v_2^2+v_3^2\right] .$$

□

Proposition 4.2. *Equations* (4.16), (4.17), (4.18), *and* (4.19) *transform under* (4.20) *to the following:*

$$0 = \epsilon^2\bar{\phi}_{\bar{x}\bar{x}} + \epsilon^2\bar{\phi}_{\bar{y}\bar{y}} + \bar{\phi}_{\bar{z}\bar{z}}$$
$$at\ \ \bar{z} = 1+\alpha H \longrightarrow \bar{\phi}_{\bar{z}} = \epsilon^2\left\{\alpha\bar{\phi}_{\bar{x}}H_{\bar{x}} + \alpha\bar{\phi}_{\bar{y}}H_{\bar{y}} + H_{\bar{t}}\right\}$$
$$at\ \ \bar{z} = 1+\alpha H \longrightarrow \bar{\phi}_{\bar{t}} + \frac{1}{2}\alpha\left\{\left(\bar{\phi}_{\bar{x}}\right)^2 + \left(\bar{\phi}_{\bar{y}}\right)^2 + \frac{1}{\epsilon^2}\left(\bar{\phi}_{\bar{z}}\right)^2\right\} + H = \frac{\tilde{B}(\bar{t}) - gh}{ga}$$
$$at\ \ \bar{z} = 0 \longrightarrow \bar{\phi}_{\bar{z}} = 0$$

for $\epsilon = \frac{x}{\lambda}$, $\alpha = \frac{a}{h}$, *and* $\tilde{B}(\bar{t}) = B\left(\frac{\lambda}{\sqrt{gh}} \cdot \bar{t}\right) = B(t)$.

Proof. Recall from (4.20) that

$$\bar{x} = \frac{x}{\lambda}\ , \quad \bar{y} = \frac{y}{\lambda}\ , \quad \bar{z} = \frac{z}{h}\ , \quad \bar{\phi} = \frac{h\ \phi}{\lambda a\sqrt{gh}} \ \text{ and } \ \bar{t} = \frac{t\sqrt{gh}}{\lambda}\ .$$

By the chain rule

$$\begin{aligned} \frac{\partial}{\partial x} = \frac{\partial}{\partial \bar{x}} \cdot \frac{\partial \bar{x}}{\partial x} = \frac{\partial}{\partial \bar{x}} \cdot \frac{1}{\lambda} \quad &, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \bar{y}} \cdot \frac{\partial \bar{y}}{\partial y} = \frac{\partial}{\partial \bar{y}} \cdot \frac{1}{\lambda} \\ \frac{\partial}{\partial z} = \frac{\partial}{\partial \bar{z}} \cdot \frac{\partial \bar{z}}{\partial z} = \frac{\partial}{\partial \bar{z}} \cdot \frac{1}{h} \quad &, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \bar{t}} \cdot \frac{\partial \bar{t}}{\partial t} = \frac{\partial}{\partial \bar{t}} \cdot \frac{\sqrt{gh}}{\lambda}\ . \end{aligned} \tag{4.56}$$

Then from (4.56) we have

$$\phi_t = \frac{\sqrt{gh}}{\lambda}\phi_{\bar{t}}\ , \ \phi_x = \frac{1}{\lambda}\phi_{\bar{x}}\ , \ \phi_y = \frac{1}{\lambda}\phi_{\bar{y}} \ \text{ and } \ \phi_z = \frac{1}{h}\phi_{\bar{z}}\ . \tag{4.57}$$

Since $\phi = \left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\bar{\phi}$ in (4.20) we have

$$\begin{aligned} \phi_t &= \frac{\sqrt{gh}}{\lambda}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\bar{\phi}_{\bar{t}} = ag\bar{\phi}_{\bar{t}} \quad , \quad \phi_x = \frac{1}{\lambda}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\bar{\phi}_{\bar{x}} \\ \phi_y &= \frac{1}{\lambda}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\bar{\phi}_{\bar{y}} \ \text{ and } \ \phi_z = \frac{1}{h}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\bar{\phi}_{\bar{z}}\ . \end{aligned} \tag{4.58}$$

Furthermore, by repeated application of (4.56),

$$\phi_{xx} = \frac{1}{\lambda^2}\phi_{\bar{x}\bar{x}}\ , \ \phi_{yy} = \frac{1}{\lambda^2}\phi_{\bar{y}\bar{y}} \ \text{ and } \ \phi_{zz} = \frac{1}{h^2}\phi_{\bar{z}\bar{z}}$$

and

$$\phi_{xx} = \frac{1}{\lambda^2}\phi_{\bar{x}\bar{x}} = \frac{1}{\lambda^2}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\bar{\phi}_{\bar{x}\bar{x}} \ ,$$
$$\phi_{yy} = \frac{1}{\lambda^2}\phi_{\bar{y}\bar{y}} = \frac{1}{\lambda^2}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\bar{\phi}_{\bar{y}\bar{y}} \ ,$$
$$\phi_{zz} = \frac{1}{h^2}\phi_{\bar{z}\bar{z}} = \frac{1}{h^2}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\bar{\phi}_{\bar{z}\bar{z}} \ .$$

Therefore, (4.16) gives

$$\begin{aligned}
& 0 = \phi_{xx} + \phi_{yy} + \phi_{zz} \\
\Rightarrow \quad & 0 = \left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\cdot\frac{1}{\lambda^2}\ \bar{\phi}_{\bar{x}\bar{x}} + \left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\cdot\frac{1}{\lambda^2}\ \bar{\phi}_{\bar{y}\bar{y}} + \left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\cdot\frac{1}{h^2}\ \bar{\phi}_{\bar{z}\bar{z}} \\
\Rightarrow \quad & 0 = \frac{1}{\lambda^2}\bar{\phi}_{\bar{x}\bar{x}} + \frac{1}{\lambda^2}\bar{\phi}_{\bar{y}\bar{y}} + \frac{1}{h^2}\bar{\phi}_{\bar{z}\bar{z}} \\
\Rightarrow \quad & 0 = \frac{h^2}{\lambda^2}\bar{\phi}_{\bar{x}\bar{x}} + \frac{h^2}{\lambda^2}\bar{\phi}_{\bar{y}\bar{y}} + \bar{\phi}_{\bar{z}\bar{z}} \ .
\end{aligned}$$

Finally (4.16) transforms to

$$0 = \epsilon^2\bar{\phi}_{\bar{x}\bar{x}} + \epsilon^2\bar{\phi}_{\bar{y}\bar{y}} + \bar{\phi}_{\bar{z}\bar{z}} \quad \text{where} \ \ \epsilon = \frac{h}{\lambda} \ ,$$

giving (4.21).

As a consequence of (4.56), we have

$$H_x = H_{\bar{x}}\frac{\partial\bar{x}}{\partial x} = H_{\bar{x}}\cdot\frac{1}{\lambda} \ , \ \ H_y = H_{\bar{y}}\frac{\partial\bar{y}}{\partial y} = H_{\bar{y}}\cdot\frac{1}{\lambda} \ \ \text{and} \ \ H_t = H_{\bar{t}}\frac{\partial\bar{t}}{\partial t} = H_{\bar{t}}\cdot\frac{\sqrt{gh}}{\lambda} \ . \qquad (4.59)$$

We can substitute (4.59) into (4.17) and use (4.58) to get

$$\begin{aligned}
\phi_z &= aH_x\phi_x + aH_y\phi_y + aH_t \\
\Rightarrow \bar{\phi}_{\bar{z}}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\frac{1}{h} &= aH_{\bar{x}}\cdot\frac{1}{\lambda}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\frac{1}{\lambda}\bar{\phi}_{\bar{x}}
\end{aligned}$$

$$
\begin{aligned}
&\qquad +aH_{\bar{y}} \cdot \frac{1}{\lambda}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\frac{1}{\lambda}\bar{\phi}_{\bar{y}} + aH_{\bar{t}}\frac{\sqrt{gh}}{\lambda} \\
\Rightarrow \bar{\phi}_{\bar{z}}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\frac{1}{h} &= a\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\frac{1}{\lambda^2}\bar{\phi}_{\bar{x}}H_{\bar{x}} + a\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\frac{1}{\lambda^2}\bar{\phi}_{\bar{y}}H_{\bar{y}} \\
&\quad + \frac{a}{\lambda}\left(\frac{h}{\lambda a\sqrt{gh}}\right)^{-1}\sqrt{gh}\left(\frac{h}{\lambda a\sqrt{gh}}\right)H_{\bar{t}} \\
\Rightarrow \qquad \bar{\phi}_{\bar{z}}\frac{1}{h} &= \frac{a}{\lambda^2}\bar{\phi}_{\bar{x}}H_{\bar{x}} + \frac{a}{\lambda^2}\bar{\phi}_{\bar{y}}H_{\bar{y}} + \sqrt{gh}\frac{h}{\lambda^2\sqrt{gh}}H_{\bar{t}} \\
\Rightarrow \qquad \bar{\phi}_{\bar{z}} &= \frac{ha}{\lambda^2}\bar{\phi}_{\bar{x}}H_{\bar{x}} + \frac{ha}{\lambda^2}\bar{\phi}_{\bar{y}}H_{\bar{y}} + \frac{h^2}{\lambda^2}H_{\bar{t}} \\
\Rightarrow \qquad \bar{\phi}_{\bar{z}} &= \frac{h^2}{\lambda^2}\left\{\frac{a}{h}\bar{\phi}_{\bar{x}}H_{\bar{x}} + \frac{a}{h}\bar{\phi}_{\bar{y}}H_{\bar{y}} + H_{\bar{t}}\right\} \\
\Rightarrow \qquad \bar{\phi}_{\bar{z}} &= \epsilon^2\left\{\alpha\bar{\phi}_{\bar{x}}H_{\bar{x}} + \alpha\bar{\phi}_{\bar{y}}H_{\bar{y}} + H_{\bar{t}}\right\} \ ,
\end{aligned}
$$

where $\alpha = \frac{a}{h}$ and $\epsilon = \frac{h}{\lambda}$. Thus (4.17) has transformed to (4.22).

