

# THE KREIN–VON NEUMANN EXTENSION REVISITED

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ABSTRACT. We revisit the Krein–von Neumann extension in the case where the underlying symmetric operator is strictly positive and apply this to derive the explicit form of the Krein–von Neumann extension for singular, general (i.e., three-coefficient) Sturm–Liouville operators on arbitrary intervals. In particular, the boundary conditions for the Krein–von Neumann extension of the strictly positive minimal Sturm–Liouville operator are explicitly expressed in terms of generalized boundary values adapted to the (possible) singularity structure of the coefficients near an interval endpoint.

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## 1. INTRODUCTION

While the principal objective of this paper is to derive the explicit form of the Krein–von Neumann extension for singular (three-coefficient) Sturm–Liouville operators on arbitrary intervals with strictly positive underlying minimal operator, we briefly pause and first describe the abstract Krein–von Neumann extension of a nonnegative symmetric operator in complex, separable Hilbert space in a nutshell.

A linear operator  $S : \text{dom}(S) \subseteq \mathcal{H} \rightarrow \mathcal{H}$  in some complex, separable Hilbert space  $\mathcal{H}$  is called *nonnegative* if

$$(u, Su)_{\mathcal{H}} \geq 0, \quad u \in \text{dom}(S) \quad (1.1)$$

(in this case  $S$  is symmetric). In addition,  $S$  is called *strictly positive*, if for some  $\varepsilon > 0$ ,  $(u, Su)_{\mathcal{H}} \geq \varepsilon \|u\|_{\mathcal{H}}^2$ ,  $u \in \text{dom}(S)$ . Next, we recall the order relation  $0 \leq A \leq B$  for two nonnegative self-adjoint operators in  $\mathcal{H}$  in the form (see, e.g., [27, Section I.6], [44, Theorem VI.2.21])

$$0 \leq A \leq B \text{ if and only if } (B + aI_{\mathcal{H}})^{-1} \leq (A + aI_{\mathcal{H}})^{-1} \text{ for all } a > 0. \quad (1.2)$$

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In the following  $0 \leq S$  is a linear, unbounded, densely defined, nonnegative operator in  $\mathcal{H}$ , and we assume that  $S$  has nonzero deficiency indices. In particular,

$$\operatorname{def}(S) = \dim(\ker(S^* - zI_{\mathcal{H}})) \in \mathbb{N} \cup \{\infty\}, \quad z \in \mathbb{C} \setminus [0, \infty), \quad (1.3)$$

is well-known to be independent of  $z$  (and if  $S \geq \varepsilon I_{\mathcal{H}}$  for some  $\varepsilon > 0$  then the independence of  $\operatorname{def}(S)$  of  $z$  extends to  $z \in \mathbb{C} \setminus [\varepsilon, \infty)$ ). Moreover, since  $S$  and its closure  $\bar{S}$  have the same self-adjoint extensions in  $\mathcal{H}$ , we will without loss of generality assume that  $S$  is closed in  $\mathcal{H}$ .

The following is a fundamental result that cements the extraordinary role played by the Friedrichs and Krein–von Neumann extensions of  $S$ , to be found in M. Krein’s celebrated 1947 paper<sup>1</sup> [46] :

**Theorem 1.1.** *Assume that  $S$  is a densely defined, closed, nonnegative operator in  $\mathcal{H}$ . Then, among all nonnegative self-adjoint extensions of  $S$ , there exist two distinguished ones,  $S_K$  and  $S_F$ , which are, respectively, the smallest and largest (in the sense of order between nonnegative self-adjoint operators) such extensions. Furthermore, a nonnegative self-adjoint operator  $\tilde{S}$  is a self-adjoint extension of  $S$  if and only if  $\tilde{S}$  satisfies*

$$S_K \leq \tilde{S} \leq S_F. \quad (1.4)$$

In particular, (1.4) determines  $S_K$  and  $S_F$  uniquely.

In addition, if  $S \geq \varepsilon I_{\mathcal{H}}$  for some  $\varepsilon > 0$ , one has  $S_F \geq \varepsilon I_{\mathcal{H}}$ , and

$$\operatorname{dom}(S_F) = \operatorname{dom}(S) \dot{+} (S_F)^{-1} \ker(S^*), \quad (1.5)$$

$$\operatorname{dom}(S_K) = \operatorname{dom}(S) \dot{+} \ker(S^*), \quad (1.6)$$

$$\begin{aligned} \operatorname{dom}(S^*) &= \operatorname{dom}(S) \dot{+} (S_F)^{-1} \ker(S^*) \dot{+} \ker(S^*) \\ &= \operatorname{dom}(S_F) \dot{+} \ker(S^*), \end{aligned} \quad (1.7)$$

in particular,

$$\ker(S_K) = \ker((S_K)^{1/2}) = \ker(S^*) = \operatorname{ran}(S)^\perp. \quad (1.8)$$

Here the operator inequalities in (1.4) are understood in the resolvent sense,

$$(S_F + aI_{\mathcal{H}})^{-1} \leq (\tilde{S} + aI_{\mathcal{H}})^{-1} \leq (S_K + aI_{\mathcal{H}})^{-1} \text{ for some (and hence for all) } a > 0 \quad (1.9)$$

(an alternative approach employs quadratic forms).

Thus,  $S_K$  and  $S_F$  are distinguished self-adjoint extensions of  $S$ , representing, in particular, extremal points of all nonnegative self-adjoint extensions  $\tilde{S} \geq 0$  of  $S$ .

We will call the operator  $S_K$  the *Krein–von Neumann extension* of  $S$ . See [46] and also the discussion in [3], [8]. It should be noted that the Krein–von Neumann extension was first considered by von Neumann [69] in 1929 in the case where  $S$  is strictly positive<sup>2</sup>, that is, if  $S \geq \varepsilon I_{\mathcal{H}}$  for some  $\varepsilon > 0$ . However, von Neumann did not isolate the extremal property of this extension as described in (1.4) and (1.9). M. Krein [46], [47] was the first to systematically treat the general case  $S \geq 0$  and to study all nonnegative self-adjoint extensions of  $S$ , illustrating the special role of the *Friedrichs extension* (i.e., the “hard” extension)  $S_F$  of  $S$  and the Krein–von Neumann (i.e., the “soft”) extension  $S_K$  of  $S$  as extremal cases when

<sup>1</sup>See also Theorems 2 and 5–7 in the English summary on page 492.

<sup>2</sup>His construction appears in the proof of Theorem 42 on pages 102–103.

considering all nonnegative extensions of  $S$ . For more results on the Krein–von Neumann extension of a strictly positive symmetric operator  $S \geq \varepsilon I_{\mathcal{H}}$  we refer to the beginning of Section 3.

However, the principal aim of this paper are (three-coefficient) generally singular Sturm–Liouville differential expressions of the type

$$\tau = \frac{1}{r(x)} \left[ -\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right] \text{ for a.e. } x \in (a, b) \subseteq \mathbb{R}, \quad (1.10)$$

on a general interval  $(a, b) \subseteq \mathbb{R}$  and their various  $L^2((a, b); rdx)$ -realizations, with the coefficients  $p, q, r$  satisfying Hypothesis 2.1. In particular, the minimal operator  $T_{min}$  associated with  $\tau$  (cf. (2.5)), assumed in addition to be strictly positive, plays the role of  $S$  above, and the corresponding maximal operator  $T_{max}$  (cf. (2.2)) represents  $S^*$ . The explicit forms of the Friedrichs and Krein extensions of  $T_{min}$  are then of the form (3.24) and (3.25), (3.26), (3.28), (3.29), respectively. Moreover, the corresponding boundary conditions are explicitly expressed in terms of generalized boundary values adapted to the (possible) singularity structure of the coefficients near an interval endpoint with the help of principal and nonprincipal solutions of the underlying Sturm–Liouville equation.

Briefly turning to a sketch of the content of each section, we note that Section 2 focuses on the basics of Sturm–Liouville operators in  $L^2((a, b); rdx)$  and the underlying Weyl–Titchmarsh–Kodaira theory, including self-adjoint extensions and generalized boundary values (and conditions) in the singular case. Section 3 then contains the bulk of the new material in this paper. After continuing a discussion of the abstract Krein–von Neumann extension of a symmetric, strictly positive operator  $S \geq \varepsilon I_{\mathcal{H}}$ , an elementary characterization of the Krein–von Neumann extension  $S_K$  as the unique self-adjoint extension of  $S$  containing  $\ker(S^*)$  in its domain is derived in Lemma 3.2. This result is then applied to derive an explicit description of the Krein–von Neumann extension of a strictly positive minimal Sturm–Liouville operator  $T_{min}$  in terms of generalized boundary values. We conclude this paper with three nontrivial and representative examples in Section 4, including a generalized Bessel operator, a singular operator relevant in the context of acoustic black holes, and the Jacobi operator.

Finally, a few remarks on the notation employed: Given a separable complex Hilbert space  $\mathcal{H}$ ,  $(\cdot, \cdot)_{\mathcal{H}}$  denotes the scalar product in  $\mathcal{H}$  (linear in the second factor), and  $I_{\mathcal{H}}$  represents the identity operator in  $\mathcal{H}$ . The domain and range of a linear operator  $T$  in  $\mathcal{H}$  are abbreviated by  $\text{dom}(T)$  and  $\text{ran}(T)$ . The closure of a closable operator  $S$  is denoted by  $\overline{S}$ . The kernel (null space) of  $T$  is denoted by  $\ker(T)$ . The spectrum, point spectrum (i.e., the set of eigenvalues), and resolvent set of a closed linear operator in  $\mathcal{H}$  will be abbreviated by  $\sigma(\cdot)$ ,  $\sigma_p(\cdot)$ , and  $\rho(\cdot)$ , respectively. If  $U_1$  and  $U_2$  are subspaces of a Banach space  $\mathcal{X}$ , their direct sum is denoted by  $U_1 + U_2$ . We also employ the shortcut  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . If the underlying  $L^2$ -space is understood, we denote the corresponding identity operator simply by  $I$ .

## 2. THE BASICS OF WEYL–TITCHMARSH–KODAIRA THEORY

In this section, following [30] and [33, Ch. 13], we summarize the singular Weyl–Titchmarsh–Kodaira theory as needed to treat the Krein–von Neumann extension for singular, general Sturm–Liouville operators in the remainder of this paper.

Throughout this section we make the following assumptions:

**Hypothesis 2.1.** *Let  $(a, b) \subseteq \mathbb{R}$  and suppose that  $p, q, r$  are (Lebesgue) measurable functions on  $(a, b)$  such that the following items (i)–(iii) hold:*

- (i)  $r > 0$  a.e. on  $(a, b)$ ,  $r \in L^1_{loc}((a, b); dx)$ .
- (ii)  $p > 0$  a.e. on  $(a, b)$ ,  $1/p \in L^1_{loc}((a, b); dx)$ .
- (iii)  $q$  is real-valued a.e. on  $(a, b)$ ,  $q \in L^1_{loc}((a, b); dx)$ .

Given Hypothesis 2.1, we study Sturm–Liouville operators associated with the general, three-coefficient differential expression

$$\tau = \frac{1}{r(x)} \left[ -\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right] \quad \text{for a.e. } x \in (a, b) \subseteq \mathbb{R}, \quad (2.1)$$

and introduce maximal and minimal operators in  $L^2((a, b); rdx)$  associated with  $\tau$  in the usual manner as follows.

**Definition 2.2.** *Assume Hypothesis 2.1. Given  $\tau$  as in (2.1), the maximal operator  $T_{max}$  in  $L^2((a, b); rdx)$  associated with  $\tau$  is defined by*

$$\begin{aligned} T_{max} f &= \tau f, \\ f \in \text{dom}(T_{max}) &= \{g \in L^2((a, b); rdx) \mid g, g^{[1]} \in AC_{loc}((a, b)); \\ &\quad \tau g \in L^2((a, b); rdx)\}. \end{aligned} \quad (2.2)$$

The preminimal operator  $\dot{T}_{min}$  in  $L^2((a, b); rdx)$  associated with  $\tau$  is defined by

$$\begin{aligned} \dot{T}_{min} f &= \tau f, \\ f \in \text{dom}(\dot{T}_{min}) &= \{g \in L^2((a, b); rdx) \mid g, g^{[1]} \in AC_{loc}((a, b)); \\ &\quad \text{supp}(g) \subset (a, b) \text{ is compact; } \tau g \in L^2((a, b); rdx)\}. \end{aligned} \quad (2.3)$$

One can prove that  $\dot{T}_{min}$  is closable, and one then defines the minimal operator  $T_{min}$  as the closure of  $\dot{T}_{min}$ .

The following facts then are well known:

$$(\dot{T}_{min})^* = T_{max}, \quad (2.4)$$

and hence  $T_{max}$  is closed and  $T_{min} = \overline{\dot{T}_{min}}$  is given by

$$\begin{aligned} T_{min} f &= \tau f, \\ f \in \text{dom}(T_{min}) &= \{g \in L^2((a, b); rdx) \mid g, g^{[1]} \in AC_{loc}((a, b)); \\ &\quad \text{for all } h \in \text{dom}(T_{max}), W(h, g)(a) = 0 = W(h, g)(b); \tau g \in L^2((a, b); rdx)\} \\ &= \{g \in \text{dom}(T_{max}) \mid W(h, g)(a) = 0 = W(h, g)(b) \text{ for all } h \in \text{dom}(T_{max})\}. \end{aligned} \quad (2.5)$$

Moreover,  $\dot{T}_{min}$  is essentially self-adjoint if and only if  $T_{max}$  is symmetric, and then  $\dot{T}_{min} = T_{min} = T_{max}$ .

Here the Wronskian of  $f$  and  $g$ , for  $f, g \in AC_{loc}((a, b))$ , is defined by

$$W(f, g)(x) = f(x)g^{[1]}(x) - f^{[1]}(x)g(x), \quad x \in (a, b), \quad (2.6)$$

with

$$y^{[1]}(x) = p(x)y'(x), \quad x \in (a, b), \quad (2.7)$$

denoting the first quasi-derivative of a function  $y \in AC_{loc}((a, b))$ .

The celebrated Weyl alternative then can be stated as follows:

**Theorem 2.3** (Weyl's Alternative).

Assume Hypothesis 2.1. Then the following alternative holds: Either,

(i) for every  $z \in \mathbb{C}$ , all solutions  $u$  of  $(\tau - z)u = 0$  are in  $L^2((a, b); rdx)$  near  $b$  (resp., near  $a$ ),

or,

(ii) for every  $z \in \mathbb{C}$ , there exists at least one solution  $u$  of  $(\tau - z)u = 0$  which is not in  $L^2((a, b); rdx)$  near  $b$  (resp., near  $a$ ). In this case, for each  $z \in \mathbb{C} \setminus \mathbb{R}$ , there exists precisely one solution  $u_b$  (resp.,  $u_a$ ) of  $(\tau - z)u = 0$  (up to constant multiples) which lies in  $L^2((a, b); rdx)$  near  $b$  (resp., near  $a$ ).

This yields the limit circle/limit point classification of  $\tau$  at an interval endpoint and links self-adjointness of  $T_{min}$  (resp.,  $T_{max}$ ) and the limit point property of  $\tau$  at both endpoints as follows.

**Definition 2.4.** Assume Hypothesis 2.1.

In case (i) in Theorem 2.3,  $\tau$  is said to be in the limit circle case at  $b$  (resp., at  $a$ ). (Frequently,  $\tau$  is then called quasi-regular at  $b$  (resp.,  $a$ ).

In case (ii) in Theorem 2.3,  $\tau$  is said to be in the limit point case at  $b$  (resp., at  $a$ ).

If  $\tau$  is in the limit circle case at  $a$  and  $b$  then  $\tau$  is also called quasi-regular on  $(a, b)$ .

**Theorem 2.5.** Assume Hypothesis 2.1, then the following items (i) and (ii) hold:

(i) If  $\tau$  is in the limit point case at  $a$  (resp.,  $b$ ), then

$$W(f, g)(a) = 0 \text{ (resp., } W(f, g)(b) = 0) \text{ for all } f, g \in \text{dom}(T_{max}). \quad (2.8)$$

(ii) Let  $T_{min} = \overline{\dot{T}_{min}}$ . Then

$$\begin{aligned} n_{\pm}(T_{min}) &= \dim(\ker(T_{max} \mp iI)) \\ &= \begin{cases} 2 & \text{if } \tau \text{ is in the limit circle case at } a \text{ and } b, \\ 1 & \text{if } \tau \text{ is in the limit circle case at } a \\ & \text{and in the limit point case at } b, \text{ or vice versa,} \\ 0 & \text{if } \tau \text{ is in the limit point case at } a \text{ and } b. \end{cases} \end{aligned} \quad (2.9)$$

In particular,  $T_{min} = T_{max}$  is self-adjoint (i.e.,  $\dot{T}_{min}$  is essentially self-adjoint) if and only if  $\tau$  is in the limit point case at  $a$  and  $b$ .

Next, we turn to a description of all self-adjoint extensions of  $T_{min}$ .

**Theorem 2.6.** Assume Hypothesis 2.1 and that  $\tau$  is in the limit circle case at  $a$  and  $b$  (i.e.,  $\tau$  is quasi-regular on  $(a, b)$ ). In addition, assume that  $v_j \in \text{dom}(T_{max})$ ,  $j = 1, 2$ , satisfy

$$W(\overline{v_1}, v_2)(a) = W(\overline{v_1}, v_2)(b) = 1, \quad W(\overline{v_j}, v_j)(a) = W(\overline{v_j}, v_j)(b) = 0, \quad j = 1, 2. \quad (2.10)$$

(E.g., real-valued solutions  $v_j$ ,  $j = 1, 2$ , of  $(\tau - \lambda)u = 0$  with  $\lambda \in \mathbb{R}$ , such that  $W(v_1, v_2) = 1$ .) For  $g \in \text{dom}(T_{max})$  we introduce the generalized boundary values

$$\begin{aligned} \tilde{g}_1(a) &= -W(v_2, g)(a), & \tilde{g}_1(b) &= -W(v_2, g)(b), \\ \tilde{g}_2(a) &= W(v_1, g)(a), & \tilde{g}_2(b) &= W(v_1, g)(b). \end{aligned} \quad (2.11)$$

Then the following items (i)–(iii) hold:

(i) All self-adjoint extensions  $T_{\gamma,\delta}$  of  $T_{min}$  with separated boundary conditions are of the form

$$\begin{aligned} T_{\gamma,\delta}f &= \tau f, \quad \gamma, \delta \in [0, \pi), \\ f \in \text{dom}(T_{\gamma,\delta}) &= \left\{ g \in \text{dom}(T_{max}) \mid \begin{aligned} \sin(\gamma)\tilde{g}_2(a) + \cos(\gamma)\tilde{g}_1(a) &= 0; \\ \sin(\delta)\tilde{g}_2(b) + \cos(\delta)\tilde{g}_1(b) &= 0 \end{aligned} \right\}. \end{aligned} \quad (2.12)$$

(ii) All self-adjoint extensions  $T_{\varphi,R}$  of  $T_{min}$  with coupled boundary conditions are of the type

$$\begin{aligned} T_{\varphi,R}f &= \tau f, \\ f \in \text{dom}(T_{\varphi,R}) &= \left\{ g \in \text{dom}(T_{max}) \mid \begin{pmatrix} \tilde{g}_1(b) \\ \tilde{g}_2(b) \end{pmatrix} = e^{i\varphi} R \begin{pmatrix} \tilde{g}_1(a) \\ \tilde{g}_2(a) \end{pmatrix} \right\}, \end{aligned} \quad (2.13)$$

where  $\varphi \in [0, 2\pi)$ , and  $R$  is a real  $2 \times 2$  matrix with  $\det(R) = 1$  (i.e.,  $R \in SL(2, \mathbb{R})$ ).

(iii) Every self-adjoint extension of  $T_{min}$  is either of type (i) (i.e., separated) or of type (ii) (i.e., coupled).

*Remark 2.7.* (i) If  $\tau$  is in the limit point case at one endpoint, say, at the endpoint  $b$ , one omits the corresponding boundary condition involving  $\delta \in [0, \pi)$  at  $b$  in (2.12) to obtain all self-adjoint extensions  $T_\gamma$  of  $T_{min}$ , indexed by  $\gamma \in [0, \pi)$ . In the case where  $\tau$  is in the limit point case at both endpoints, all boundary values and boundary conditions become superfluous as in this case  $T_{min} = T_{max}$  is self-adjoint.

(ii) Assume the special case where  $\tau$  is regular on the finite interval  $[a, b]$ , that is, suppose that Hypothesis 2.1 is replaced by the more stringent set of assumptions:

**Hypothesis** ( $\tau$  regular on  $[a, b]$ .)

Let  $(a, b) \subset \mathbb{R}$  be a finite interval and suppose that  $p, q, r$  are (Lebesgue) measurable functions on  $(a, b)$  such that the following items (i')–(iii') hold:

(i')  $r > 0$  a.e. on  $(a, b)$ ,  $r \in L^1((a, b); dx)$ .

(ii')  $p > 0$  a.e. on  $(a, b)$ ,  $1/p \in L^1((a, b); dx)$ .

(iii')  $q$  is real-valued a.e. on  $(a, b)$ ,  $q \in L^1((a, b); dx)$ .

In this case one chooses  $v_j \in \text{dom}(T_{max})$ ,  $j = 1, 2$ , such that

$$v_1(x) = \begin{cases} \theta_0(\lambda, x, a), & \text{for } x \text{ near } a, \\ \theta_0(\lambda, x, b), & \text{for } x \text{ near } b, \end{cases} \quad v_2(x) = \begin{cases} \phi_0(\lambda, x, a), & \text{for } x \text{ near } a, \\ \phi_0(\lambda, x, b), & \text{for } x \text{ near } b, \end{cases} \quad (2.14)$$

where  $\phi_0(\lambda, \cdot, d)$ ,  $\theta_0(\lambda, \cdot, d)$ ,  $d \in \{a, b\}$ , are real-valued solutions of  $(\tau - \lambda)u = 0$ ,  $\lambda \in \mathbb{R}$ , satisfying the boundary conditions

$$\begin{aligned} \phi_0(\lambda, a, a) &= \theta_0^{[1]}(\lambda, a, a) = 0, & \theta_0(\lambda, a, a) &= \phi_0^{[1]}(\lambda, a, a) = 1, \\ \phi_0(\lambda, b, b) &= \theta_0^{[1]}(\lambda, b, b) = 0, & \theta_0(\lambda, b, b) &= \phi_0^{[1]}(\lambda, b, b) = 1. \end{aligned} \quad (2.15)$$

Then one verifies that

$$\tilde{g}_1(a) = g(a), \quad \tilde{g}_1(b) = g(b), \quad \tilde{g}_2(a) = g^{[1]}(a), \quad \tilde{g}_2(b) = g^{[1]}(b), \quad (2.16)$$

and hence Theorem 2.6 in the special regular case recovers the well-known situation of separated self-adjoint boundary conditions for three-coefficient regular Sturm–Liouville operators in  $L^2((a, b); rdx)$ .