Next, we can substitute into (4.18) using the change of variables described in (4.57) and (4.58) and get

$$
\begin{aligned}
&\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) + gaH = B(t) - gh \\
\Rightarrow \quad & ag\bar{\phi}_{\bar{t}} + \frac{1}{2}\left(\left[\frac{a\sqrt{gh}}{h}\bar{\phi}_{\bar{x}}\right]^2 + \left[\frac{a\sqrt{gh}}{h}\bar{\phi}_{\bar{y}}\right]^2 + \left[\frac{\lambda a\sqrt{gh}}{h^2}\bar{\phi}_{\bar{z}}\right]^2\right) + gaH = B(t) - gh \ .
\end{aligned}
$$

Now $\tilde{B}(\ \bar{t}\ ) \equiv B\left(\frac{\lambda}{\sqrt{gh}}\left[t\frac{\sqrt{gh}}{\lambda}\right]\right) = B(t)$

$$
\begin{aligned}
\Rightarrow \quad & ag\bar{\phi}_{\bar{t}} + \frac{1}{2}\left\{\frac{a^2gh}{h^2}\left[\bar{\phi}_{\bar{x}}\right]^2 + \frac{a^2gh}{h^2}\left[\bar{\phi}_{\bar{y}}\right]^2 + \frac{\lambda^2a^2gh}{h^4}\left[\bar{\phi}_{\bar{z}}\right]^2\right\} + gaH = \tilde{B}(\bar{t}) - gh \\
\Rightarrow \quad & \bar{\phi}_{\bar{t}} + \frac{1}{2}\left\{\frac{a}{h}\left(\bar{\phi}_{\bar{x}}\right)^2 + \frac{a}{h}\left(\bar{\phi}_{\bar{y}}\right)^2 + \frac{a}{h}\frac{\lambda^2}{h^2}\left(\bar{\phi}_{\bar{z}}\right)^2\right\} + H = \frac{\tilde{B}(\bar{t}) - gh}{ga} \\
\Rightarrow \quad & \bar{\phi}_{\bar{t}} + \frac{1}{2}\alpha\left\{\left(\bar{\phi}_{\bar{x}}\right)^2 + \left(\bar{\phi}_{\bar{y}}\right)^2 + \frac{1}{\epsilon^2}\left(\bar{\phi}_{\bar{z}}\right)^2\right\} + H = \frac{\tilde{B}(\bar{t}) - gh}{ga} \ .
\end{aligned}
$$

Thus (4.18) has transformed to (4.23). Next, from (4.19) using (4.58) we get

$$
\begin{aligned}
\phi_z &= 0 \qquad \text{at } z = 0 \\
\Rightarrow \qquad \frac{\lambda a\sqrt{gh}}{h^2}\bar{\phi}_{\bar{z}} &= 0 \qquad \text{at } \bar{z} = 0
\end{aligned}
$$

$$\Rightarrow \qquad \bar{\phi}_{\bar{z}} = 0 \qquad \text{at } \bar{z} = 0 \ .$$

Thus (4.19) has transformed to (4.24). □

Proposition 4.3. *Equations* (4.25), (4.26), (4.27), *and* (4.28) *transform under* (4.29) *to*

$$\begin{aligned} 0 &= \alpha\psi_{XX} + \alpha^2\psi_{YY} + \psi_{ZZ} \\ \psi_Z &= \alpha^2\psi_X H_X + \alpha^3\psi_Y H_Y + (\alpha^2-\alpha)H_X + \alpha^2 H_\tau \qquad \textit{at} \quad Z = 1+\alpha H \\ \alpha\psi_X - \psi_X + \alpha\psi_Z + \frac{1}{2}\{\alpha\psi_X^2 + \alpha^2\psi_Y^2 + \psi_Z^2\} + H &= 0 \qquad \textit{at} \quad Z = 1+\alpha H \\ \psi_Z &= 0 \qquad \textit{at} \quad Z = 0 \end{aligned}$$

Proof. Recall from (4.29) that

$$X = \frac{\alpha^{1/2}}{\epsilon}\left(\bar{x} + (\alpha-1)\bar{t}\right), \quad Y = \frac{\alpha}{\epsilon}\bar{y}, \quad Z = \bar{z}, \ \tau = \frac{\alpha^{3/2}}{\epsilon}\bar{t}, \text{ and } \quad \psi = \frac{\alpha^{1/2}}{\epsilon}\bar{\phi} \ .$$

Assume $\alpha^{1/2} = k\epsilon$ for some $k \in \mathbb{R}$. Then

$$\frac{\partial}{\partial\bar{x}} = \frac{\partial}{\partial X}\frac{\partial X}{\partial\bar{x}} = \frac{\partial}{\partial X}\frac{\alpha^{1/2}}{\epsilon} \ , \quad \frac{\partial}{\partial\bar{y}} = \frac{\partial}{\partial Y}\frac{\partial Y}{\partial\bar{y}} = \frac{\partial}{\partial Y}\frac{\alpha}{\epsilon} \text{ and } \quad \frac{\partial}{\partial\bar{z}} = \frac{\partial}{\partial Z}\frac{\partial Z}{\partial\bar{z}} = \frac{\partial}{\partial Z} \ , \tag{4.60}$$

also

$$\frac{\partial}{\partial\bar{t}} = \frac{\partial}{\partial X}\frac{\partial X}{\partial\bar{t}} + \frac{\partial}{\partial\tau}\frac{\partial\tau}{\partial\bar{t}} = \frac{\partial}{\partial X}\frac{\alpha^{1/2}}{\epsilon}(\alpha-1) + \frac{\partial}{\partial\tau}\frac{\alpha^{3/2}}{\epsilon} \ . \tag{4.61}$$

Then from (4.25), (4.60) and (4.61)

$$\begin{aligned} 0 &= \epsilon^2\bar{\phi}_{\ \bar{x}\bar{x}} + \epsilon^2\bar{\phi}_{\ \bar{y}\bar{y}} + \bar{\phi}_{\ \bar{z}\bar{z}} \\ 0 &= \epsilon^2\frac{\partial}{\partial\bar{x}}\frac{\partial}{\partial\bar{x}}\bar{\phi} + \epsilon^2\frac{\partial}{\partial\bar{y}}\frac{\partial}{\partial\bar{y}}\bar{\phi} + \frac{\partial}{\partial\bar{z}}\frac{\partial}{\partial\bar{z}}\bar{\phi} \\ 0 &= \epsilon^2\frac{\alpha}{\epsilon^2}\frac{\partial}{\partial X}\frac{\partial}{\partial X}\bar{\phi} + \epsilon^2\frac{\alpha^2}{\epsilon^2}\frac{\partial}{\partial Y}\frac{\partial}{\partial Y}\bar{\phi} + \frac{\partial}{\partial Z}\frac{\partial}{\partial Z}\bar{\phi} \\ 0 &= \epsilon^2\frac{\alpha}{\epsilon^2}\bar{\phi}_{XX} + \epsilon^2\frac{\alpha^2}{\epsilon^2}\bar{\phi}_{YY} + \bar{\phi}_{ZZ} \\ 0 &= \alpha\frac{\epsilon}{\alpha^{1/2}}\psi_{XX} + \alpha^2\frac{\epsilon}{\alpha^{1/2}}\psi_{YY} + \frac{\epsilon}{\alpha^{1/2}}\psi_{ZZ} \end{aligned}$$

$$0 \;=\; \alpha\psi_{XX} + \alpha^2\psi_{YY} + \psi_{ZZ} \ .$$

which gives (4.30). Next, substitute into (4.26) to get

$$\begin{aligned}
\bar{\phi}_{\ \bar{z}} &= \epsilon^2 \left\{\alpha\bar{\phi}_{\ \bar{x}} H_{\bar{x}} + \alpha\bar{\phi}_{\ \bar{y}} H_{\bar{y}} + H_{\bar{t}}\right\} \\
\bar{\phi}_{\ Z} &= \epsilon^2 \left\{\alpha\left(\frac{\alpha^{1/2}}{\epsilon}\bar{\phi}_X \frac{\alpha^{1/2}}{\epsilon} H_X\right) + \alpha\left(\frac{\alpha}{\epsilon}\bar{\phi}_Y \frac{\alpha}{\epsilon} H_Y\right) + \frac{\alpha^{1/2}}{\epsilon}(\alpha-1)H_X + \frac{\alpha^{3/2}}{\epsilon} H_\tau\right\} \\
\bar{\phi}_{\ Z} &= \left\{\alpha^2\bar{\phi}_X H_X + \alpha^3\bar{\phi}_Y H_Y + \left(\alpha^{3/2} - \alpha^{1/2}\right)\epsilon H_X + \alpha^{3/2}\epsilon H_\tau\right\} \\
\frac{\epsilon}{\alpha^{1/2}}\psi_{\ Z} &= \left\{\alpha^2 \frac{\epsilon}{\alpha^{1/2}}\psi_X H_X + \alpha^3 \frac{\epsilon}{\alpha^{1/2}}\psi_Y H_Y + \left(\alpha^{3/2} - \alpha^{1/2}\right)\epsilon H_X + \alpha^{3/2}\epsilon H_\tau\right\} \\
\psi_{\ Z} &= \alpha^2\psi_X H_X + \alpha^3\psi_Y H_Y + \left(\alpha^2 - \alpha\right) H_X + \alpha^2 H_\tau \qquad \text{when } Z = 1 + \alpha H \ ,
\end{aligned}$$

which gives (4.31).

Now from (4.27) we can obtain the Bernoulli equation as follows:

$$\begin{aligned}
& \bar{\phi}_{\ \bar{t}} + \frac{1}{2}\alpha\left\{\left(\bar{\phi}_{\ \bar{x}}\right)^2 + \left(\bar{\phi}_{\ \bar{y}}\right)^2 + \frac{1}{\epsilon^2}\left(\bar{\phi}_{\ \bar{z}}\right)^2\right\} + H = 0 \\
\Rightarrow \quad & \frac{\alpha^{1/2}}{\epsilon}(\alpha-1)\bar{\phi}_X + \frac{\alpha^{3/2}}{\epsilon}\bar{\phi}_\tau + \frac{1}{2}\alpha\left\{\left(\frac{\alpha^{1/2}}{\epsilon}\bar{\phi}_{\bar{x}}\right)^2 + \left(\frac{\alpha}{\epsilon}\bar{\phi}_{\bar{y}}\right)^2 + \frac{1}{\epsilon^2}\left(\bar{\phi}_Z\right)^2\right\} + H = 0 \\
\Rightarrow \quad & \frac{\epsilon}{\alpha^{1/2}}\left(\frac{\alpha^{1/2}}{\epsilon}(\alpha-1)\psi_X\right) + \frac{\alpha^{3/2}}{\epsilon}\frac{\epsilon}{\alpha^{1/2}}\psi_\tau \\
& \quad + \frac{1}{2}\alpha\left\{\left(\frac{\alpha^{1/2}}{\epsilon}\frac{\epsilon}{\alpha^{1/2}}\psi_X\right)^2 + \left(\frac{\alpha}{\epsilon}\frac{\epsilon}{\alpha^{1/2}}\psi_Y\right)^2 + \frac{1}{\epsilon^2}\left(\frac{\epsilon}{\alpha^{1/2}}\psi_Z\right)^2\right\} + H = 0 \\
\Rightarrow \quad & (\alpha-1)\psi_X + \alpha\psi_\tau + \frac{1}{2}\alpha\left\{\left(\psi_X\right)^2 + \alpha\left(\psi_Y\right)^2 + \frac{1}{\alpha}\left(\psi_Z\right)^2\right\} + H = 0 \\
\Rightarrow \quad & \alpha\psi_X - \psi_X + \alpha\psi_\tau + \frac{1}{2}\left\{+\alpha\psi_X^2 + \alpha^2\psi_Y^2 + \psi_Z^2\right\} + H = 0 \quad \text{at} \quad Z = 1 + \alpha H \ .
\end{aligned}$$

This gives (4.32). We can look at (4.28) and obtain the following:

$$\begin{aligned}
& \bar{\phi}_{\ \bar{z}} = 0 & & \text{at } \bar{z} = 0 \\
\Rightarrow \quad & \bar{\phi}_{\ Z} = 0 & & \text{at } Z = 0 \\
\Rightarrow \quad & \frac{\epsilon}{\alpha^{1/2}}\psi_{\ Z} = 0 & & \text{at } Z = 0 \\
\Rightarrow \quad & \psi_{\ Z} = 0 & & \text{at } Z = 0
\end{aligned}$$

This gives (4.33). □

Proposition 4.4. *Equations* (4.36), (4.37), *and* (4.38) *in conjunction with the bottom boundary condition* (4.33) *imply the following:*

$$
\begin{aligned}
(4.36)\ \textit{and}\ (4.33) &\Rightarrow \psi_0 = B_0(X,Y,\tau) \\
(4.37)\ \textit{and}\ (4.33) &\Rightarrow \psi_1 = -\frac{Z^2}{2}B_{0XX} + B_1(X,Y,\tau) \\
(4.38)\ \textit{and}\ (4.33) &\Rightarrow \psi_2 = \frac{Z^4}{4!}B_{0XXXX} - \frac{Z^2}{2}B_{1XX} - \frac{Z^2}{2}\psi_{0YY} + B_2(X,Y,\tau)
\end{aligned}
$$

Proof. Antidifferentiate (4.36) once with respect to Z to get $\psi_{0Z} = C_0(X,\tau)$ and again to get

$$\psi_0 = B_0(X,Y,\tau) + ZC_0(X,Y,\tau)\ . \tag{4.62}$$

Now (4.33) gives

$$0 = \psi_Z = \psi_{0Z} + \alpha\psi_{1Z} + \alpha^2\psi_{2Z} + O(\alpha^3) \qquad \text{for} \quad Z = 0\ ,$$

so for $Z = 0$

$$0 = \psi_{0Z} = \psi_{1Z} = \psi_{2Z}\ . \tag{4.63}$$

Thus by (4.62) $0 = \psi_{0Z} = C_0(X,Y,\tau)$ for all Z. Thus, (4.36), (4.62), and (4.63) imply that

$$\psi_0 = B_0(X,Y,\tau) \quad .$$

Note that ψ_0 does not depend on Z, nor does $\psi_{0XX} = B_{0XX}$. Next anti-differentiating (4.37) once with respect to Z gives

$$\psi_{1Z} = (-\psi_{0XX})\,Z + D(X,Y,\tau)\ . \tag{4.64}$$

Then, from (4.63) we get

$$\psi_{1Z} = 0 = 0 + D(X,Y,\tau) \qquad \text{at} \quad Z = 0$$

so $D(X,Y,\tau) = 0$ for all Z. Thus,

$$\psi_{1Z} = -Z\psi_{0XX} \ . \tag{4.65}$$

Anti-differentiating again with respect to Z gives

$$\psi_1 = -\frac{Z^2}{2}\psi_{0XX} + B_1(X,Y,\tau) \tag{4.66}$$

and (4.39) implies $\psi_{0XX} = B_{0XX}$, so

$$\psi_1 = -\frac{Z^2}{2}B_{0XX} + B_1 \ ,$$

Next (4.38) gives $\psi_{2ZZ} = -\psi_{1XX} - \psi_{0YY}$ so, by (4.66) one obtains

$$\psi_{2ZZ} = \frac{Z^2}{2}B_{0XXXX} - B_{1XX} - \psi_{0YY} \ . \tag{4.67}$$

Anti-differentiating with respect to Z gives

$$\psi_{2Z} = \frac{Z^3}{3!}B_{0XXXX} - ZB_{1XX} - Z\psi_{0YY} + E(X,Y,\tau) \ . \tag{4.68}$$

Now, from (4.63) at $Z = 0$ we get

$$\psi_{2Z}|_{Z=0} = 0 = 0 - 0 + E$$

and so we get that $E = 0$ for all Z. Thus

$$\psi_{2Z} = \frac{Z^3}{3!} B_{0XXXX} - Z B_{1XX} - Z \psi_{0YY} \ .$$

Anti-differentiating once more with respect to Z, we get

$$\psi_2 = \frac{Z^4}{4!} B_{0XXXX} - \frac{\bar{Z}^2}{2} B_{1XX} - \frac{\bar{Z}^2}{2} \psi_{0YY} + B_2(X, Y, \tau) \ .$$

□

Proposition 4.5. *The leading orders* $O(\alpha^0)$ *and* $O(\alpha^1)$ *of equation* (4.32) *at the surface give*

$$\begin{aligned} O(\alpha^0): &\qquad H_0 = \psi_{0X} = B_{0X} \\ O(\alpha^1): &\qquad 0 = H_1 + B_{0X} + \frac{1}{2} B_{0XXX} - B_{1X} + B_{0\tau} + \frac{1}{2} B_{0X}^2 \end{aligned}$$

Proof. At $Z = 1 + \alpha H$ (4.32) gives

$$H + \alpha \psi_X - \psi_X + \alpha \psi_\tau + \frac{1}{2} \left\{ \alpha \psi_X^2 + \alpha^2 \psi_Y^2 + \psi_Z^2 \right\} = 0 \ .$$

Then (4.34) and (4.35) give

$$\begin{aligned} 0 =& H_0 + \alpha H_1 + O(\alpha^2) + \alpha \psi_{0X} + \alpha^2 \psi_{1X} + \alpha^3 \psi_{2X} + O(\alpha^4) \\ & - \psi_{0X} - \alpha \psi_{1X} - \alpha^2 \psi_{2X} - O(\alpha^3) \\ & + \alpha \psi_{0\tau} + \alpha^2 \psi_{1\tau} + \alpha^3 \psi_{2\tau} + O(\alpha^4) \\ & + \alpha \frac{1}{2} \left[\psi_{0X} + \alpha \psi_{1X} + \alpha^2 \psi_{2X} + O(\alpha^3) \right]^2 \\ & + \alpha^2 \frac{1}{2} \left[\psi_{0Y} + \alpha \psi_{1Y} + \alpha^2 \psi_{2Y} + O(\alpha^3) \right]^2 \\ & + \frac{1}{2} \left[\psi_{0Z} + \alpha \psi_{1Z} + \alpha^2 \psi_{2Z} + O(\alpha^3) \right]^2 \ , \end{aligned}$$

and we get

$$\begin{aligned}0 = H_0 + \alpha H_1 + O(\alpha^2) \\ + \alpha\psi_{0X} + \alpha^2\psi_{1X} + \alpha^3\psi_{2X} + O(\alpha^4) \\ -\psi_{0X} - \alpha\psi_{1X} - \alpha^2\psi_{2X} - O(\alpha^3) \\ + \alpha\psi_{0\tau} + \alpha^2\psi_{1\tau} + \alpha^3\psi_{2\tau} + O(\alpha^4) \\ + \alpha\frac{1}{2}\psi_{0X}^2 + \alpha^2\psi_{0X}\psi_{1X} + O(\alpha^3) \\ + \alpha^2\psi_{0Y} + O(\alpha^3) \\ +\frac{1}{2}\psi_{0Z}^2 + \alpha\psi_{0Z}\psi_{1Z} + O(\alpha^2) \ .\end{aligned}$$

Therefore

$$\begin{aligned}0 &= \left[H_0 - \psi_{0X} + \frac{1}{2}\psi_{0Z}^2\right] \\ &+ \alpha\left[H_1 + \psi_{0X} - \psi_{1X} + \psi_{0\tau} + \frac{1}{2}\psi_{0X}^2 + \psi_{0Z}\psi_{1Z}\right] \\ &+ O(\alpha^2) \ .\end{aligned} \tag{4.69}$$

Thus, from the α^0 term of (4.69), we get

$$0 = H_0 - \psi_{0X} + \frac{1}{2}\psi_{0Z}^2 \tag{4.70}$$

and from the α^1 term of (4.69),

$$0 = H_1 + \psi_{0X} - \psi_{1X} + \psi_{0\tau} + \frac{1}{2}\psi_{0X}^2 + \psi_{0Z}\psi_{1Z} \tag{4.71}$$

at $Z = 1 + \alpha H$.

Thus from (4.70), (4.63), and (4.39) we get

$$\begin{aligned} 0 &= H_0 - B_{0X} + \frac{1}{2}0^2 \\ &= H_0 - B_{0X} \ . \end{aligned} \tag{4.72}$$

Thus,

$$H_0 = \psi_{0X} = B_{0X} \ .$$

Furthermore,

$$\psi_{0\tau} = B_{0\tau}$$

from (4.39). Also (4.40) implies

$$\psi_1 = -\frac{Z^2}{2} B_{0XX} + B_1 \ ,$$

and differentiating with respect to X we have

$$\psi_{1X} = -\frac{Z^2}{2} B_{0XXX} + B_{1X} \ .$$

Thus

$$\psi_{1X}|_{Z=1+\alpha H} = -\frac{1}{2}(1 + 2\alpha H + \alpha^2 H^2) B_{0XXX} + B_{1X} \ . \tag{4.73}$$

Now the α and α^2 terms in (4.73) create $O(\alpha^2)$ and $O(\alpha^3)$ terms in (4.69), or higher order terms in (4.71) so we can drop both terms in these settings to obtain

$$\psi_{1X}|_{Z=1+\alpha H} = -\frac{1}{2} B_{0XXX} + B_{1X} \ . \tag{4.74}$$

Next, from (4.71), (4.63), (4.39), and (4.74), we get

$$0 = H_1 + B_{0X} - \left(-\frac{1}{2} B_{0XXX} + B_{1X} \right) + B_{0\tau} + \frac{1}{2} B_{0X}^2 + 0$$

so,

$$0 = H_1 + B_{0X} + \frac{1}{2} B_{0XXX} - B_{1X} + B_{0\tau} + \frac{1}{2} B_{0X}^2 \ .$$

□

Proposition 4.6. *The leading orders $O(\alpha^1)$ and $O(\alpha^2)$ of equation* (4.31) *at the surface give:*

$$O(\alpha^1) : H_{0X} = B_{0XX}$$

$$O(\alpha^2) : -H_0 B_{0XX} + \frac{1}{6} B_{0XXXX} - B_{1XX} - \psi_{0YY}$$
$$= -H_{1X} + H_{0X} + H_{02} + B_{0X} H_{0X}$$

Proof. At $Z = 1 + \alpha H = 1 + \alpha H_0 + \alpha^2 H_1 + O(\alpha^3)$ the kinematic equation (4.31) gives

$$\psi_Z = \alpha^2 \psi_X H_X + \alpha^2 H_X - \alpha H_X + \alpha^2 H_\tau + \alpha^3 H_Y \psi_Y \ .$$

Thus substituting in (4.34) and (4.35) gives

$$
\begin{aligned}
\psi_{0Z} + \alpha\psi_{1Z} + & \alpha^2\psi_{2Z} + O(\alpha^3) \\
&= -\alpha[H_{0X} + \alpha H_{1X} + O(\alpha^2)] \\
&\quad + \alpha^2[H_{0X} + \alpha H_{1X} + O(\alpha^2)] \\
&\quad + \alpha^2[H_{0\tau} + \alpha H_{1\tau} + O(\alpha^2)] \\
&\quad + \alpha^3[\psi_{0Y} + \alpha\psi_{1Y} + \alpha^2\psi_{2Y} + O(\alpha^3)][H_{0Y} + \alpha H_{1Y} + O(\alpha^2)] \\
&\quad + \alpha^2[\psi_{0X} + \alpha\psi_{1X} + \alpha^2\psi_{2X} + O(\alpha^3)][H_{0X} + \alpha H_{1X} + O(\alpha^2)] \; .
\end{aligned} \tag{4.75}
$$

Now from (4.40) and (4.41) we have,

$$
\begin{aligned}
\psi_{0Z} + & \alpha\psi_{1Z} + \alpha^2\psi_{2Z} + O(\alpha^3) \\
&= 0 + \alpha(-ZB_{0XX}) + \alpha^2\left(\frac{1}{6}Z^3B_{0XXXX} - Z\psi_{0YY} - ZB_{1XX}\right) + O(\alpha^3)
\end{aligned}
$$

At $Z = 1 + \alpha H$, we expand using (4.35) and get

$$
\begin{aligned}
\psi_{0Z} + \alpha\psi_{1Z} + & \alpha^2\psi_{2Z} + O(\alpha^3) \\
&= -\alpha[1 + \alpha H_0 + \alpha^2 H_1 + O(\alpha^3)]B_{0XX} \\
&\quad + \alpha^2\frac{1}{6}[1 + \alpha H_0 + \alpha^2 H_1 + O(\alpha^3)]^3 B_{0XXXX} \\
&\quad - \alpha^2[1 + \alpha H_0 + \alpha^2 H_1 + O(\alpha^3)]\psi_{0YY} \\
&\quad - \alpha^2[1 + \alpha H_0 + \alpha^2 H_1 + O(\alpha^3)]B_{1XX} + 0 + O(\alpha^3) \; .
\end{aligned}
$$