(iii) In connection with (2.11), an explicit calculation demonstrates that for  $g, h \in \text{dom}(T_{max})$ ,

$$\tilde{g}_1(d)\tilde{h}_2(d) - \tilde{g}_2(d)\tilde{h}_1(d) = W(g, h)(d), \quad d \in \{a, b\}, \quad (2.17)$$

interpreted in the sense that either side in (2.17) has a finite limit as  $d \downarrow a$  and  $d \uparrow b$ . Of course, for (2.17) to hold at  $d \in \{a, b\}$ , it suffices that  $g$  and  $h$  lie locally in  $\text{dom}(T_{max})$  near  $x = d$ .  $\diamond$

In the special case where  $T_{min}$  is bounded from below, one can further analyze the generalized boundary values (2.11) in the singular context by invoking principal and nonprincipal solutions of  $\tau u = \lambda u$  for appropriate  $\lambda \in \mathbb{R}$ . This leads to natural analogs of (2.16) also in the singular case, and we will turn to this topic next.

We start by reviewing some oscillation theory with particular emphasis on principal and nonprincipal solutions, a notion originally due to Leighton and Morse [49], Rellich [57], [58], and Hartman and Wintner [38, Appendix] (see also [19], [24, Sects 13.6, 13.9, 13.0], [37, Ch. XI], [52], [72, Chs. 4, 6–8]).

**Definition 2.8.** *Assume Hypothesis 2.1.*

(i) Fix  $c \in (a, b)$  and  $\lambda \in \mathbb{R}$ . Then  $\tau - \lambda$  is called *nonoscillatory at a* (resp., *b*), if every real-valued solution  $u(\lambda, \cdot)$  of  $\tau u = \lambda u$  has finitely many zeros in  $(a, c)$  (resp.,  $(c, b)$ ). Otherwise,  $\tau - \lambda$  is called *oscillatory at a* (resp., *b*).

(ii) Let  $\lambda_0 \in \mathbb{R}$ . Then  $T_{min}$  is called *bounded from below by  $\lambda_0$* , and one writes  $T_{min} \geq \lambda_0 I$ , if

$$(u, [T_{min} - \lambda_0 I]u)_{L^2((a,b);rdx)} \geq 0, \quad u \in \text{dom}(T_{min}). \quad (2.18)$$

The following is a key result.

**Theorem 2.9.** *Assume Hypothesis 2.1. Then the following items (i)–(iii) are equivalent:*

(i)  $T_{min}$  (and hence any symmetric extension of  $T_{min}$ ) is bounded from below.

(ii) There exists a  $\nu_0 \in \mathbb{R}$  such that for all  $\lambda < \nu_0$ ,  $\tau - \lambda$  is nonoscillatory at  $a$  and  $b$ .

(iii) For fixed  $c, d \in (a, b)$ ,  $c \leq d$ , there exists a  $\nu_0 \in \mathbb{R}$  such that for all  $\lambda < \nu_0$ ,  $\tau u = \lambda u$  has (real-valued) nonvanishing solutions  $u_a(\lambda, \cdot) \neq 0$ ,  $\hat{u}_a(\lambda, \cdot) \neq 0$  in the neighborhood  $(a, c]$  of  $a$ , and (real-valued) nonvanishing solutions  $u_b(\lambda, \cdot) \neq 0$ ,  $\hat{u}_b(\lambda, \cdot) \neq 0$  in the neighborhood  $[d, b)$  of  $b$ , such that

$$W(\hat{u}_a(\lambda, \cdot), u_a(\lambda, \cdot)) = 1, \quad u_a(\lambda, x) = o(\hat{u}_a(\lambda, x)) \text{ as } x \downarrow a, \quad (2.19)$$

$$W(\hat{u}_b(\lambda, \cdot), u_b(\lambda, \cdot)) = 1, \quad u_b(\lambda, x) = o(\hat{u}_b(\lambda, x)) \text{ as } x \uparrow b, \quad (2.20)$$

$$\int_a^c dx p(x)^{-1} u_a(\lambda, x)^{-2} = \int_d^b dx p(x)^{-1} u_b(\lambda, x)^{-2} = \infty, \quad (2.21)$$

$$\int_a^c dx p(x)^{-1} \hat{u}_a(\lambda, x)^{-2} < \infty, \quad \int_d^b dx p(x)^{-1} \hat{u}_b(\lambda, x)^{-2} < \infty. \quad (2.22)$$

**Definition 2.10.** *Assume Hypothesis 2.1, suppose that  $T_{min}$  is bounded from below, and let  $\lambda \in \mathbb{R}$ . Then  $u_a(\lambda, \cdot)$  (resp.,  $u_b(\lambda, \cdot)$ ) in Theorem 2.9(iii) is called a *principal* (or *minimal*) *solution of  $\tau u = \lambda u$  at  $a$  (resp.,  $b$ )*. A real-valued solution  $\tilde{u}_a(\lambda, \cdot)$  (resp.,  $\tilde{u}_b(\lambda, \cdot)$ ) of  $\tau u = \lambda u$  linearly independent of  $u_a(\lambda, \cdot)$  (resp.,  $u_b(\lambda, \cdot)$ ) is called *nonprincipal at  $a$  (resp.,  $b$ )*.*

Principal and nonprincipal solutions are well-defined due to Lemma 2.11 below.

**Lemma 2.11.** *Assume Hypothesis 2.1 and suppose that  $T_{min}$  is bounded from below. Then  $u_a(\lambda, \cdot)$  and  $u_b(\lambda, \cdot)$  in Theorem 2.9 (iii) are unique up to (nonvanishing) real constant multiples. Moreover,  $u_a(\lambda, \cdot)$  and  $u_b(\lambda, \cdot)$  are minimal solutions of  $\tau u = \lambda u$  in the sense that*

$$u(\lambda, x)^{-1}u_a(\lambda, x) = o(1) \text{ as } x \downarrow a, \quad (2.23)$$

$$u(\lambda, x)^{-1}u_b(\lambda, x) = o(1) \text{ as } x \uparrow b, \quad (2.24)$$

for any other solution  $u(\lambda, \cdot)$  of  $\tau u = \lambda u$  (which is nonvanishing near  $a$ , resp.,  $b$ ) with  $W(u_a(\lambda, \cdot), u(\lambda, \cdot)) \neq 0$ , respectively,  $W(u_b(\lambda, \cdot), u(\lambda, \cdot)) \neq 0$ .

Given these oscillation theoretic preparations, one can now revisit and complement Theorem 2.6 as follows:

**Theorem 2.12.** *Assume Hypothesis 2.1 and that  $\tau$  is in the limit circle case at  $a$  and  $b$  (i.e.,  $\tau$  is quasi-regular on  $(a, b)$ ). In addition, assume that  $T_{min} \geq \lambda_0 I$  for some  $\lambda_0 \in \mathbb{R}$ , and denote by  $u_a(\lambda_0, \cdot)$  and  $\widehat{u}_a(\lambda_0, \cdot)$  (resp.,  $u_b(\lambda_0, \cdot)$  and  $\widehat{u}_b(\lambda_0, \cdot)$ ) principal and nonprincipal solutions of  $\tau u = \lambda_0 u$  at  $a$  (resp.,  $b$ ), satisfying*

$$W(\widehat{u}_a(\lambda_0, \cdot), u_a(\lambda_0, \cdot)) = W(\widehat{u}_b(\lambda_0, \cdot), u_b(\lambda_0, \cdot)) = 1. \quad (2.25)$$

Then the following items (i)–(iii) hold:

(i) Introducing  $v_j \in \text{dom}(T_{max})$ ,  $j = 1, 2$ , via

$$v_1(x) = \begin{cases} \widehat{u}_a(\lambda_0, x), & \text{for } x \text{ near } a, \\ \widehat{u}_b(\lambda_0, x), & \text{for } x \text{ near } b, \end{cases} \quad v_2(x) = \begin{cases} u_a(\lambda_0, x), & \text{for } x \text{ near } a, \\ u_b(\lambda_0, x), & \text{for } x \text{ near } b, \end{cases} \quad (2.26)$$

one obtains for all  $g \in \text{dom}(T_{max})$ ,

$$\begin{aligned} \widetilde{g}(a) &= -W(v_2, g)(a) = \widetilde{g}_1(a) = -W(u_a(\lambda_0, \cdot), g)(a) \\ &= \lim_{x \downarrow a} \frac{g(x)}{\widehat{u}_a(\lambda_0, x)}, \\ \widetilde{g}(b) &= -W(v_2, g)(b) = \widetilde{g}_1(b) = -W(u_b(\lambda_0, \cdot), g)(b) \\ &= \lim_{x \uparrow b} \frac{g(x)}{\widehat{u}_b(\lambda_0, x)}, \end{aligned} \quad (2.27)$$

$$\begin{aligned} \widetilde{g}'(a) &= W(v_1, g)(a) = \widetilde{g}_2(a) = W(\widehat{u}_a(\lambda_0, \cdot), g)(a) \\ &= \lim_{x \downarrow a} \frac{g(x) - \widetilde{g}(a)\widehat{u}_a(\lambda_0, x)}{u_a(\lambda_0, x)}, \\ \widetilde{g}'(b) &= W(v_1, g)(b) = \widetilde{g}_2(b) = W(\widehat{u}_b(\lambda_0, \cdot), g)(b) \\ &= \lim_{x \uparrow b} \frac{g(x) - \widetilde{g}(b)\widehat{u}_b(\lambda_0, x)}{u_b(\lambda_0, x)}. \end{aligned} \quad (2.28)$$

In particular, the limits on the right-hand sides in (2.27), (2.28) exist.

(ii) All self-adjoint extensions  $T_{\gamma, \delta}$  of  $T_{min}$  with separated boundary conditions are of the form

$$\begin{aligned} T_{\gamma, \delta} f &= \tau f, \quad \gamma, \delta \in [0, \pi), \\ f \in \text{dom}(T_{\gamma, \delta}) &= \{g \in \text{dom}(T_{max}) \mid \sin(\gamma)\widetilde{g}'(a) + \cos(\gamma)\widetilde{g}(a) = 0; \\ &\quad \sin(\delta)\widetilde{g}'(b) + \cos(\delta)\widetilde{g}(b) = 0\}. \end{aligned} \quad (2.29)$$

Moreover,  $\sigma(T_{\gamma,\delta})$  is simple.

(iii) All self-adjoint extensions  $T_{\varphi,R}$  of  $T_{min}$  with coupled boundary conditions are of the type

$$\begin{aligned} T_{\varphi,R}f &= \tau f, \\ f \in \text{dom}(T_{\varphi,R}) &= \left\{ g \in \text{dom}(T_{max}) \mid \begin{pmatrix} \tilde{g}(b) \\ \tilde{g}'(b) \end{pmatrix} = e^{i\varphi} R \begin{pmatrix} \tilde{g}(a) \\ \tilde{g}'(a) \end{pmatrix} \right\}, \end{aligned} \quad (2.30)$$

where  $\varphi \in [0, 2\pi)$ , and  $R \in SL(2, \mathbb{R})$ .

Moreover, under the hypotheses of Theorem 2.12, relation (2.5) implies that the minimal operator takes on the form

$$\begin{aligned} T_{min}f &= \tau f, \\ f \in \text{dom}(T_{min}) &= \{g \in \text{dom}(T_{max}) \mid \tilde{g}(a) = \tilde{g}'(a) = 0 = \tilde{g}(b) = \tilde{g}'(b)\}. \end{aligned} \quad (2.31)$$

The Friedrichs extension  $T_F$  of  $T_{min}$  now permits a particularly simple characterization in terms of the generalized boundary values  $\tilde{g}(a), \tilde{g}(b)$  as derived by Kalf [42] and subsequently by Niessen and Zettl [52] (see also [58], [59] and the extensive literature cited in [30], [33, Ch. 13]):

**Theorem 2.13.** *Assume Hypothesis 2.1 and that  $\tau$  is in the limit circle case at  $a$  and  $b$  (i.e.,  $\tau$  is quasi-regular on  $(a, b)$ ). In addition, assume that  $T_{min} \geq \lambda_0 I$  for some  $\lambda_0 \in \mathbb{R}$ . Then the Friedrichs extension  $T_F$  of  $T_{min}$  is characterized by*

$$T_F f = \tau f, \quad f \in \text{dom}(T_F) = \{g \in \text{dom}(T_{max}) \mid \tilde{g}(a) = 0 = \tilde{g}(b)\}. \quad (2.32)$$

In particular,  $T_F = T_{0,0}$ .