Moving the α^3 terms into $O(\alpha^3)$, we have

$$
\begin{aligned}
\psi_{0Z} + \alpha\psi_{1Z} + \alpha^2\psi_{2Z} + O(\alpha^3) & \\
&= -\alpha B_{0XX} - \alpha^2 H_0 B_{0XX} + O(\alpha^3) \\
&\qquad + \frac{1}{6}\alpha^2 B_{0XXXX} + O(\alpha^3) \\
&\qquad - \alpha^2 \psi_{0YY} + O(\alpha^3) \\
&\qquad - \alpha^2 B_{1XX} + O(\alpha^3)\ .
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\psi_{0Z} + \alpha\psi_{1Z} + &\alpha^2\psi_{2Z} + O(\alpha^3) \\
&= -\alpha B_{0XX} + \alpha^2\left[-H_0 B_{0XX} + \frac{1}{6}B_{0XXXX} - \psi_{0YY} - B_{1XX}\right] + O(\alpha^3)\ .
\end{aligned}
\tag{4.76}
$$

Now substituting (4.76) into (4.75) we get

$$
\begin{aligned}
-\alpha B_{0XX} + \alpha^2 &\left[-H_0 B_{0XX} + \frac{1}{6}B_{0XXXX} - \psi_{0YY} - B_{1XX}\right] + O(\alpha^3) \\
&= -\alpha H_{0X} - \alpha^2 H_{1X} + O(\alpha^3) \\
&\qquad + \alpha^2 H_{0X} + O(\alpha^3) \\
&\qquad + \alpha^2 H_{0\tau} + O(\alpha^3) \\
&\qquad + \alpha^2 \psi_{0X} H_{0X} + O(\alpha^3)\ .
\end{aligned}
$$

So matching α^1 terms implies $B_{0XX} = H_{0X}$ which is consistent with (4.72). Matching α^2 terms implies

$$-H_0 B_{0XX} + \frac{1}{6} B_{0XXXX} - \psi_{0YY} - B_{1XX} = -H_{1X} + H_{0X} + H_{0\tau} + \psi_{0X} H_{0X} \ .$$

Substituting $\psi_{0X} = B_{0X}$ from (4.39) gives

$$-H_0 B_{0XX} + \frac{1}{6} B_{0XXXX} - \psi_{0YY} - B_{1XX} = -H_{1X} + H_{0X} + H_{0\tau} + B_{0X} H_{0X} \ . \quad (4.77)$$

□

Proposition 4.7. (4.52) *transforms into*

$$\left[\hat{H}_{\hat{\tau}} + \hat{A}\hat{H}_{\hat{X}} + \frac{3}{2}\hat{B}\hat{H}\hat{H}_{\hat{X}} + \frac{1}{6}\hat{C}\hat{H}_{\hat{X}\hat{X}\hat{X}} \right]_{\hat{X}} + \frac{1}{2}\hat{D}\hat{H}_{\hat{Y}\hat{Y}} = 0 \quad (4.78)$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ are arbitrary constants with the stipulation that the sign of $\hat{C}$ and the sign of $\hat{D}$ must be the same.

Proof. From (4.50) we have

$$0 = 2H_\tau + 3HH_X + \frac{1}{3} H_{XXX} + 2H_{0X} + \psi_{YY}$$

or

$$0 = H_\tau + \frac{3}{2} HH_X + \frac{1}{6} H_{XXX} + H_X + \frac{1}{2}\psi_{YY} \ .$$

We can use the following to have a coordinate transformation:

$$\hat{X} = BX \ , \ \hat{Y} = DY \ , \ \hat{Z} = Z \ \text{ and } \hat{\tau} = C\tau$$

along with $\hat{h} = L^{-1}H + K$, or $L\hat{H} - LK = H$, to obtain

$$0 = C(L\hat{H} - LK)_{\hat{\tau}} + 3(L\hat{H} - LK)B(L\hat{H} - LK)_{\hat{X}} \\ + \frac{1}{3}B^3(L\hat{H} - LK)_{\hat{X}\hat{X}\hat{X}} + 2B(L\hat{H} - LK)_{\hat{X}} + D^2\psi_{\hat{Y}\hat{Y}} \ .$$

Therefore,

$$0 = BL\hat{H}_{\hat{X}} + CL\hat{H}_{\tau} + \frac{3}{2}\left(L^2B\hat{H}\hat{H}_{\hat{X}} - L^2KB\hat{H}_{\hat{X}}\right) \\ + \frac{1}{6}LB^3\hat{H}_{\hat{X}\hat{X}\hat{X}} + \frac{1}{2}D^2\psi_{\bar{Y}\bar{Y}}$$

or

$$0 = BL\left\{1 - \frac{3}{2}KL\right\}\hat{H}_{\hat{X}} + CL\hat{H}_{\tau} + \frac{3}{2}L^2B\hat{H}\hat{H}_{\hat{X}} \\ + \frac{1}{6}LB^3\hat{H}_{\hat{X}\hat{X}\hat{X}} + \frac{1}{2}D^2\psi_{\bar{Y}\bar{Y}} \ .$$

Thus, dividing by CL yields

$$0 = \frac{B}{C}\left\{1 - \frac{3}{2}KL\right\}\hat{H}_{\hat{X}} + \hat{H}_{\tau} + \frac{3}{2}L\frac{B}{C}\hat{H}\hat{H}_{\hat{X}} \\ + \frac{1}{6}\frac{B}{C}B^2\hat{H}_{\hat{X}\hat{X}\hat{X}} + \frac{1}{2}\frac{D^2}{CL}\psi_{\hat{Y}\hat{Y}} \ . \tag{4.79}$$

Next from (4.42) we have $H_0 = \psi_{0X}$, so for α small we have $B_X = H$. Thus, we get

$$\begin{aligned} \left\{L\hat{H} - KL\right\} &= \psi_x \\ \left\{L\hat{H} - KL\right\} &= B\psi_{\hat{X}} \end{aligned}$$

so

$$\left\{L\hat{H} - KL\right\}_{\hat{Y}\hat{Y}} = B\psi_{\hat{X}\hat{Y}\hat{Y}}$$

or

$$L\hat{H}_{\hat{Y}\hat{Y}} = B\psi_{\hat{X}\hat{Y}\hat{Y}} \ . \tag{4.80}$$

Differentiating (4.79) with respect to $\hat{X}$ gives

$$0 = \left[\frac{B}{C}\left\{1 - \frac{3}{2}KL\right\}\hat{H}_{\hat{X}} + \hat{H}_{\tau} + \frac{3}{2}L\frac{B}{C}\hat{H}\hat{H}_{\hat{X}} + \frac{1}{6}\frac{B}{C}B^2\hat{H}_{\hat{X}\hat{X}\hat{X}}\right]_{\hat{X}} + \frac{1}{2}\frac{D^2}{CL}\psi_{\hat{Y}\hat{Y}\hat{X}} \ . \tag{4.81}$$

Now substituting (4.80) into (4.79) gives

$$0 = \left[\frac{B}{C}\left\{1 - \frac{3}{2}KL\right\}\hat{H}_{\hat{X}} + \hat{H}_{\tau} + \frac{3}{2}L\frac{B}{C}\hat{H}\hat{H}_{\hat{X}} + \frac{1}{6}\frac{B}{C}B^2\hat{H}_{\hat{X}\hat{X}\hat{X}}\right]_{\hat{X}} + \frac{1}{2}\frac{D^2}{CB}\hat{H}_{\hat{Y}\hat{Y}} \ .$$

Next

1. Choose C
2. Choose B to determine the coefficient of $\hat{H}_{\bar{X}\bar{X}\bar{X}}$, $\hat{C} = \frac{B}{C}B^2$
3. Choose L to determine the coefficient of $\hat{H}\hat{H}_{\bar{X}}$, $\hat{B} = L\frac{B}{C}$
4. Choose K to determine the coefficient of $\bar{H}_{\bar{X}}$, $\hat{A} = \frac{B}{C}\left\{1 - \frac{3}{2}KL\right\}$
5. Choose D to determine the coefficient of $\bar{H}_{\bar{Y}\bar{Y}}$, $\hat{D} = \frac{d^2}{CB}$. Note that $\hat{D}$ must have the same sign as $\hat{C}$.

Thus

$$0 = \left[\hat{A}\hat{H}_{\hat{X}} + \hat{H}_{\tau} + \frac{3}{2}\hat{B}\hat{H}\hat{H}_{\hat{X}} + \frac{1}{6}\hat{C}\hat{H}_{\hat{X}\hat{X}\hat{X}}\right]_{\hat{X}} + \frac{1}{2}\hat{D}\hat{H}_{\hat{Y}\hat{Y}\hat{X}}$$

for $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ arbitrary but where the sign of $\hat{C}$ is the same as the sign of $\hat{D}$.

$\square$

CHAPTER 5: Run-Up Equation

The last part of a tsunami model is the run-up stage, where the depth of the ocean begins to become shallow as the tsunami approaches land. For this final stage we can use a variant of the KdV equation where the depth of the ocean h is no longer constant, but rather a function of the horizontal directions. From (3.53) we have, for the 1 dimensional KdV equation,

$$0 = \hat{H}_{\hat{\tau}} + \sqrt{gh}\hat{H}_{\hat{X}} + \frac{3}{2}\alpha\hat{H}\hat{H}_{\hat{X}} + \frac{1}{6}\epsilon^2\hat{H}_{\hat{X}\hat{X}\hat{X}}$$

To model the run-up phase of a wave we will let the depth of the water h be a function $h(\hat{X})$ of $\hat{X}$ as shown in [1]. Since h is now a function of $\hat{X}$, there is also an aditional term, $\frac{1}{2}\left(\sqrt{gh(\hat{X})}\right)_{\hat{X}}\hat{H}$. Thus, we can model a wave approaching some sort of land where the depth of the water h is no longer constant [9] and get,

$$0 = \hat{H}_{\hat{\tau}} + \sqrt{gh(\hat{X})}\hat{H}_{\hat{X}} + \frac{3}{2}\alpha\hat{H}\hat{H}_{\hat{X}} + \frac{1}{6}\epsilon^2\hat{H}_{\hat{X}\hat{X}\hat{X}} + \frac{1}{2}\left(\sqrt{gh(\hat{X})}\right)_{\hat{X}}\hat{H} \tag{5.1}$$

We use (5.1) to simulate the run-up stage in our model.

CHAPTER 6: Numerical Approximations

6.1 Lax-Wendroff

Before we can make numerical approximations for the first stage of a tsunami, we need to introduce the Lax-Wendroff correction term. Even though the wave equation is linear, there are instabilities associated with it. To adjust for these instabilities, we add in the Lax-Wendroff correction term. We now derive the Lax-Wendroff term for each half wave equation. We choose Δt and Δx sufficiently small to meet any stability criteria.

We begin with (2.4):

$$H_{tt} - c^2 H_{xx} = f \ .$$

We re-write this as

$$\begin{aligned} f &= \left(\frac{\partial}{\partial t}\right)^2 H - c^2 \left(\frac{\partial}{\partial x}\right)^2 H \\ f &= \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) H \ . \end{aligned} \tag{6.1}$$

We can now let $\tilde{H} = \left(\frac{\partial}{\partial t} + \gamma\frac{\partial}{\partial x}\right) H$. Thus (6.1) has become the coupled (paired) PDEs, or half wave equations,

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\tilde{H} = f \tag{6.2}$$

and

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) H = \tilde{H} \ , \tag{6.3}$$

where (6.2) is first solved for $\tilde{H}$ and then (6.3) is solved for H.

We first solve for the Lax-Wendroff term for half wave equations of the form

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\tilde{H} = f \ , \tag{6.4}$$

which can be re-written as

$$\frac{\partial}{\partial t}\tilde{H} = c\frac{\partial}{\partial x}\tilde{H} + f \ . \tag{6.5}$$

Therefore

$$\begin{aligned}\frac{\partial^2 \tilde{H}}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \tilde{H}}{\partial t}\right) &= \frac{\partial}{\partial t}\left(c\frac{\partial \tilde{H}}{\partial x} + f\right) \\ &= c\frac{\partial}{\partial x}\left(\frac{\partial \tilde{H}}{\partial t}\right) + \frac{\partial f}{\partial t} \\ &= c\frac{\partial}{\partial x}\left(c\frac{\partial \tilde{H}}{\partial x} + f\right) + \frac{\partial f}{\partial t}\end{aligned}$$

and thus

$$\frac{\partial^2 \tilde{H}}{\partial t^2} = c^2\frac{\partial^2 \tilde{H}}{\partial x^2} + c\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} \ . \tag{6.6}$$

Here we introduce the notation G_i^k to be a function G evaluated on the grid at the i^{th} stage of x, $i\Delta x$, and the k^{th} stage of t, $k\Delta t$. We can then get a Taylor series expansion of $\tilde{H}$, namely,

$$\tilde{H}_i^{k+1} = \tilde{H}_i^k + \left(\frac{\partial \tilde{H}}{\partial t}\right)_i^k \Delta t + \frac{1}{2}\left(\frac{\partial^2 \tilde{H}}{\partial t^2}\right)_i^k \Delta t^2 + O(\Delta t^3) \ . \tag{6.7}$$

Now using (6.5) and (6.6) we get

$$\begin{aligned}\tilde{H}_i^{k+1} = \tilde{H}_i^k &+ \Delta t\left[c\left(\frac{\partial \tilde{H}}{\partial x}\right)_i^k + f_i^k\right] \\ &+ \frac{(\Delta t^2)}{2}\left[c^2\left(\frac{\partial^2 \tilde{H}}{\partial x^2}\right)_i^k + c\left(\frac{\partial f}{\partial x}\right)_i^k + \left(\frac{\partial f}{\partial t}\right)_i^k\right] + O(\Delta t^3)\end{aligned} \tag{6.8}$$

Now,

$$\left(\frac{\partial \tilde{H}}{\partial x_i}\right)_i^k = \frac{\tilde{H}_{i+1}^k - \tilde{H}_{i-1}^k}{2\Delta x} \tag{6.9}$$

and

$$\left(\frac{\partial^2 \tilde{H}}{\partial x^2}\right)_i^k = \frac{\tilde{H}_{i+1}^k - 2\tilde{H}_i^k + \tilde{H}_{i-1}^k}{(\Delta x)^2}\ . \tag{6.10}$$

So substituting (6.9) and (6.10) into (6.8) yields

$$\begin{aligned}\tilde{H}_i^{k+1} - \tilde{H}_i^k =& c\Delta t\left[\frac{\tilde{H}_{i+1}^k - \tilde{H}_{i-1}^k}{2\Delta x}\right] \\ &+ \Delta t f_i^k + \frac{(c\Delta t)^2}{2}\left[\frac{\tilde{H}_{i+1}^k - 2\tilde{H}_i^k + \tilde{H}_{i-1}^k}{(\Delta x)^2}\right] \\ &+ \frac{(\Delta t)^2}{2}\left[c\left(\frac{\partial f}{\partial x}\right)_i^k + \left(\frac{\partial f}{\partial t}\right)_i^k\right]\ ,\end{aligned}$$

which can be written as

$$\begin{aligned}\tilde{H}_i^{k+1} =& \tilde{H}_i^k + \Delta t\left[c\frac{\tilde{H}_{i+1}^k - \tilde{H}_{i-1}^k}{2\Delta x} + f_i^k\right] \\ &+ \frac{(c\Delta t)^2}{2}\left[\frac{\tilde{H}_{i+1}^k - 2\tilde{H}_i^k + \tilde{H}_{i-1}^k}{(\Delta x)^2}\right] \\ &+ \frac{(\Delta t)^2}{2}\left[c\left(\frac{f_{i+1}^k - f_{i-1}^k}{2\Delta x}\right) + \left(\frac{f_i^{k+1} - f_i^{k-1}}{2\Delta t}\right)\right]\ .\end{aligned} \tag{6.11}$$

In the case of an earthquake acting as a forcing, the higher order forcing terms

$$\frac{(\Delta t)^2}{2}\left[c\left(\frac{f_{i+1}^k - f_{i-1}^k}{2\Delta x}\right) + \left(\frac{f_i^{k+1} - f_i^{k-1}}{2\Delta t}\right)\right]$$

are considered to be negligible, and so we obtain the Lax-Wendroff term for $\tilde{H}$, namely

$$\frac{(c\Delta t)^2}{2}\left[\frac{\tilde{H}_{i+1}^k - 2\tilde{H}_i^k + \tilde{H}_{i-1}^k}{(\Delta x)^2}\right]\ . \tag{6.12}$$

That is the Lax-Wendroff term (6.12) can be seen as a second order correction term to Euler's Method:

$$\tilde{H}_i^{k+1} = \tilde{H}_i^k + \Delta t \left[c\frac{\tilde{H}_{i+1}^k - \tilde{H}_{i-1}^k}{2\Delta x} + f_i^k \right] + \frac{(c\Delta t)^2}{2} \left[\frac{\tilde{H}_{i+1}^k - 2\tilde{H}_i^k + \tilde{H}_{i-1}^k}{(\Delta x)^2} \right] \tag{6.13}$$

We use (6.13) as our numerical approximation scheme.

We next solve for the Lax-Wendroff term for half wave equations of the form

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x} \right) H = \tilde{H} \tag{6.14}$$

which can be re-written as

$$\frac{\partial}{\partial t} H = -c\frac{\partial}{\partial x} H + \tilde{H} \ . \tag{6.15}$$

Therefore

$$\begin{aligned} \frac{\partial^2 H}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial H}{\partial t}\right) &= \frac{\partial}{\partial t}\left(-c\frac{\partial H}{\partial x} + \tilde{H}\right) \\ &= -c\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial t}\right) + \frac{\partial \tilde{H}}{\partial t} \\ &= -c\frac{\partial}{\partial x}\left(-c\frac{\partial H}{\partial x} + \tilde{H}\right) + \frac{\partial \tilde{H}}{\partial t} \end{aligned}$$

thus

$$\frac{\partial^2 H}{\partial t^2} = c^2\frac{\partial^2 H}{\partial x^2} - c\frac{\partial \tilde{H}}{\partial x} + \frac{\partial \tilde{H}}{\partial t} \ . \tag{6.16}$$

We can now express H as a Taylor series expansion, namely,

$$H_i^{k+1} = H_i^k + \left(\frac{\partial H}{\partial t}\right)_i^k \Delta t + \frac{1}{2}\left(\frac{\partial^2 H}{\partial t^2}\right)_i^k \Delta t^2 + O(\Delta t^3) \ . \tag{6.17}$$

Now using (6.15) and (6.16) we get

$$\begin{aligned} H_i^{k+1} = H_i^k + \Delta t \left[-c \left(\frac{\partial H}{\partial x} \right)_i^k + \tilde{H}_i^k \right] \\ + \frac{(\Delta t^2)}{2} \left[c^2 \left(\frac{\partial^2 H}{\partial x^2} \right)_i^k - c \left(\frac{\partial \tilde{H}}{\partial x} \right)_i^k + \left(\frac{\partial \tilde{H}}{\partial t} \right)_i^k \right] + O(\Delta t^3) \ . \end{aligned} \tag{6.18}$$

Now,

$$\left(\frac{\partial H}{\partial x_i} \right)_i^k = \frac{H_{i+1}^k - H_{i-1}^k}{2\Delta x} \tag{6.19}$$

and

$$\left(\frac{\partial^2 H}{\partial x^2} \right)_i^k = \frac{H_{i+1}^k - 2H_i^k + H_{i-1}^k}{(\Delta x)^2} \ . \tag{6.20}$$

Substituting (6.19) and (6.20) into (6.18) yields

$$\begin{aligned} H_i^{k+1} - H_i^k = & - c\Delta t \left[\frac{H_{i+1}^k - H_{i-1}^k}{2\Delta x} \right] \\ & + \Delta t \tilde{H}_i^k + \frac{(c\Delta t)^2}{2} \left[\frac{H_{i+1}^k - 2H_i^k + H_{i-1}^k}{(\Delta x)^2} \right] \\ & + \frac{(\Delta t)^2}{2} \left[-c \left(\frac{\partial \tilde{H}}{\partial x} \right)_i^k + \left(\frac{\partial \tilde{H}}{\partial t} \right)_i^k \right] \ , \end{aligned}$$

which can be written as

$$\begin{aligned} H_i^{k+1} = H_i^k + \Delta t \left[-c \frac{H_{i+1}^k - H_{i-1}^k}{2\Delta x} + \tilde{H}_i^k \right] \\ + \frac{(c\Delta t)^2}{2} \left[\frac{H_{i+1}^k - 2H_i^k + H_{i-1}^k}{(\Delta x)^2} \right] \\ + \frac{(\Delta t)^2}{2} \left[-c \left(\frac{\tilde{H}_{i+1}^k - \tilde{H}_{i-1}^k}{2\Delta x} \right) + \left(\frac{\tilde{H}_i^{k+1} - \tilde{H}_i^{k-1}}{2\Delta t} \right) \right] \ . \end{aligned}$$

Now the higher order terms of $\tilde{H}$,

$$\frac{(\Delta t)^2}{2}\left[-c\left(\frac{\tilde{H}_{i+1}^k-\tilde{H}_{i-1}^k}{2\Delta x}\right)+\left(\frac{\tilde{H}_i^{k+1}-\tilde{H}_i^{k-1}}{2\Delta t}\right)\right]$$

are considered to be negligible, so we obtain the Lax-Wendroff term for H, namely

$$\frac{(c\Delta t)^2}{2}\left[\frac{H_{i+1}^k-2H_i^k+H_{i-1}^k}{(\Delta x)^2}\right] . \tag{6.21}$$

That is, the Lax-Wendroff term (6.21) can be seen as a second order correction term to Euler's Method:

$$H_i^{k+1}=H_i^k+\Delta t\left[-c\frac{H_{i+1}^k-H_{i-1}^k}{2\Delta x}+\tilde{H}_i^k\right]+\frac{(c\Delta t)^2}{2}\left[\frac{H_{i+1}^k-2H_i^k+H_{i-1}^k}{(\Delta x)^2}\right] \tag{6.22}$$

We use (6.22) as our approximation scheme.