*Remark 2.14.* (i) As in (2.17), one readily verifies for  $g, h \in \text{dom}(T_{max})$ ,

$$\tilde{g}(d)\tilde{h}'(d) - \tilde{g}'(d)\tilde{h}(d) = W(g, h)(d), \quad d \in \{a, b\}, \quad (2.33)$$

again interpreted in the sense that either side in (2.33) has a finite limit as  $d \downarrow a$  and  $d \uparrow b$ . In particular, if  $\tau$  is regular at an endpoint then Remark 2.7 (ii) shows that the generalized boundary values in (2.27), (2.28) reduce to the canonical ones in (2.16).

(ii) While the principal solution at an endpoint is unique up to constant multiples (which we will ignore), nonprincipal solutions differ by additive constant multiples of the principal solution. As a result, if

$$\begin{aligned} \hat{u}_a(\lambda_0, \cdot) &\longrightarrow \hat{u}_a(\lambda_0, \cdot) + C u_a(\lambda_0, \cdot), \quad C \in \mathbb{R}, \\ \text{then } \tilde{g}(a) &\longrightarrow \tilde{g}(a), \quad \tilde{g}'(a) \longrightarrow \tilde{g}'(a) - C \tilde{g}(a), \end{aligned} \quad (2.34)$$

and analogously at the endpoint  $b$ . Hence, generalized boundary values  $\tilde{g}'(d)$  at the endpoint  $d \in \{a, b\}$  depend on the choice of nonprincipal solution  $\hat{u}_d(\lambda_0, \cdot)$  of  $\tau u = \lambda_0 u$  at  $d$ . However, the Friedrichs boundary conditions  $\tilde{g}(a) = 0 = \tilde{g}(b)$  are clearly independent of the choice of nonprincipal solution.

(iii) As always in this context, if  $\tau$  is in the limit point case at one or both interval endpoints, the corresponding boundary conditions at that endpoint are dropped and only a separated boundary condition at the other end point (if the latter is a limit circle endpoint for  $\tau$ ) has to be imposed in Theorems 2.12 and 2.13. In other words, the generalized boundary values (2.11) and (2.27), (2.28) are only relevant if the endpoint in question is of the limit circle type. In the case where  $\tau$  is in the

limit point case at both endpoints, all boundary values and boundary conditions become superfluous as in this case  $T_{min} = T_{max}$  is self-adjoint.  $\diamond$

All results surveyed in this section can be found in [30] and [33, Ch. 13] which contain very detailed lists of references to the basics of Weyl–Titchmarsh theory. Here we just mention a few additional and classical sources such as [2, Sect. 129], [21, Chs. 8, 9], [24, Sects. 13.6, 13.9, 13.10], [41, Ch. III], [50, Ch. V], [52], [55, Ch. 6], [60, Ch. 9], [70, Sect. 8.3], [71, Ch. 13], [72, Chs. 4, 6–8].

### 3. THE KREIN–VON NEUMANN EXTENSION OF $T_{min} > 0$

In this section we derive the Krein–von Neumann extension for  $T_{min}$  under the assumption  $T_{min} \geq \varepsilon I$  for some  $\varepsilon > 0$ . We continue with some more abstract facts on the Krein–von Neumann extension of strictly positive symmetric operators in a complex Hilbert space and refer to [2, Sect. 109], [3], [4], [7], [8], [9], [10], [11], [12], [13], [15, Sect. 5.4], [16], [22], [23], [27, Part III], [34], [35, Sect. 13.2], [39], [40], [46], [47], [51], [56], [61], [62], [63], [64], [65], [67], [68], [69], and the references cited therein for some of the basic literature in this context.

Denote by

$$n_{\pm}(T) = \dim(\ker(T^* \mp iI_{\mathcal{H}})) = \dim(\operatorname{ran}(T \pm iI_{\mathcal{H}})^{\perp}) \in \mathbb{N}_0 \cup \{\infty\}, \quad (3.1)$$

the deficiency indices of a densely defined, closed, symmetric operator  $T$  in the complex, separable Hilbert space  $\mathcal{H}$ .

The Krein–von Neumann and Friedrichs extension of a densely defined, closed, symmetric operator  $S$  with  $n_{\pm}(S) > 0$ , satisfying

$$S \geq 0, \quad (3.2)$$

are denoted by  $S_K$  and  $S_F$ , respectively. If, in addition,

$$S \geq \varepsilon I_{\mathcal{H}} \quad (3.3)$$

for some  $\varepsilon > 0$ , then one also has

$$n_{\pm}(S) = \dim(\ker(S^*)), \quad (3.4)$$

$$S_F \geq \varepsilon I_{\mathcal{H}}, \quad (3.5)$$

and

$$\begin{aligned} \operatorname{dom}(S_K) &= \operatorname{dom}(S) \dot{+} \ker(S^*), \\ S_K f &= S^* f, \quad f \in \operatorname{dom}(S_K). \end{aligned} \quad (3.6)$$

For completeness we also recall that under hypothesis (3.3)

$$\operatorname{dom}(S_F) = \operatorname{dom}(S) \dot{+} (S_F)^{-1} \ker(S^*), \quad S_F f = S^* f, \quad f \in \operatorname{dom}(S_F), \quad (3.7)$$

$$\operatorname{dom}(S^*) = \operatorname{dom}(S) \dot{+} (S_F)^{-1} \ker(S^*) \dot{+} \ker(S^*), \quad (3.8)$$

$$\ker(S_K) = \ker(S^*). \quad (3.9)$$

Here the notation  $\dot{+}$  addresses the direct (not orthogonal direct) sum in the sense that if  $X_1, X_2$  are linear subspaces of a Banach space  $X$ , then  $X_1 \dot{+} X_2$  denotes the subspace of  $X$  given by

$$X_1 \dot{+} X_2 = \{x \in X \mid x = x_1 + x_2, \ x_j \in X_j, \ j = 1, 2\}, \quad (3.10)$$

assuming

$$X_1 \cap X_2 = \{0\}. \quad (3.11)$$

*Remark 3.1.* If  $S \geq \varepsilon I_{\mathcal{H}}$ ,  $\varepsilon > 0$ , then  $\text{dom}(S) \dot{+} \ker(S^*)$  is well-defined since

$$f \in \text{dom}(S) \cap \ker(S^*) \quad (3.12)$$

implies  $0 = S^*f = Sf$ , and hence  $f = 0$  as  $S \geq \varepsilon I_{\mathcal{H}}$ ,  $\varepsilon > 0$ .  $\diamond$

**Lemma 3.2.** *Suppose  $S$  is densely defined, symmetric, and for some  $\varepsilon > 0$ ,  $S \geq \varepsilon I_{\mathcal{H}}$ . If  $\tilde{S}$  is a self-adjoint extension of  $S$  such that  $\text{dom}(\tilde{S}) \supset \ker(S^*)$ , then  $\tilde{S} = S_K$ .*

*Proof.* Without loss of generality we may assume that  $S$  is closed. By definition,  $\text{dom}(\tilde{S}) \supset \text{dom}(S)$ , and by assumption,  $\text{dom}(\tilde{S}) \supset \ker(S^*)$ , hence  $\text{dom}(\tilde{S}) \supseteq \text{dom}(S) \dot{+} \ker(S^*) = \text{dom}(S_K)$ . Utilizing the facts  $\tilde{S} = S^*|_{\text{dom}(\tilde{S})}$  and  $S_K = S^*|_{\text{dom}(S_K)}$ , one infers that

$$\tilde{S}|_{\text{dom}(S_K)} = S^*|_{\text{dom}(\tilde{S}) \cap \text{dom}(S_K)} = S^*|_{\text{dom}(S_K)} = S_K, \quad (3.13)$$

that is,  $\tilde{S} \supseteq S_K$ . Since self-adjoint operators are maximal in the sense that  $S_K$  has no proper symmetric extensions, one concludes that  $\tilde{S} = S_K$ .  $\square$

*Remark 3.3.* (i) From the outset, (3.6) implies (3.9), that is,  $\ker(S_K) = \ker(S^*)$ . Lemma 3.2 now implies a converse in the sense that if  $\ker(\tilde{S}) = \ker(S^*)$  (and hence lies in  $\text{dom}(\tilde{S})$ ), then  $\tilde{S} = S_K$ . One notes that Lemma 3.2 does not *a priori* assume that  $\tilde{S}$  is bounded from below.

(ii) The fact (3.9) has been isolated in [3] as uniquely identifying the Krein-von Neumann extension (under the hypothesis  $S \geq \varepsilon I_{\mathcal{H}}$ ) (see also [10, eq. (2.39)]), as a byproduct of an entirely different quadratic form approach that characterizes all nonnegative self-adjoint extensions of  $S$ . Here we derive this uniqueness aspect with entirely elementary means only.

(iii) Applications to  $2m$ th order regular differential operators: Suppose  $T_{min} = \tau_{2m}|_{C_0^\infty((a,b))}$ , then  $T_{min}^* = T_{max}$ , that is, no boundary conditions are necessary in  $\text{dom}(T_{max})$ . Then  $\tilde{T}$ , a self-adjoint extension of  $T_{min}$  in  $L^2((a,b); rdx)$  equals  $T_{min,K}$ , the Krein extension of  $T_{min}$  in  $L^2((a,b); rdx)$ , if and only if

$$\dim(\ker(\tilde{T})) = 2m. \quad (3.14)$$

Of course,  $\dim(\ker(T_{max})) = 2m$  since we assumed  $T_{min}$  to be regular.

(iv) Relation (3.14) only holds if  $T_{min}$  is indeed minimally defined, that is, as the closure of  $\tau_{2m}|_{C_0^\infty((a,b))}$ . If some of the possible zero boundary conditions are missing in  $\text{dom}(T_{min})$  then they will reappear in  $\text{dom}(T_{max} = T_{min}^*)$  and hence  $2m$  in (3.14) has to be diminished accordingly.

(v) The  $2m$  solutions giving rise to (3.14) (in the regular case) are simply generated by solving the ordinary differential equation of  $2m$ th order

$$\tau_{2m}y = 0 \quad (3.15)$$

in the distributional sense.  $\diamond$

To demonstrate that  $\varepsilon > 0$  is necessary for Lemma 3.2 to hold, we recall the following (counter) example.