6.2 Exposition of Wave Equation Numerical Approximation

We can now look at the one-dimensional wave equation and solve for a numerical solution. We first solve (6.2):

$$\left(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\right)\tilde{H}=f$$

which can be re-written as

$$\frac{\partial}{\partial t}\tilde{H}=c\frac{\partial}{\partial x}\tilde{H}+f .$$

We now look at our position function at each step in time Δt, and let x values be chosen on a Δx grid. For this we use Euler's method with initial conditions at $t=0$ of $f(x,0)=0$ and $\tilde{H}(x,0)=0$. The initial condition gives $\tilde{H}(i\Delta x,0)=0$, for all i, and for $k\Delta t = 0$. We then deploy the numerical scheme from (6.13) recursively. Once $\tilde{H}(i\Delta x, k\Delta t)$ is determined for all i and for fixed k, by (6.13) one obtains

$$\begin{aligned}
&\tilde{H}(i\Delta x,(k+1)\Delta t) = \tilde{H}(i\Delta x, k\Delta t) \\
&+ \left[c \left\{ \frac{\tilde{H}((i+1)\Delta x, k\Delta t) - \tilde{H}((i-1)\Delta x, k\Delta t)}{2\Delta x} \right\} + f(i\Delta x, k\Delta t) \right] \Delta t \\
&+ \frac{(c\Delta t)^2}{2} \left[\frac{\tilde{H}((i+1)\Delta x, k\Delta t) - 2\tilde{H}(i\Delta x, k\Delta t) + \tilde{H}((i-1)\Delta x, k\Delta t)}{(\Delta x)^2} \right] .
\end{aligned} \tag{6.23}$$

Thus one obtains $\tilde{H}(i\Delta x, k\Delta t)$ for all i and for all k. With $\tilde{H}$ determined the initial condition on H is $H(i\Delta x, 0) = 0$ for all i.

The next step is to solve (6.3) for H by deploying (6.22) recursively to

$$\left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) H = \tilde{H} \ ,$$

$$\begin{aligned}
&H(i\Delta x,(k+1)\Delta t) = H(i\Delta x, k\Delta t) \\
&+ \left[-c \left\{ \frac{H((i+1)\Delta x, k\Delta t) - H((i-1)\Delta x, k\Delta t)}{2\Delta x} \right\} + \tilde{H}(i\Delta x, k\Delta t) \right] \Delta t \\
&+ \frac{(c\Delta t)^2}{2} \left[\frac{H((i+1)\Delta x, k\Delta t) - 2H(i\Delta x, k\Delta t) + H((i-1)\Delta x, k\Delta t)}{(\Delta x)^2} \right] ,
\end{aligned} \tag{6.24}$$

which produces H for all steps of time.

6.3 Exposition of KdV Equation Numerical Approximation

For the numerical solution of the KdV equation in one dimension, we begin with the equation from (3.53)

$$0 = \hat{H}_{\hat{\tau}} + \sqrt{gh}\hat{H}_{\hat{X}} + \frac{3}{2}\alpha\hat{H}\hat{H}_{\hat{X}} + \frac{1}{6}\epsilon^2\hat{H}_{\hat{X}\hat{X}\hat{X}} \quad .$$

We can then rewrite this as

$$\hat{H}_{\hat{\tau}} = -\sqrt{gh}\hat{H}_{\hat{X}} - \frac{3}{2}\alpha\hat{H}\hat{H}_{\hat{X}} - \frac{1}{6}\epsilon^2\hat{H}_{\hat{X}\hat{X}\hat{X}} \ . \tag{6.25}$$

Once again, we look at the KdV equation throughout steps in time in a grid pattern. First we get an estimation of each component of the KdV equation. We first start off with $\hat{H}_{\hat{X}}$:

$$\hat{H}_{\hat{X}}(\hat{X}, k\Delta\hat{\tau}) \approx \frac{\hat{H}(\hat{X} + \Delta\hat{X}, k\Delta\hat{\tau}) - \hat{H}(\hat{X} - \Delta\hat{X}, k\Delta\hat{\tau})}{2\Delta\hat{X}} \tag{6.26}$$

Next, we obtain $\hat{H}_{\hat{X}\hat{X}}$:

$$\hat{H}_{\hat{X}\hat{X}}(\hat{X}, k\Delta\hat{\tau}) \approx \frac{\frac{1}{\Delta\hat{X}}\left(\hat{H}_{\hat{X}}(\hat{X} + \Delta\hat{X}, k\Delta\hat{\tau}) - \hat{H}_{\hat{X}}(\hat{X}, k\Delta\hat{\tau})\right) + \frac{1}{\Delta\hat{X}}\left(\hat{H}_{\hat{X}}(\hat{X}, k\Delta\hat{\tau}) - \hat{H}_{\hat{X}}(\hat{X} - \Delta\hat{X}, k\Delta\hat{\tau})\right)}{2}$$

and simplifying gives

$$\hat{H}_{\hat{X}\hat{X}}(\hat{X}, k\Delta\hat{\tau}) \approx \frac{\left(\hat{H}_{\hat{X}}(\hat{X} + \Delta\hat{X}, k\Delta\hat{\tau}) - \hat{H}_{\hat{X}}(\hat{X} - \Delta\hat{X}, k\Delta\hat{\tau})\right)}{2\Delta\hat{X}} \ .$$

Now we can apply (6.26) and get

$$\hat{H}_{\hat{X}\hat{X}}(\hat{X}, k\Delta\hat{\tau}) \approx$$

$$\frac{\frac{1}{2\Delta\hat{X}}\left(\hat{H}(\hat{X}+2\Delta\hat{X}, k\Delta\hat{\tau}) - \hat{H}(\hat{X}, k\Delta\hat{\tau})\right) - \frac{1}{2\Delta x}\left(\hat{H}(\hat{X}, k\Delta\hat{\tau}) - \hat{H}(\hat{X}-2\Delta\hat{X}, k\Delta\hat{\tau})\right)}{2\Delta\hat{X}}$$

which simplifies to

$$\hat{H}_{\hat{X}\hat{X}}(\hat{X}, k\Delta\hat{\tau}) \approx \frac{\hat{H}(\hat{X}+2\Delta\hat{X}, k\Delta\hat{\tau}) - 2\hat{H}(\hat{X}, k\Delta\hat{\tau}) + \hat{H}(\hat{X}-2\Delta\hat{X}, k\Delta\hat{\tau})}{(2\Delta\hat{X})^2} \ . \quad (6.27)$$

Lastly we obtain $H_{\hat{X}\hat{X}\hat{X}}$:

$$\hat{H}_{\hat{X}\hat{X}\hat{X}}(\hat{X}, k\Delta\hat{\tau}) \approx \frac{\frac{1}{\Delta\hat{X}}\left(\hat{H}_{\hat{X}\hat{X}}(\hat{X}+\Delta\hat{X}, k\Delta\hat{\tau}) - \hat{H}_{\hat{X}\hat{X}}(\hat{X}, k\Delta\hat{\tau})\right)}{2} + \frac{\frac{1}{\Delta\hat{X}}\left(\hat{H}_{\hat{X}\hat{X}}(\hat{X}, k\Delta\hat{\tau}) - \hat{H}_{\hat{X}\hat{X}}(\hat{X}-\Delta\hat{X}, k\Delta\hat{\tau})\right)}{2} \ .$$

Simplifying gives

$$\hat{H}_{\hat{X}\hat{X}\hat{X}}(\hat{X}, k\Delta\hat{\tau}) \approx \frac{\hat{H}_{\hat{X}\hat{X}}(\hat{X}+\Delta\hat{X}, k\Delta\hat{\tau}) - \hat{H}_{\hat{X}\hat{X}}(\hat{X}-\Delta\hat{X}, k\Delta\hat{\tau})}{2\Delta\hat{X}} \ .$$

Now, substituting with (6.27) we get

$$\hat{H}_{\hat{X}\hat{X}\hat{X}}(\hat{X}, k\Delta\hat{\tau}) \approx$$
$$\frac{\frac{1}{(2\Delta\hat{X})^2}\left(\hat{H}(\hat{X}+3\Delta\hat{X}, k\Delta\hat{\tau}) - 2\hat{H}(\hat{X}+\Delta\hat{X}, k\Delta\hat{\tau}) + \hat{H}(\hat{X}-\Delta\hat{X}, k\Delta\hat{\tau})\right)}{2\Delta\hat{X}}$$
$$-\frac{\frac{1}{(2\Delta\hat{X})^2}\left(\hat{H}(\hat{X}+\Delta\hat{X}, k\Delta\hat{\tau}) - 2\hat{H}(\hat{X}-\Delta\hat{X}, k\Delta\hat{\tau}) + \hat{H}(\hat{X}-3\Delta\hat{X}, k\Delta\hat{\tau})\right)}{2\Delta\hat{X}} \ ,$$

which simplifies to

$$\hat{H}_{\hat{X}\hat{X}\hat{X}}(\hat{X}, k\Delta\hat{\tau}) \approx$$
$$\frac{\hat{H}(\hat{X}+3\Delta\hat{X}, k\Delta\hat{\tau}) - 3\hat{H}(\hat{X}+\Delta\hat{X}, k\Delta\hat{\tau}) + 3\hat{H}(\hat{X}-\Delta\hat{X}, k\Delta\hat{\tau}) - \hat{H}(\hat{X}-3\Delta\hat{X}, k\Delta\hat{\tau})}{(2\Delta\hat{X})^3} \ . \tag{6.28}$$

We can now look at (6.25) and substitute in (6.26), (6.28) along with

$$H_{\hat{\tau}} \doteq \frac{H(\hat{X}, (k+1)\Delta\hat{\tau}) - H(\hat{X}, k\Delta\hat{\tau})}{\Delta\hat{\tau}} \ . \tag{6.29}$$

Solving the resulting equation for $H(\hat{X}, (k+1)\Delta\hat{\tau})$ yields

$$\begin{aligned}\hat{H}(\hat{X},(k+1)\Delta\hat{\tau}) =& \hat{H}(\hat{X},k\Delta t)\\ &+\Delta\hat{\tau}\left\{-\sqrt{gh}\left(\frac{\hat{H}(\hat{X}+\Delta\hat{X},k\Delta\hat{\tau})-\hat{H}(\hat{X}-\Delta\hat{X},k\Delta\hat{\tau})}{2\Delta\hat{X}}\right)\right.\\ &-\frac{3}{2}\alpha\hat{H}(\hat{X},k\Delta\hat{\tau})\left(\frac{\hat{H}(\hat{X}+\Delta\hat{X},k\Delta\hat{\tau})-\hat{H}(\hat{X}-\Delta\hat{X},k\Delta\hat{\tau})}{2\Delta\hat{X}}\right)\\ &-\frac{1}{6}\epsilon^2\left(\frac{\hat{H}(\hat{X}+3\Delta\hat{X},k\Delta\hat{\tau})-3\hat{H}(\hat{X}+\Delta\hat{X},k\Delta\hat{\tau})}{(2\Delta\hat{X})^3}\right.\\ &\left.\left.+\ \frac{3\hat{H}(\hat{X}-\Delta\hat{X},k\Delta\hat{\tau})-\hat{H}(\hat{X}-3\Delta\hat{X},k\Delta\hat{\tau})}{(2\Delta\hat{X})^3}\right)\right\}\end{aligned} \tag{6.30}$$

which produces $\hat{H}(\hat{X},(k+1)\Delta\hat{\tau})$ for each step of time in the second stage, or traveling stage of our model. Substituting $\hat{X} = i\Delta x$ in (6.32) gives $\hat{H}(i\Delta x, k\Delta t)$ for all i and k. The initial condition for $\hat{H}(i\Delta\hat{X},\bar{k}\Delta\bar{\tau})$ is $\hat{H}(i\Delta\hat{X},\bar{k}\Delta t) = H(i\Delta x,\bar{k}\Delta\bar{\tau})$ where $\bar{k}$ is the last time step of solving the numerical wave equation in Section 6.2.