**Example 3.4.** Let  $\mathcal{H} = L^2((0, \infty); dx)$ , and

$$\begin{aligned} T_{min}^{(0)} f &= -f'', \\ f \in \text{dom}(T_{min}^{(0)}) &= \{g \in L^2((0, \infty); dx) \mid \text{for all } R > 0: g, g' \in AC([0, R]); \\ &g(0) = g'(0) = 0; g'' \in L^2((0, \infty); dx)\}, \end{aligned} \quad (3.16)$$

$$T_{min}^{(0)} = \overline{-d^2/dx^2|_{C_0^\infty((0, \infty))}}. \quad (3.17)$$

Then

$$T_{min}^{(0)} \geq 0 \quad (3.18)$$

but there is no  $\varepsilon > 0$  such that  $T_{min}^{(0)} \geq \varepsilon I$ . Moreover,

$$\begin{aligned} T_{max}^{(0)} f &= (T_{min}^{(0)})^* f = -f'', \\ f \in \text{dom}(T_{max}^{(0)}) &= \{g \in L^2((0, \infty); dx) \mid \text{for all } R > 0: g, g' \in AC([0, R]); \\ &g'' \in L^2((0, \infty); dx)\}, \end{aligned} \quad (3.19)$$

and

$$\begin{aligned} T_{min, K}^{(0)} f &= T_N^{(0)} f = -f'', \\ f \in \text{dom}(T_{min, K}^{(0)} = T_N^{(0)}) &= \{g \in \text{dom}(T_{max}^{(0)}) \mid g'(0) = 0\}. \end{aligned} \quad (3.20)$$

Furthermore, consider self-adjoint extensions  $T_\alpha^{(0)}$  of  $T_{min}^{(0)}$  given by

$$\begin{aligned} T_\alpha^{(0)} f &= -f'', \quad \alpha \in [0, \infty) \cup \{\infty\}, \\ f \in \text{dom}(T_\alpha^{(0)}) &= \{g \in \text{dom}(T_{max}^{(0)}) \mid g'(0) = \alpha g(0)\}. \end{aligned} \quad (3.21)$$

Then

$$\ker(T_\alpha^{(0)}) = \ker(T_{min, K}^{(0)} = T_N^{(0)} \equiv T_{\alpha=0}^{(0)}) = \ker((T_{min}^{(0)})^* = T_{max}^{(0)}) = \{0\}, \quad (3.22)$$

and

$$\sigma(T_\alpha^{(0)}) = [0, \infty), \quad \alpha \in [0, \infty) \cup \{\infty\}. \quad (3.23)$$

Thus, Lemma 3.2 requires  $\varepsilon > 0$ . For the fact (3.20) see, for example, [29, Corollary 5.6] (choose  $\gamma = \pi/2$ ,  $q(x) = 0$ , and note that  $m_0^W(z) = iz^{1/2}$ , hence  $m_0^W(0) = 0$ ).

Of course,

$$\begin{aligned} T_{\alpha=\infty}^{(0)} f &= T_{min, F}^{(0)} f = T_D^{(0)} f = -f'', \\ f \in \text{dom}(T_D^{(0)}) &= \{g \in \text{dom}(T_{max}^{(0)}) \mid g(0) = 0\}, \end{aligned} \quad (3.24)$$

represents the Friedrichs (resp., Dirichlet) extension of  $T_{min}^{(0)}$ .

Combining [20] and [30] one can now extend the description of the Krein extension from the known regular case to the singular case as follows:

**Theorem 3.5.** In addition to Hypothesis 2.1, suppose that  $T_{min} \geq \varepsilon I$  for some  $\varepsilon > 0$ . Then the following items (i) and (ii) hold:

(i) Assume that  $n_\pm(T_{min}) = 1$  and denote the principal solutions of  $\tau u = 0$  at  $a$  and  $b$  by  $u_a(0, \cdot)$  and  $u_b(0, \cdot)$ , respectively. If  $\tau$  is in the limit circle case at  $a$  and

in the limit point case at  $b$ , then the Krein-von Neumann extension  $T_{\gamma_K}$  of  $T_{min}$  is given by

$$\begin{aligned} T_{\gamma_K} f &= \tau f, \\ f \in \text{dom}(T_{\gamma_K}) &= \{g \in \text{dom}(T_{max}) \mid \sin(\gamma_K)\tilde{g}'(a) + \cos(\gamma_K)\tilde{g}(a) = 0\}, \\ \cot(\gamma_K) &= -\tilde{u}'_b(0, a)/\tilde{u}_b(0, a), \quad \gamma_K \in (0, \pi). \end{aligned} \quad (3.25)$$

Similarly, if  $\tau$  is in the limit circle case at  $b$  and in the limit point case at  $a$ , then the Krein-von Neumann extension  $T_{\delta_K}$  of  $T_{min}$  is given by

$$\begin{aligned} T_{\delta_K} f &= \tau f, \\ f \in \text{dom}(T_{\delta_K}) &= \{g \in \text{dom}(T_{max}) \mid \sin(\delta_K)\tilde{g}'(b) + \cos(\delta_K)\tilde{g}(b) = 0\}, \\ \cot(\delta_K) &= -\tilde{u}'_a(0, b)/\tilde{u}_a(0, b), \quad \delta_K \in (0, \pi). \end{aligned} \quad (3.26)$$

(ii) Assume that  $n_{\pm}(T_{min}) = 2$ , that is,  $\tau$  is in the limit circle case at  $a$  and  $b$ . Then, introducing a basis for  $\ker(T_{max})$ , denoted by  $u_1(0, \cdot)$ ,  $u_2(0, \cdot)$  as follows,

$$\begin{aligned} \tau u_j(0, \cdot) &= 0, \quad j = 1, 2, \\ \tilde{u}_1(0, a) &= 0, \quad \tilde{u}_1(0, b) = 1, \\ \tilde{u}_2(0, a) &= 1, \quad \tilde{u}_2(0, b) = 0, \end{aligned} \quad (3.27)$$

the Krein-von Neumann extension  $T_{0, R_K}$  of  $T_{min}$  is given by

$$\begin{aligned} T_{0, R_K} f &= \tau f, \\ f \in \text{dom}(T_{0, R_K}) &= \left\{ g \in \text{dom}(T_{max}) \mid \begin{pmatrix} \tilde{g}(b) \\ \tilde{g}'(b) \end{pmatrix} = R_K \begin{pmatrix} \tilde{g}(a) \\ \tilde{g}'(a) \end{pmatrix} \right\}, \end{aligned} \quad (3.28)$$

where

$$R_K = \frac{1}{\tilde{u}'_1(0, a)} \begin{pmatrix} -\tilde{u}'_2(0, a) & 1 \\ \tilde{u}'_1(0, a)\tilde{u}'_2(0, b) - \tilde{u}'_1(0, b)\tilde{u}'_2(0, a) & \tilde{u}'_1(0, b) \end{pmatrix}. \quad (3.29)$$

Alternatively, employing the nonprincipal solutions  $\hat{u}_a(0, \cdot)$  and  $\hat{u}_b(0, \cdot)$  of  $\tau u = 0$  at  $a$  and  $b$ , respectively, satisfying (2.25) and used to introduce the generalized boundary values (2.27), (2.28),  $R_K$  can be characterized in terms of principal and nonprincipal solutions by

$$R_K = \begin{pmatrix} \tilde{u}_a(0, b) & \tilde{u}_a(0, b) \\ \tilde{u}'_a(0, b) & \tilde{u}'_a(0, b) \end{pmatrix}, \quad (3.30)$$

equivalently, by

$$R_K = \begin{pmatrix} \tilde{u}'_b(0, a) & -\tilde{u}_b(0, a) \\ -\tilde{u}'_b(0, a) & \tilde{u}_b(0, a) \end{pmatrix}. \quad (3.31)$$

*Proof.* (i) It suffices to prove (3.25), the proof for (3.26) being entirely analogous. Since  $\ker(T_{max})$  is one-dimensional, and  $\tau$  is in the limit point case at  $b$ , one concludes that only the smallest solution of  $\tau u = 0$  near  $b$ , that is, the principal solution,  $u_b(0, \cdot)$ , can lie in  $\ker(T_{max})$ ,

$$\ker(T_{max}) = \text{lin. span}\{u_b(0, \cdot)\}. \quad (3.32)$$

One also notes that by (2.27), (2.28),

$$\tilde{u}_b(0, b) = 0, \quad \tilde{u}'_b(0, b) = 1. \quad (3.33)$$

Since the Krein extension is now necessarily associated with separated boundary conditions, in fact, a self-adjoint boundary condition at the endpoint  $a$  only, (3.9) implies that  $u_b(0, \cdot)$  must necessarily satisfy the boundary condition at  $a$  which defines the Krein extension of  $T_{min}$  in this case. Thus, the underlying boundary condition parameter  $\gamma_K \in [0, \pi)$  is determined via

$$\sin(\gamma_K)\tilde{u}'_b(0, a) + \cos(\gamma_K)\tilde{u}_b(0, a) = 0. \quad (3.34)$$

If  $\tilde{u}_b(0, a) = 0$ , then together with (2.32) and (3.5) this would imply that  $u_b(0, \cdot) \in \text{dom}(T_F = T_0)$  in the notation of (2.29) (one notes that no boundary condition is required at  $b$  due to the limit point property of  $\tau$  at  $b$ ). Thus,  $0 \in \sigma_p(T_F)$ , contradicting  $T_F \geq \varepsilon I$ . Hence  $\gamma_K > 0$ , completing the proof of (3.25).

(ii) Since the limit circle case is assumed at the endpoints  $a$  and  $b$ , all solutions  $u(0, \cdot)$  of  $\tau u = 0$  satisfy  $u(0, \cdot) \in \text{dom}(T_{max})$  and hence the generalized boundary values (2.27), (2.28) with  $g = u(0, \cdot)$  are well-defined. Introducing a basis for the null space of  $T_{max}$  as in (3.27), one notes via (2.33) that

$$W(u_1(0, \cdot), u_2(0, \cdot)) = -\tilde{u}'_1(0, a) = \tilde{u}'_2(0, b). \quad (3.35)$$

Next, introducing  $R_K$  as in (3.29) and employing (3.27) and (3.35) one computes that

$$\det(R_K) = -\tilde{u}'_2(0, b)/\tilde{u}'_1(0, a) = 1, \text{ that is, } R_K \in SL(2, \mathbb{R}). \quad (3.36)$$

Thus, (3.28) represents one of the self-adjoint extensions of  $T_{min}$  characterized by  $\varphi = 0$  and  $R = R_K$  according to Theorem 2.12 (iii), and hence it remains to show that this extension is precisely the Krein–von Neumann extension of  $T_{min}$ .

For this purpose we turn to Lemma 3.2 next: Since

$$\text{dom}(T_{0, R_K}) = \text{dom}(T_{min}) \dot{+} \ker(T_{min}^*) = \text{dom}(T_{min}) \dot{+} \ker(T_{max}), \quad (3.37)$$

and since  $T_{min}$  is characterized by the vanishing of all generalized boundary values as depicted in (2.31), it suffices to verify that  $u_1(0, \cdot)$  and  $u_2(0, \cdot)$  both satisfy the boundary conditions in (3.28), given (3.29) (see also Remark 3.3 (iii) with  $m = 1$ ). This verification reduces to the following elementary computations, employing (3.27) and (3.35) once more,

$$\begin{aligned} R_K \begin{pmatrix} \tilde{u}_1(0, a) \\ \tilde{u}'_1(0, a) \end{pmatrix} &= \begin{pmatrix} 1 \\ \tilde{u}'_1(0, b) \end{pmatrix} = \begin{pmatrix} \tilde{u}_1(0, b) \\ \tilde{u}'_1(0, b) \end{pmatrix}, \\ R_K \begin{pmatrix} \tilde{u}_2(0, a) \\ \tilde{u}'_2(0, a) \end{pmatrix} &= \begin{pmatrix} 0 \\ \tilde{u}'_2(0, b) \end{pmatrix} = \begin{pmatrix} \tilde{u}_2(0, b) \\ \tilde{u}'_2(0, b) \end{pmatrix}, \end{aligned} \quad (3.38)$$

completing the proof of (3.28), (3.29).

Next, choosing

$$\begin{aligned} u_1(0, x) &= u_a(0, x)/\tilde{u}_a(0, b), \\ u_2(0, x) &= \hat{u}_a(0, x) - [\tilde{u}_a(0, b)/\tilde{u}_a(0, b)]u_a(0, x), \end{aligned} \quad (3.39)$$

noting that  $\tilde{u}_a(0, b) \neq 0$ , one easily verifies that (3.27) is satisfied since

$$\begin{aligned} \tilde{u}_1(0, a) &= \tilde{u}_a(0, a)/\tilde{u}_a(0, b) = 0, \\ \tilde{u}_1(0, b) &= \tilde{u}_a(0, b)/\tilde{u}_a(0, b) = 1, \\ \tilde{u}_2(0, a) &= \tilde{u}_a(0, a) = 1, \\ \tilde{u}_2(0, b) &= \tilde{u}_a(0, b) - \tilde{u}_a(0, b) = 0, \end{aligned} \quad (3.40)$$

proving (3.30). Expression (3.31) is proved analogously by using the principal and nonprincipal solutions near  $x = b$  and choosing

$$\begin{aligned} u_1(0, x) &= \widehat{u}_b(0, x) - [\widetilde{u}_b(0, a)/\widetilde{u}_b(0, a)]u_b(0, x), \\ u_2(0, x) &= u_b(0, x)/\widetilde{u}_b(0, a). \end{aligned} \quad (3.41)$$

□

Relations (3.28), (3.29) extend [20, Example 3.3] in the regular context to the singular one.

#### 4. THREE EXAMPLES

In this section we illustrate Theorem 3.5 with three examples, including a generalized Bessel and Jacobi-type operators.

We start with the generalized Bessel operator following the analysis in [32, Section 6].

**Example 4.1** (A Generalized Bessel Operator). *Let  $a = 0$  and  $b \in (0, \infty)$  in (2.1), and consider the concrete example*

$$\begin{aligned} p(x) &= x^\beta, \quad r(x) = x^\alpha, \quad q(x) = \frac{(2 + \alpha - \beta)^2 \gamma^2 - (1 - \beta)^2}{4} x^{\beta-2}, \\ &\alpha > -1, \quad \beta < 1, \quad \gamma \geq 0, \quad x \in (0, b). \end{aligned} \quad (4.1)$$

Then

$$\begin{aligned} \tau_{\alpha, \beta, \gamma} &= x^{-\alpha} \left[ -\frac{d}{dx} x^\beta \frac{d}{dx} + \frac{(2 + \alpha - \beta)^2 \gamma^2 - (1 - \beta)^2}{4} x^{\beta-2} \right], \\ &\alpha > -1, \quad \beta < 1, \quad \gamma \geq 0, \quad x \in (0, b), \end{aligned} \quad (4.2)$$

is singular at the endpoint 0 (since the potential,  $q$  is not integrable near  $x = 0$ ) and is regular at  $x = b$ . Furthermore,  $\tau_{\alpha, \beta, \gamma}$  is in the limit circle case at  $x = 0$  if  $0 \leq \gamma < 1$  and in the limit point case at  $x = 0$  when  $\gamma \geq 1$ .

Solutions of  $\tau_{\alpha, \beta, \gamma} u = zu$  are given by (cf. [43, No. 2.162, p. 440])

$$\begin{aligned} y_{1, \alpha, \beta, \gamma}(z, x) &= x^{(1-\beta)/2} J_\gamma(2z^{1/2} x^{(2+\alpha-\beta)/2} / (2 + \alpha - \beta)), \quad \gamma \geq 0, \\ y_{2, \alpha, \beta, \gamma}(z, x) &= \begin{cases} x^{(1-\beta)/2} J_{-\gamma}(2z^{1/2} x^{(2+\alpha-\beta)/2} / (2 + \alpha - \beta)), & \gamma \notin \mathbb{N}_0, \\ x^{(1-\beta)/2} Y_\gamma(2z^{1/2} x^{(2+\alpha-\beta)/2} / (2 + \alpha - \beta)), & \gamma \in \mathbb{N}_0, \end{cases} \quad \gamma \geq 0, \end{aligned} \quad (4.3)$$

where  $J_\nu(\cdot), Y_\nu(\cdot)$  are the standard Bessel functions of order  $\nu \in \mathbb{R}$  (cf. [1, Ch. 9]).

In the following we assume that

$$\gamma \in [0, 1) \quad (4.4)$$

to ensure the limit circle case at  $x = 0$ . In this case it suffices to focus on the generalized boundary values at the singular endpoint  $x = 0$ . For this purpose we introduce principal and nonprincipal solutions  $u_{0, \alpha, \beta, \gamma}(0, \cdot)$  and  $\widehat{u}_{0, \alpha, \beta, \gamma}(0, \cdot)$  of  $\tau_{\alpha, \beta, \gamma} u = 0$  at  $x = 0$  by

$$\begin{aligned} u_{0, \alpha, \beta, \gamma}(0, x) &= (1 - \beta)^{-1} x^{[1-\beta+(2+\alpha-\beta)\gamma]/2}, \quad \gamma \in [0, 1), \\ \widehat{u}_{0, \alpha, \beta, \gamma}(0, x) &= \begin{cases} (1 - \beta)[(2 + \alpha - \beta)\gamma]^{-1} x^{[1-\beta-(2+\alpha-\beta)\gamma]/2}, & \gamma \in (0, 1), \\ (1 - \beta)x^{(1-\beta)/2} \ln(1/x), & \gamma = 0, \end{cases} \\ &\alpha > -1, \quad \beta < 1, \quad x \in (0, 1). \end{aligned} \quad (4.5)$$

The generalized boundary values for  $g \in \text{dom}(T_{max,\alpha,\beta,\gamma})$  at  $x = 0$  are then of the form

$$\begin{aligned} \tilde{g}(0) &= -W(u_{0,\alpha,\beta,\gamma}(0, \cdot), g)(0) \\ &= \begin{cases} \lim_{x \downarrow 0} g(x) / [(1-\beta)[(2+\alpha-\beta)\gamma]^{-1} x^{[1-\beta-(2+\alpha-\beta)\gamma/2]}], & \gamma \in (0, 1), \\ \lim_{x \downarrow 0} g(x) / [(1-\beta)x^{(1-\beta)/2} \ln(1/x)], & \gamma = 0, \end{cases} \end{aligned} \quad (4.6)$$

$$\begin{aligned} \tilde{g}'(0) &= W(\hat{u}_{0,\alpha,\beta,\gamma}(0, \cdot), g)(0) \\ &= \begin{cases} \lim_{x \downarrow 0} [g(x) - \tilde{g}(0)(1-\beta)[(2+\alpha-\beta)\gamma]^{-1} x^{[1-\beta-(2+\alpha-\beta)\gamma/2]} \\ \quad \times [(1-\beta)^{-1} x^{[1-\beta+(2+\alpha-\beta)\gamma/2]}]^{-1}], & \gamma \in (0, 1), \\ \lim_{x \downarrow 0} [g(x) - \tilde{g}(0)(1-\beta)x^{(1-\beta)/2} \ln(1/x)] \\ \quad \times [(1-\beta)^{-1} x^{(1-\beta)/2}]^{-1}], & \gamma = 0. \end{cases} \end{aligned} \quad (4.7)$$

Choosing

$$\begin{aligned} u_1(0, x) &= u_{0,\alpha,\beta,\gamma}(0, x) / u_{0,\alpha,\beta,\gamma}(0, b), & \gamma \in [0, 1), \\ u_2(0, x) &= \begin{cases} \hat{u}_{0,\alpha,\beta,\gamma}(0, x) - (1-\beta)^2 [(2+\alpha-\beta)\gamma]^{-1} b^{-(2+\alpha-\beta)\gamma} u_{0,\alpha,\beta,\gamma}(0, x), & \gamma \in (0, 1), \\ \hat{u}_{0,\alpha,\beta,0}(0, x) - (1-\beta)^2 \ln(1/b) u_{0,\alpha,\beta,0}(0, x), & \gamma = 0, \end{cases} \\ & \quad \alpha > -1, \beta < 1, x \in (0, b), \end{aligned} \quad (4.8)$$

in (3.27)–(3.29) yields the Krein–von Neumann extension  $T_{0,R_K,\alpha,\beta,\gamma}$  of  $T_{min,\alpha,\beta,\gamma}$  in the form

$$T_{0,R_K,\alpha,\beta,\gamma} f = \tau_{\alpha,\beta,\gamma} f, \quad (4.9)$$

$$f \in \text{dom}(T_{0,R_K,\alpha,\beta,\gamma}) = \left\{ g \in \text{dom}(T_{max,\alpha,\beta,\gamma}) \mid \begin{pmatrix} \tilde{g}(b) \\ \tilde{g}'(b) \end{pmatrix} = R_{K,\alpha,\beta,\gamma} \begin{pmatrix} \tilde{g}(0) \\ \tilde{g}'(0) \end{pmatrix} \right\},$$

where

$$R_{K,\alpha,\beta,\gamma} = \begin{cases} \begin{pmatrix} b^{[\beta-1-(2+\alpha-\beta)\gamma/2]} \\ \times \begin{pmatrix} \frac{1-\beta}{(2+\alpha-\beta)\gamma} b^{1-\beta} & \frac{1}{1-\beta} b^{1-\beta+(2+\alpha-\beta)\gamma} \\ \frac{(1-\beta)^2}{2(2+\alpha-\beta)\gamma} - \frac{1-\beta}{2} & \left[ \frac{1}{2} + \frac{(2+\alpha-\beta)\gamma}{2(1-\beta)} \right] b^{(2+\alpha-\beta)\gamma} \end{pmatrix} \end{pmatrix}, & \gamma \in (0, 1), \\ \begin{pmatrix} (1-\beta) \ln(1/b) b^{(1-\beta)/2} & \frac{1}{1-\beta} b^{(1-\beta)/2} \\ \frac{(1-\beta)^2 \ln(1/b) - 2(1-\beta)}{2} b^{(\beta-1)/2} & \frac{1}{2} b^{(\beta-1)/2} \end{pmatrix}, & \gamma = 0. \end{cases} \quad (4.10)$$

One verifies that  $\det(R_{K,\alpha,\beta,\gamma}) = 1$ .

For the Krein extension of the standard Bessel operator on the half-line  $(0, \infty)$  (i.e.,  $\alpha = \beta = 0$ ,  $a = 0$ ,  $b = \infty$ ) we also refer to [17] (see also [5], [6]).

Next, we turn to a singular operator relevant to the phenomenon of acoustic black holes, following [14].