6.4 Exposition of Run-up Equation Numerical Approximation

We are left to find a numerical scheme for the run-up stage of our model. To obtain this, we begin with the run-up equation from (5.1),

$$0 = \hat{H}_{\hat{\tau}} + \sqrt{gh(\hat{X})}\hat{H}_{\hat{X}} + \frac{3}{2}\alpha\hat{H}\hat{H}_{\hat{X}} + \frac{1}{6}\epsilon^2\hat{H}_{\hat{X}\hat{X}\hat{X}} + \frac{1}{2}\left(\sqrt{gh(\hat{X})}\right)_{\hat{X}}\hat{H}\ , \tag{6.31}$$

where we note that the only difference between this equation and the KdV equation is that h is a function of $\hat{X}$ and the addition of the term $\frac{1}{2}\left(\sqrt{gh(\hat{X})}\right)_{\hat{X}}\hat{H}$. We use the same numerical scheme from the KdV numerics from (6.32), namely

$$
\begin{aligned}
\hat{H}(\hat{X},(k+1)\Delta\hat{\tau}) =& \hat{H}(\hat{X},k\Delta t) \\
&+\Delta\hat{\tau}\left\{-\sqrt{gh}\left(\frac{\hat{H}(\hat{X}+\Delta\hat{X},k\Delta\hat{\tau})-\hat{H}(\hat{X}-\Delta\hat{X},k\Delta\hat{\tau})}{2\Delta\hat{X}}\right)\right. \\
&-\frac{3}{2}\alpha\hat{H}(\hat{X},k\Delta\hat{\tau})\left(\frac{\hat{H}(\hat{X}+\Delta\hat{X},k\Delta\hat{\tau})-\hat{H}(\hat{X}-\Delta\hat{X},k\Delta\hat{\tau})}{2\Delta\hat{X}}\right) \\
&-\frac{1}{6}\epsilon^2\left(\frac{\hat{H}(\hat{X}+3\Delta\hat{X},k\Delta\hat{\tau})-3\hat{H}(\hat{X}+\Delta\hat{X},k\Delta\hat{\tau})}{(2\Delta\hat{X})^3}\right. \\
&+\left.\left.\frac{3\hat{H}(\hat{X}-\Delta\hat{X},k\Delta\hat{\tau})-\hat{H}(\hat{X}-3\Delta\hat{X},k\Delta\hat{\tau})}{(2\Delta\hat{X})^3}\right)\right\} \ ,
\end{aligned}
\tag{6.32}
$$

along with the numerical approximation for the additional term

$$
\frac{1}{2}\left(\sqrt{gh(\hat{X})}\right)_{\hat{X}}\hat{H} = \frac{1}{2}\left(\sqrt{gh(\hat{X})}\right)_{\hat{X}}\hat{H}(\hat{X},k\Delta\tau)\ . \tag{6.33}
$$

Combining (6.32) and (6.33) and letting h be a function of $\hat{X}$, we obtain

$$
\begin{aligned}
\hat{H}(\hat{X},(k+1)\Delta\hat{\tau}) =& \hat{H}(\hat{X},k\Delta t) \\
&+\Delta\hat{\tau}\left\{-\sqrt{gh(\hat{X})}\left(\frac{\hat{H}(\hat{X}+\Delta\hat{X},k\Delta\hat{\tau})-\hat{H}(\hat{X}-\Delta\hat{X},k\Delta\hat{\tau})}{2\Delta\hat{X}}\right)\right. \\
&-\frac{3}{2}\alpha\hat{H}(\hat{X},k\Delta\hat{\tau})\left(\frac{\hat{H}(\hat{X}+\Delta\hat{X},k\Delta\hat{\tau})-\hat{H}(\hat{X}-\Delta\hat{X},k\Delta\hat{\tau})}{2\Delta\hat{X}}\right) \\
&-\frac{1}{6}\epsilon^2\left(\frac{\hat{H}(\hat{X}+3\Delta\hat{X},k\Delta\hat{\tau})-3\hat{H}(\hat{X}+\Delta\hat{X},k\Delta\hat{\tau})}{(2\Delta\hat{X})^3}\right. \\
&+\left.\frac{3\hat{H}(\hat{X}-\Delta\hat{X},k\Delta\hat{\tau})-\hat{H}(\hat{X}-3\Delta\hat{X},k\Delta\hat{\tau})}{(2\Delta\hat{X})^3}\right) \\
&\left.+\frac{1}{2}\left(\sqrt{gh(\hat{X})}\right)_{\hat{X}}\hat{H}(\hat{X},k\Delta\tau)\right\}
\end{aligned}
\tag{6.34}
$$

.

Substituting $\hat{X} = i\Delta x$ in (6.34) gives $\hat{H}(i\Delta x, k\Delta t)$ for all i and k. We use (6.34) to model the run-up stage of our model.

CHAPTER 7: Transitioning Between Stages

We have now derived each stage of our model. It remains to discuss how these models fit together. When an earthquake begins underwater, it produces a forcing on the environment around it. This forcing produces a wave which can be modeled using the forced wave equation. We can recall and analyze both half wave equations from (6.2) and (6.3) namely

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\tilde{H} = f \tag{7.1}$$

and

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)H = \tilde{H}\ . \tag{7.2}$$

Note that f in (7.1) is the main forcing due to an earthquake. This forcing creates a half wave $\tilde{H}$ that acts as the forcing in the other half wave equation (7.2). As the forcing f dies down the wave $\tilde{H}$ proceeds on, and as it moves away its impact dissipates as the force in (7.2). Once the earthquake retires as a forcing and the produced waves move away from each other, natural dispersion and self-focusing terms previously dismissed start to build up and have more impact on the effect of the wave. The dispersion and self-focusing effects are what keeps the moving wave from dissipating or enlarging. This is where the first transition from the wave equation to the KdV equation takes place, as the wave equation does not have the dispersion and self-focusing terms in it.

The KdV equation models the wave as it travels through the ocean, up to the point where it hits land. As the KdV equation is a model for waves traveling along a constant bottom, it makes sense for it to model a wave traveling in the relatively constant depths of the ocean. As the wave approaches land where the depth of the ocean is no long constant, it is clear there is need to switch to a different equation that does not require the bottom to be constant. At this stage we switch to the run-up equation that allows for the depth of the ocean to vary. The three equations we will deploy to model tsunamis are:

$$\begin{aligned}
\text{Wave:}\quad & H_t + cH_x && = \tilde{H} \\
\text{KdV:}\quad & H_t + cH_x + \frac{3}{2}\alpha H H_x + \frac{1}{6}\epsilon^2 H_{xxx} && = 0 \\
\text{Run-up:}\quad & H_t + cH_x + \frac{3}{2}\alpha H H_x + \frac{1}{6}\epsilon^2 H_{xxx} + \frac{1}{2}c_x H && = 0\ .
\end{aligned}$$

We can see that we initially use the wave equation when there is a forcing $\tilde{H}$. Once this forcing dies down the non-linear (self-focusing) and dispersion terms start to build up and have a greater impact on the wave. These terms are added on to the wave equation to form the KdV equation. The KdV equation is then used while the ocean floor remains relatively constant. As the wave approaches land, the ocean floor drastically changes and thus an additional term is used to compensate for the change. For this we add the additional term onto the KdV equation to obtain the run-up equation. This shows a smooth transition between each stage of the model.

CHAPTER 8: Special q-Advanced Functions

Here we discuss the special q-advanced functions, where the parameter q satisfies $q > 1$, see [6], [7], [8]. We start with

$$K_q(t) \;\equiv\; \sum_{k=-\infty}^{\infty} (-1)^k e^{-tq^k} \big/ q^{k(k+1)/2} \ , \quad \text{for } t \geq 0,\ q > 1 \ , \tag{8.1}$$

where $K_q(t) \equiv 0$ for $t < 0$. K_q depends on the parameter $q > 1$. $K_q(t)$ is a wavelet satisfying the multiplicatively advanced differential equation, or MADE, $K_q'(t) = K_q(qt)$ [6]. At any time t such that $t \leq 0$, both $K_q(t)$ and its derivatives $K_q^{(p)}(t)$ are flat. For values of $t \geq 0$, both $K_q(t)$ and its derivatives oscillate by first going negative into a trough, then positive forming a crest before dampening down (see Figure 8.1). Note the resemblance that K_q has to a tsunami profile. This indicates that this function and its derivatives will be very useful in modeling tsunamis.

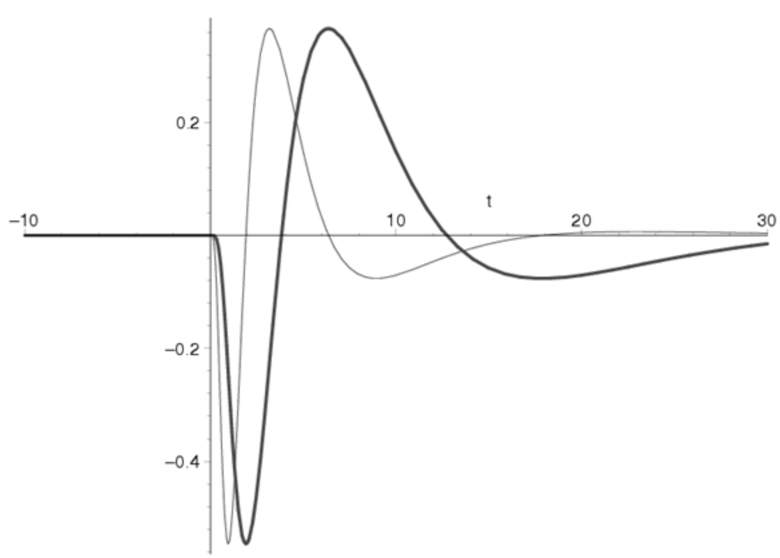

Figure 8.1: K_q (dark) and its derivative (light) for q=2

We describe two more wavelets (as in [8]) that will also be useful in modeling tsunamis, namely,

$$ {}_qCos(t) \;=\; N_q \cdot \int_0^\infty K_q(u)K_q(u-t)\, du \;=\; N_q \sum_{j=-\infty}^{\infty} (-1)^j \; e^{-q^j|t|}/q^{j^2} \;, \tag{8.2} $$

$$ {}_qSin(t) \;=\; -N_q \cdot \int_0^\infty K_q(u)K_q(qu-qt)\, du \;=\; \left(\frac{t}{|t|}\right) N_q \sum_{j=-\infty}^{\infty} (-1)^j \; e^{-q^j|t|}/q^{j(j-1)} \;, \tag{8.3} $$

where

$$ N_q = \left(\sum_{j=-\infty}^{\infty} \frac{(-1)^j}{q^{j^2}} \right)^{-1} \; . $$

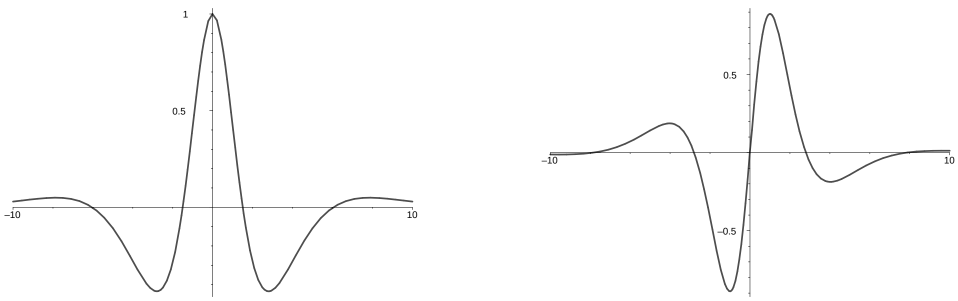

Figure 8.2: **Left:** $y = {}_qCos(t)$ for $q = 1.5$; **Right:** $y = {}_qSin(t)$ for $q = 1.5$.