**Example 4.2** (Acoustic Black Hole). *Let  $a = 0$  and  $b \in (0, \infty)$  in (2.1) and consider*

$$p(x) = p_0(x)x^\alpha, \quad r(x) = r_0(x)x^\beta, \quad q(x) = 0, \quad x \in (0, b), \quad (4.11)$$

where  $\alpha, \beta \in \mathbb{R}$  are fixed and  $p_0$  and  $r_0$  are continuous real-valued functions on  $(0, b)$  that satisfy for some  $m, M \in (0, \infty)$ ,

$$m \leq p_0(x) \leq M, \quad m \leq r_0(x) \leq M, \quad x \in (0, b). \quad (4.12)$$

Then

$$\tau_{p_0, r_0, \alpha, \beta} = -\frac{1}{r_0(x)x^\beta} \frac{d}{dx} p_0(x)x^\alpha \frac{d}{dx}, \quad \alpha, \beta \in \mathbb{R}, \quad x \in (0, b). \quad (4.13)$$

Linearly independent solutions  $y_j(0, \cdot)$ ,  $j \in \{1, 2\}$ , of the differential equation  $\tau_{p_0, r_0, \alpha, \beta} y = 0$  are given by

$$y_{1, p_0, r_0, \alpha, \beta}(0, x) = 1, \quad y_{2, p_0, r_0, \alpha, \beta}(0, x) = \int_x^b \frac{dt}{p_0(t)t^\alpha}, \quad \alpha, \beta \in \mathbb{R}, \quad x \in (0, b), \quad (4.14)$$

and they satisfy

$$W(y_{2, p_0, r_0, \alpha, \beta}(0, \cdot), y_{1, p_0, r_0, \alpha, \beta}(0, \cdot)) = 1. \quad (4.15)$$

Explicit calculations reveal that

$$y_{1, p_0, r_0, \alpha, \beta}(0, \cdot), y_{2, p_0, r_0, \alpha, \beta}(0, \cdot) \in L^2((0, b); r_0(x)x^\beta dx), \quad (4.16)$$

that is,  $\tau_{p_0, r_0, \alpha, \beta}$  is in the limit circle case at  $x = 0$ , if and only if  $\beta > \max\{-1, 2\alpha - 3\}$ . In addition,  $\tau_{p_0, r_0, \alpha, \beta}$  is regular at  $x = b$ . To avoid the scenario where  $\tau_{p_0, r_0, \alpha, \beta}$  is in the limit point case at  $x = 0$ , we thus assume that  $\alpha$  and  $\beta$  satisfy  $\beta > \max\{-1, 2\alpha - 3\}$ . To avoid that  $\tau_{p_0, r_0, \alpha, \beta}$  is also regular at  $x = 0$  (and hence regular on  $(0, b)$ ), we now also assume that  $\alpha \geq 1$ . (There is no need to discuss the Krein–von Neumann extensions in the regular case as that can be found in [20, Example 3.3]). Altogether, this means we are assuming

$$\alpha \geq 1 \quad \text{and} \quad \beta > 2\alpha - 3. \quad (4.17)$$

Under the assumption  $\alpha \geq 1$ ,  $y_1(0, \cdot)$  and  $y_2(0, \cdot)$  are principal and nonprincipal solutions, respectively, of  $\tau_{p_0, r_0, \alpha, \beta} u = 0$  at  $x = 0$ . Thus, in accordance with Theorem 2.9, one chooses

$$u_{0, p_0, r_0, \alpha, \beta}(0, x) = 1, \quad \widehat{u}_{0, p_0, r_0, \alpha, \beta}(0, x) = \int_x^b \frac{dt}{p_0(t)t^\alpha}, \quad (4.18)$$

$$\alpha \geq 1, \quad \beta > 2\alpha - 3, \quad x \in (0, b).$$

The generalized boundary values for  $g \in \text{dom}(T_{\max, p_0, r_0, \alpha, \beta})$  are then of the form

$$\widetilde{g}(0) = \lim_{x \downarrow 0} g(x) \left[ \int_x^b \frac{dt}{p_0(t)t^\alpha} \right]^{-1}, \quad (4.19)$$

$$\widetilde{g}'(0) = \lim_{x \downarrow 0} \left[ g(x) - \widetilde{g}(0) \int_x^b \frac{dt}{p_0(t)t^\alpha} \right], \quad (4.20)$$

$$\widetilde{g}(b) = \lim_{x \uparrow b} g(x) = g(b), \quad (4.21)$$

$$\widetilde{g}'(b) = \lim_{x \uparrow b} [pg'](x) = [pg'](b) = p_0(b)b^\alpha g'(b). \quad (4.22)$$

A basis for  $\ker(T_{max,p_0,r_0,\alpha,\beta})$  which satisfies (3.27) is given by

$$\begin{aligned} u_{1,p_0,r_0,\alpha,\beta}(0,x) &= 1, & u_{2,p_0,r_0,\alpha,\beta}(0,x) &= \int_x^b \frac{dt}{p_0(t)t^\alpha}, \\ & & \alpha &\geq 1, \beta > 2\alpha - 3, x \in (0,b), \end{aligned} \quad (4.23)$$

and explicit calculations using (4.18), (4.20), and (4.22) reveal

$$\begin{aligned} \tilde{u}'_{1,p_0,r_0,\alpha,\beta}(0,0) &= 1, & \tilde{u}'_{2,p_0,r_0,\alpha,\beta}(0,0) &= 0, \\ \tilde{u}'_{1,p_0,r_0,\alpha,\beta}(0,b) &= 0, & \tilde{u}'_{2,p_0,r_0,\alpha,\beta}(0,b) &= -1. \end{aligned} \quad (4.24)$$

Using (4.24) in (3.29) then yields

$$R_{K,p_0,r_0,\alpha,\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \alpha \geq 1, \beta > 2\alpha - 3, \quad (4.25)$$

and hence the Krein-von Neumann extension  $T_{0,R_K,p_0,r_0,\alpha,\beta}$  of  $T_{min,p_0,r_0,\alpha,\beta}$  is characterized by

$$\begin{aligned} T_{0,R_K,p_0,r_0,\alpha,\beta}f &= \tau_{p_0,r_0,\alpha,\beta}f, \\ \text{dom}(T_{0,R_K,p_0,r_0,\alpha,\beta}) &= \{g \in \text{dom}(T_{max,p_0,r_0,\alpha,\beta}) \mid g(b) = \tilde{g}'(0), g^{[1]}(b) = -\tilde{g}(0)\}. \end{aligned} \quad (4.26)$$

Finally, we turn to the Jacobi operator referring to [31] for a much more detailed analysis.

**Example 4.3** (Jacobi Operator). *Let*

$$\begin{aligned} a &= -1, & b &= 1, \\ p(x) &= p_{\alpha,\beta}(x) = (1-x)^{\alpha+1}(1+x)^{\beta+1}, & q(x) &= q_{\alpha,\beta}(x) = 0, \\ r(x) &= r_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta, & x &\in (-1,1), \quad \alpha, \beta \in \mathbb{R} \end{aligned} \quad (4.27)$$

(see, e.g., [1, Ch. 22], [18], [25, Sect. 23], [26], [28], [36], [45], [48], [54, Ch. 18], [66, Ch. IV]) and consider the Jacobi differential expression

$$\begin{aligned} \tau_{\alpha,\beta} &= -(1-x)^{-\alpha}(1+x)^{-\beta}(d/dx)((1-x)^{\alpha+1}(1+x)^{\beta+1})(d/dx), \\ & x \in (-1,1), \quad \alpha, \beta \in \mathbb{R}. \end{aligned} \quad (4.28)$$

To decide the limit point/limit circle classification of  $\tau_{\alpha,\beta}$  at the interval endpoints  $\pm 1$ , it suffices to note that if  $y_1$  is a given solution of  $\tau y = 0$ , then a 2nd linearly independent solution  $y_2$  of  $\tau y = 0$  is obtained via the standard formula

$$y_2(x) = y_1(x) \int_c^x dx' p(x')^{-1} y_1(x')^{-2}, \quad c, x \in (a,b). \quad (4.29)$$

Returning to the concrete Jacobi case at hand, one notices that

$$\begin{aligned} y_1(x) &= 1, & x &\in (-1,1), \\ y_2(x) &= \int_0^x dx' (1-x')^{-1-\alpha}(1+x')^{-1-\beta}, & x &\in (-1,1), \end{aligned} \quad (4.30)$$

and hence

$$y_2(x) \quad (4.31)$$

$$= \begin{cases} 2^{-1-\alpha}\beta^{-1}(1+x)^{-\beta}[1+O(1+x)]+O(1), & \alpha \in \mathbb{R}, \beta \in \mathbb{R} \setminus \{0\}, \text{ as } x \downarrow -1, \\ -2^{-1-\alpha}\ln(1+x)+O(1), & \alpha \in \mathbb{R}, \beta = 0, \text{ as } x \downarrow -1, \\ 2^{-1-\beta}\alpha^{-1}(1-x)^{-\alpha}[1+O(1-x)]+O(1), & \alpha \in \mathbb{R} \setminus \{0\}, \beta \in \mathbb{R}, \text{ as } x \uparrow +1, \\ -2^{-1-\beta}\ln(1-x)+O(1), & \alpha = 0, \beta \in \mathbb{R}, \text{ as } x \uparrow +1. \end{cases}$$

Thus, an application of Theorem 2.3, Definition 2.4, and Remark 2.7 (ii) implies the classification,

$$\tau_{\alpha,\beta} \text{ is } \begin{cases} \text{regular at } -1 \text{ if and only if } \alpha \in \mathbb{R}, \beta \in (-1, 0), \\ \text{in the limit circle case at } -1 \text{ if and only if } \alpha \in \mathbb{R}, \beta \in [0, 1), \\ \text{in the limit point case at } -1 \text{ if and only if } \alpha \in \mathbb{R}, \beta \in \mathbb{R} \setminus (-1, 1), \\ \text{regular at } +1 \text{ if and only if } \alpha \in (-1, 0), \beta \in \mathbb{R}, \\ \text{in the limit circle case at } +1 \text{ if and only if } \alpha \in [0, 1), \beta \in \mathbb{R}, \\ \text{in the limit point case at } +1 \text{ if and only if } \alpha \in \mathbb{R} \setminus (-1, 1), \beta \in \mathbb{R}. \end{cases} \quad (4.32)$$

The fact (4.30) naturally leads to principal and nonprincipal solutions  $u_{\pm 1, \alpha, \beta}(0, x)$  and  $\hat{u}_{\pm 1, \alpha, \beta}(0, x)$  of  $\tau_{\alpha, \beta} y = 0$  near  $\pm 1$  as follows:

$$\begin{aligned} u_{-1, \alpha, \beta}(0, x) &= \begin{cases} -2^{-\alpha-1}\beta^{-1}(1+x)^{-\beta}[1+O(1+x)], & \beta \in (-\infty, 0), \\ 1, & \beta \in [0, \infty), \end{cases} \\ \hat{u}_{-1, \alpha, \beta}(0, x) &= \begin{cases} 1, & \beta \in (-\infty, 0), \\ -2^{-\alpha-1}\ln((1+x)/2), & \beta = 0, \\ 2^{-\alpha-1}\beta^{-1}(1+x)^{-\beta}[1+O(1+x)], & \beta \in (0, \infty), \end{cases} \end{aligned} \quad \alpha \in \mathbb{R}, \quad (4.33)$$

and

$$\begin{aligned} u_{+1, \alpha, \beta}(0, x) &= \begin{cases} 2^{-\beta-1}\alpha^{-1}(1-x)^{-\alpha}[1+O(1-x)], & \alpha \in (-\infty, 0), \\ 1, & \alpha \in [0, \infty), \end{cases} \\ \hat{u}_{+1, \alpha, \beta}(0, x) &= \begin{cases} 1, & \alpha \in (-\infty, 0), \\ 2^{-\beta-1}\ln((1-x)/2), & \alpha = 0, \\ -2^{-\beta-1}\alpha^{-1}(1-x)^{-\alpha}[1+O(1-x)], & \alpha \in (0, \infty), \end{cases} \end{aligned} \quad \beta \in \mathbb{R}. \quad (4.34)$$

Combining the fact (4.32) with Theorem 2.5, the preminimal operator  $\dot{T}_{\min, \alpha, \beta}$  corresponding to  $\tau_{\alpha, \beta}$  is essentially self-adjoint in  $L^2((-1, 1); r_{\alpha, \beta} dx)$  if and only if  $\alpha, \beta \in \mathbb{R} \setminus (-1, 1)$ . Thus, boundary values for the maximal operator  $T_{\max, \alpha, \beta}$  associated with  $\tau_{\alpha, \beta}$  at  $-1$  exist if and only if  $\alpha \in \mathbb{R}, \beta \in (-1, 1)$ , and similarly, boundary values for  $T_{\max, \alpha, \beta}$  at  $+1$  exist if and only if  $\alpha \in (-1, 1), \beta \in \mathbb{R}$ .