The constant N_q is a normalization constant chosen such that ${}_qCos(0) = 1$. In addition to also being wavelets, ${}_qCos(t)$ and ${}_qSin(t)$ also satisfy the MADE criterion [8]:

$$ {}_qCos''(t) \;\equiv\; -q \; {}_qCos(qt) \;, \quad {}_qSin''(t) \;\equiv\; -q^2 \; {}_qSin(qt) \; . \tag{8.4} $$

For q near 1, the MADEs in (8.4) approach $f''(t) = -f(t)$ and, in fact, ${}_qCos(t)$ approaches $\cos(t)$ and ${}_qSin(t)$ approaches $\sin(t)$ uniformly on compact sets as $q \to 1$.

CHAPTER 9: Tsunami modeling with q-advanced functions

The q-advanced wavelets $K_q(t)$ and ${}_qSin(t)$ have been used to construct wave height $H_q(x,t)$ and forcing terms $F_q(x,t)$ in tsunami models, see [9], where

$$H_q(x,t) \;\equiv\; A \cdot K_q\left(\frac{t}{\tau}\right) {}_qSin(\gamma \cdot x) \tag{9.1}$$

and, by differentiating (9.1) twice with respect to t,

$$F_q(x,t) \equiv \rho \frac{\partial^2 H_q}{\partial t^2} \;=\; \rho \frac{A \cdot q}{\tau^2}\; K_q\left(\frac{q^2 t}{\tau}\right) {}_qSin(\gamma \cdot x)\;. \tag{9.2}$$

Notice the similarity of the forcing function F_q to the wave height H_q.

The models discussed in this thesis have a clear difference to the ones previously used. For example in [12], the following function used to model forcing from a landslide:

$$H_{TS} \;\equiv\; \frac{A}{\gamma} \tanh\left(\gamma x - \frac{t}{\tau}\right)\;. \tag{9.3}$$

This suggests that the landslide will not only provide the forcing associated with it, but the landslide will continue happening for all time.

A second example is one in [14] which models earthquakes, namely,

$$H_{ZWL} \;\equiv\; (2A\gamma)\,\mathrm{sech}^2(\gamma x)\tanh(\gamma x)\sin\left(\frac{\pi t}{2\tau}\right)\;. \tag{9.4}$$

This earthquake model again simulates an earthquake that quakes on forever. Both of these models, despite their limitations, are still used to model the initial forcing of tsunamis.

An advantage of (9.1) and (9.2) is that F_q is an $\mathcal{L}^2$ function with the consequence that the forcing terms damp down, a more realistic scenario. Another big advantage of (9.1) and (9.2) is that the wave height function H_q is very similar to the forcing function F_q. This implies that the forcing that creates a tsunami is very similar to the tsunami itself. The previous models definitely do not have this attribute. As we shall see, this H_q, F_q similarity endows our model with an early warning potential for ocean shores sufficiently distant from the epicenter of the earthquake.

CHAPTER 10: Approximation of Precursor Wave and Forcing terms via Special Functions

In the event of an earthquake, a vibration emanates from the epicenter disturbing the ground in every direction. As the earthquake disturbance spreads out underwater, the disturbance of the land creates a small forcing on the water above, creating what we will refer to as a precursor wave that precedes the actual tsunami.

We will first take the height function used in [9], and modify it to produce a more accurate forcing. We begin with the height equation $H_q(x,t)$ from the precursor wave. From (9.1):

$$H_q(x,t) \equiv A \cdot K_q\left(\frac{t}{\tau}\right)_q Sin(\gamma \cdot x)$$

which for the appropriate choice of A, q, and τ produces the following graph,

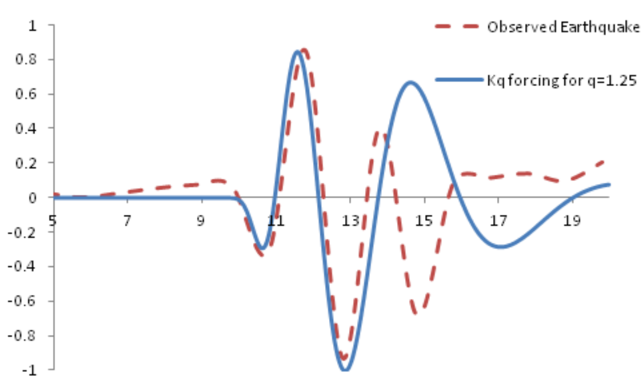

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Figure 10.1: Precursor wave (red dashed) vs. height function $H_q(x,t)$ (blue solid) where the horizontal variable is time (min) and the vertical variable is height (m).

where the dashed red line is the precursor wave for the 2011 Japanese tsunami, and the solid blue line is the H_q model. We clearly see that H_q matches the precursor wave very well for the first and second troughs and the first crest. It then starts to diverge at the second trough and continues to get worse until both dampen down almost completely.

We can now modify and improve the height equation for the precursor wave by adding an additional term and delaying the time to produce a function that has a better fit when compared to the precursor wave. We add in the a term that is a scaled version of the original, along with a delay in time, namely,

$$H_q(x,t) \;\equiv\; A \cdot K_q\left(\frac{t}{\tau}\right){}_qSin(\gamma \cdot x) + B \cdot K_q\left(\frac{t-t_0}{\tau}\right){}_qSin(\gamma \cdot x)\ . \qquad (10.1)$$

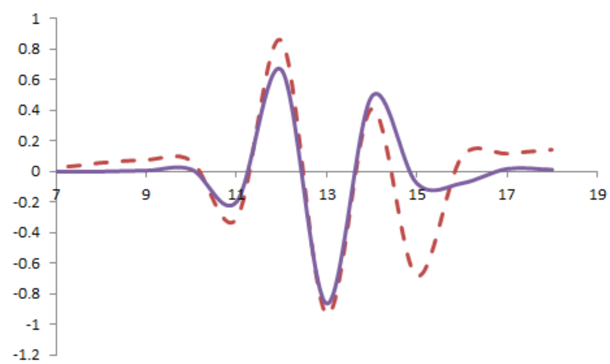

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Figure 10.2: Precursor wave (red dashed) vs. new height function $H_q(x,t)$ (purple solid) where the horizontal variable is time (min) and the vertical variable is height (m).

We can see by comparison of the 2011 Japanese tsunami precursor wave (represented as the red dashed line) and the new height function (represented as the purple solid line) that, similar to the previous model, we have a great match on the first wave, but where the previous data diverged, we now have equation (10.1) matching better. Thus we have gained more accuracy in matching the height term to the precursor wave.

This new height function, (10.1), then gives a new forcing function that we use to model the creation of the 2011 Japanese tsunami, namely

$$F_q(x,t) \;\equiv\; \rho\frac{\partial^2 H_q}{\partial t^2} \;=\; \rho\frac{A\cdot q}{\tau^2}\,K_q\left(\frac{q^2 t}{\tau}\right)_q Sin(\gamma\cdot x)+\rho\frac{B\cdot q}{\tau^2}\,K_q\left(\frac{q^2 t - t_0}{\tau}\right)_q Sin(\gamma\cdot x)\;. \tag{10.2}$$

When this improved forcing term is entered into the numerical approximation schemes, we see improved accuracy in comparing the run-up model to actual data.

CHAPTER 11: Program Modeling of Special Functions in the Run-Up Stage

We now have the capability to choose a forcing, and model a tsunami through each stage of its existence. First, we can look at the model from [9] that uses the forcing function (9.2)

$$F_q(x,t) \equiv \rho \frac{\partial^2 H_q}{\partial t^2} = \rho \frac{A \cdot q}{\tau^2} K_q \left(\frac{q^2 t}{\tau} \right) {}_q Sin(\gamma \cdot x) \; .$$

As we saw in the previous chapter, this forcing was modeled from the height function (9.1). This function, when compared to the precursor wave of the 2011 Japanese tsunami, matched very closely up to the second trough before it diverged. The resulting model after being run through a computer program that simulates each stage of a wave's existence produced data that simulated a tsunami run-up onto Wake Island. This model gave an extremely accurate depiction of the tsunami that actually hit Wake Island in 2011. The figure below compares actual data collected from Wake Island during the tsunami (depicted in dashed red), and simulated data obtained by the model in [9] (in solid blue).

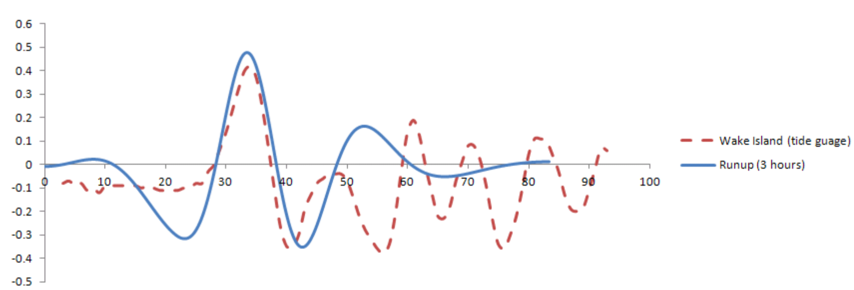

Figure 11.1: Data showing *tsunami* that hit Wake Island (red dashed) and the run-up model (blue solid) for Wake Island shoal.

Note that just like the precursor wave, the model matches very closely to the actual tsunami data up to the second trough. In particular, the initial wave of the tsunami is very accurately depicted in the model.

We can now use the new forcing from (10.2) to create a new model of the 2011 Japanese tsunami. We substitute the new forcing function,

$$F_q(x,t) \equiv \rho \frac{\partial^2 H_q}{\partial t^2} = \rho \frac{A \cdot q}{\tau^2} K_q \left(\frac{q^2 t}{\tau} \right) {}_q Sin(\gamma \cdot x) + \rho \frac{B \cdot q}{\tau^2} K_q \left(\frac{q^2 (t - t_0)}{\tau} \right) {}_q Sin(\gamma \cdot x) \, ,$$

into the forced wave equation, and run each stage as a program to obtain the simulated Wake Island tsunami run-up.

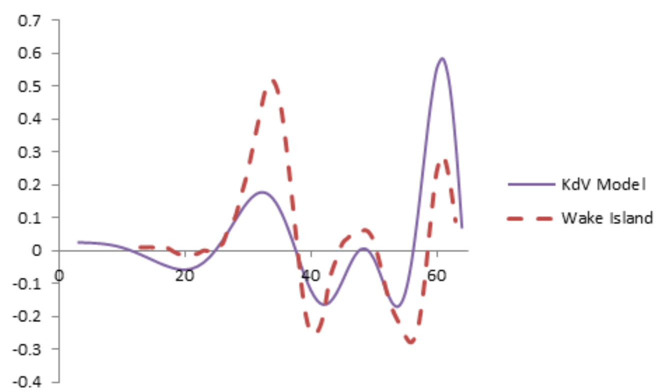

Figure 11.2: Data showing *tsunami* that hit Wake Island (red dashed) and the new run-up model (purple solid) for Wake Island shoal.

We can clearly see in figure 11.2 that in addition to matching the collected tsunami data well up to the second trough, the new forcing function matches well up to the third crest. So, we have obtained a more accurate model of the run-up stage of the Japanese 2011 tsunami.

CHAPTER 12: Conclusions

We have gained a clear understanding of how each stage of a tsunami is modeled. We have seen that a tsunami model can be produced from a precursor wave that travels from its source at a much faster speed than the tsunami itself. We have also seen that the special functions studied in [9] have clear advantages over previous models due the more realistic characteristics of the functions and from the similarities that both the special functions and their derivatives share with the shape of tsunamis. The models that are produced from these precursor waves produce an accurate model for the subsequent tsunami. The new, modified forms of the special functions studied in this thesis were shown to produce an even more accurate model for a tsunami.

Since we can predict a tsunami with a precursor wave that travels at speeds far greater then the following tsunami, this is an indication of a potential early warning system for coastal cities sufficiently far from the epicenter of an earthquake creating a tsunami.

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