Employing the principal and nonprincipal solutions (4.33), (4.34) at  $\pm 1$ , according to (2.27), (2.28), generalized boundary values for  $g \in \text{dom}(T_{\max, \alpha, \beta})$  are of the

form

$$\begin{aligned} \tilde{g}(-1) &= \begin{cases} g(-1), & \beta \in (-1, 0), \\ -2^{\alpha+1} \lim_{x \downarrow -1} g(x) / \ln((1+x)/2), & \beta = 0, \\ \beta 2^{\alpha+1} \lim_{x \downarrow -1} (1+x)^\beta g(x), & \beta \in (0, 1), \end{cases} \\ \tilde{g}'(-1) &= \begin{cases} g^{[1]}(-1), & \beta \in (-1, 0), \\ \lim_{x \downarrow -1} [g(x) + \tilde{g}(-1) 2^{-\alpha-1} \ln((1+x)/2)], & \beta = 0, \\ \lim_{x \downarrow -1} [g(x) - \tilde{g}(-1) 2^{-\alpha-1} \beta^{-1} (1+x)^{-\beta}], & \beta \in (0, 1), \end{cases} \end{aligned} \quad \alpha \in \mathbb{R}, \quad (4.35)$$

$$\begin{aligned} \tilde{g}(1) &= \begin{cases} g(1), & \alpha \in (-1, 0), \\ 2^{\beta+1} \lim_{x \uparrow 1} g(x) / \ln((1-x)/2), & \alpha = 0, \\ -\alpha 2^{\beta+1} \lim_{x \uparrow 1} (1-x)^\alpha g(x), & \alpha \in (0, 1), \end{cases} \\ \tilde{g}'(1) &= \begin{cases} g^{[1]}(1), & \alpha \in (-1, 0), \\ \lim_{x \uparrow 1} [g(x) - \tilde{g}(1) 2^{-\beta-1} \ln((1-x)/2)], & \alpha = 0, \\ \lim_{x \uparrow 1} [g(x) + \tilde{g}(1) 2^{-\beta-1} \alpha^{-1} (1-x)^{-\alpha}], & \alpha \in (0, 1), \end{cases} \end{aligned} \quad \beta \in \mathbb{R}. \quad (4.36)$$

For a detailed treatment of solutions of the Jacobi differential equation and the associated hypergeometric differential equations we refer to [31, Appendix A].

To shorten the presentation of this example and hence avoid case (i) in Theorem 3.5 where  $\tau_{\alpha,\beta}$  is in the limit point case at  $-1$  or  $+1$  and in the limit circle case at the opposite endpoint (i.e., the cases where  $n_\pm(T_{\min,\alpha,\beta}) = 1$ ), we now assume that  $\tau_{\alpha,\beta}$  is in the limit circle case at  $\pm 1$  (i.e.,  $n_\pm(T_{\min,\alpha,\beta}) = 2$  as in case (ii) of Theorem 3.5). Thus, we assume that

$$\alpha, \beta \in (-1, 1) \quad (4.37)$$

in the following. Next, we consider two linearly independent solutions of  $\tau_{\alpha,\beta}y = 0$  near  $x = -1$  given by

$$\begin{aligned} y_{1,\alpha,\beta,-1}(0, x) &= 1, \\ y_{2,\alpha,\beta,-1}(0, x) &= (1+x)^{-\beta} F(1+\alpha, -\beta; 1-\beta; (1+x)/2), \quad \beta \in (-1, 1) \setminus \{0\}, \\ &\quad \alpha \in (-1, 1), \quad x \in (-1, 1). \end{aligned} \quad (4.38)$$

Furthermore, using the connection formulas found in [1, Eq. 15.3.6, 15.3.10] yields the behavior of  $y_{2,\alpha,\beta,-1}(0, x)$  near  $x = 1$ ,

$$y_{2,\alpha,\beta,-1}(0, x) = \begin{cases} (1+x)^{-\beta} \frac{\Gamma(1-\beta)\Gamma(-\alpha)}{\Gamma(-\alpha-\beta)} F(1+\alpha, -\beta; 1+\alpha; (1-x)/2) \\ - (1+x)^{-\beta} (1-x)^{-\alpha} 2^\alpha \alpha^{-1} \beta F(-\alpha-\beta, 1; 1-\alpha; (1-x)/2), \\ \quad \alpha \in (-1, 1) \setminus \{0\}, \\ - (1+x)^{-\beta} \beta \sum_{n=0}^{\infty} \frac{(-\beta)_n}{2^n (n!)^2} [\psi(n+1) - \psi(n-\beta) \\ - \ln((1-x)/2)] (1-x)^n, \quad \alpha = 0, \\ \quad \beta \in (-1, 1) \setminus \{0\}, \quad x \in (-1, 1). \end{cases} \quad (4.39)$$

Here  $F(\cdot, \cdot; \cdot; \cdot)$  denotes the hypergeometric function (see, e.g., [1, Ch. 15]),  $\psi(\cdot) = \Gamma'(\cdot)/\Gamma(\cdot)$  the Digamma function,  $\gamma_E = -\psi(1) = 0.57721\dots$  represents

Euler's constant, and

$$(\zeta)_0 = 1, \quad (\zeta)_n = \Gamma(\zeta + n)/\Gamma(\zeta), \quad n \in \mathbb{N}, \quad \zeta \in \mathbb{C} \setminus (-\mathbb{N}_0), \quad (4.40)$$

abbreviates Pochhammer's symbol (see, e.g., [1, Ch. 6]).

Similarly, we consider linearly independent solutions of  $\tau_{\alpha,\beta}y = 0$  near  $x = 1$ ,

$$\begin{aligned} y_{1,\alpha,\beta,1}(0, x) &= 1, \\ y_{2,\alpha,\beta,1}(0, x) &= (1-x)^{-\alpha} F(1+\beta, -\alpha; 1-\alpha; (1-x)/2), \quad \alpha \in (-1, 1) \setminus \{0\}, \\ &\quad \beta \in (-1, 1), \quad x \in (-1, 1), \end{aligned} \quad (4.41)$$

noting one can show that

$$y_{2,\alpha,\beta,1}(0, x) = \begin{cases} (1-x)^{-\alpha} \frac{\Gamma(1-\alpha)\Gamma(-\beta)}{\Gamma(-\alpha-\beta)} F(1+\beta, -\alpha; 1+\beta; (1+x)/2) \\ \quad - (1-x)^{-\alpha} (1+x)^{-\beta} 2^\beta \beta^{-1} \alpha F(-\alpha-\beta, 1; 1-\beta; (1+x)/2), & \beta \in (-1, 1) \setminus \{0\}, \\ - (1-x)^{-\alpha} \alpha \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{2^n (n!)^2} [\psi(n+1) - \psi(n-\alpha) \\ \quad - \ln((1+x)/2)] (1+x)^n, & \beta = 0, \end{cases} \quad \alpha \in (-1, 1) \setminus \{0\}, \quad x \in (-1, 1). \quad (4.42)$$

For  $\alpha, \beta \in (-1, 1)$ , the following five cases are associated with a strictly positive minimal operator  $T_{\min, \alpha, \beta}$  (see, [31]) and we now provide the corresponding choices  $u_1, u_2$  in (3.27)–(3.29) that yield  $R_{K, \alpha, \beta}$  and the Krein-von Neumann extension  $T_{0, R_K, \alpha, \beta}$  of  $T_{\min, \alpha, \beta}$ ,

$$T_{0, R_K, \alpha, \beta} f = \tau_{\alpha, \beta} f, \quad (4.43)$$

$$f \in \text{dom}(T_{0, R_K, \alpha, \beta}) = \left\{ g \in \text{dom}(T_{\max, \alpha, \beta}) \mid \begin{pmatrix} \tilde{g}(1) \\ \tilde{g}'(1) \end{pmatrix} = R_{K, \alpha, \beta} \begin{pmatrix} \tilde{g}(-1) \\ \tilde{g}'(-1) \end{pmatrix} \right\}.$$

**(I) The regular case  $\alpha, \beta \in (-1, 0)$ : Choosing**

$$u_1(0, x) = 2^\beta \frac{\Gamma(-\alpha-\beta)}{\Gamma(-\alpha)\Gamma(1-\beta)} y_{2,\alpha,\beta,-1}(0, x), \quad u_2(0, x) = 1 - u_1(0, x), \quad (4.44)$$

yields

$$R_{K, \alpha, \beta} = \begin{pmatrix} 1 & 2^{-\alpha-\beta-1} \frac{\Gamma(-\alpha)\Gamma(-\beta)}{\Gamma(-\alpha-\beta)} \\ 0 & 1 \end{pmatrix}, \quad \alpha, \beta \in (-1, 0). \quad (4.45)$$

**(II) The case  $\alpha \in (-1, 0)$ ,  $\beta \in (0, 1)$ : Choosing**

$$u_1(0, x) = 1, \quad u_2(0, x) = \beta^{-1} 2^{-\alpha-1} y_{2,\alpha,\beta,-1}(0, x) + 2^{-\alpha-\beta-1} \frac{\Gamma(-\alpha)\Gamma(-\beta)}{\Gamma(-\alpha-\beta)}, \quad (4.46)$$

yields

$$R_{K, \alpha, \beta} = \begin{pmatrix} -2^{-\alpha-\beta-1} \frac{\Gamma(-\alpha)\Gamma(-\beta)}{\Gamma(-\alpha-\beta)} & 1 \\ -1 & 0 \end{pmatrix}, \quad \alpha \in (-1, 0), \quad \beta \in (0, 1). \quad (4.47)$$

**(III) The case  $\alpha \in (0, 1)$ ,  $\beta \in (-1, 0)$ : Choosing**

$$u_1(0, x) = \beta^{-1} 2^{-\alpha-1} y_{2,\alpha,\beta,-1}(0, x), \quad u_2(0, x) = 1, \quad (4.48)$$

yields

$$R_{K,\alpha,\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 2^{-\alpha-\beta-1} \frac{\Gamma(-\alpha)\Gamma(-\beta)}{\Gamma(-\alpha-\beta)} \end{pmatrix}, \quad \alpha \in (0, 1), \beta \in (-1, 0). \quad (4.49)$$

(In cases **(II)** and **(III)** we interpret  $1/\Gamma(0) = 0$ .)

**(IV) The case  $\alpha = 0$ ,  $\beta \in (-1, 0)$ : Choosing**

$$u_1(0, x) = \beta^{-1} 2^{-1} y_{2,0,\beta,-1}(0, x), \quad u_2(0, x) = 1, \quad (4.50)$$

yields

$$R_{K,0,\beta} = \begin{pmatrix} 0 & -1 \\ 1 & -2^{-\beta-1} [\gamma_E + \psi(-\beta)] \end{pmatrix}, \quad \alpha = 0, \beta \in (-1, 0). \quad (4.51)$$

**(V) The case  $\alpha \in (-1, 0)$ ,  $\beta = 0$ : Choosing**

$$u_1(0, x) = 1, \quad u_2(0, x) = -\alpha^{-1} 2^{-1} y_{2,\alpha,0,1}(0, x), \quad (4.52)$$

yields

$$R_{K,\alpha,0} = \begin{pmatrix} 2^{-\alpha-1} [\gamma_E + \psi(-\alpha)] & 1 \\ -1 & 0 \end{pmatrix}, \quad \alpha \in (-1, 0), \beta = 0. \quad (4.53)$$

Obviously,  $\det(R_{K,\alpha,\beta}) = 1$  in all five cases.

*Remark 4.4.* In the remaining four cases in Example 4.3, given by all combinations of  $\alpha = 0$ ,  $\beta = 0$ ,  $\alpha \in (0, 1)$ , and  $\beta \in (0, 1)$ , one observes that Theorem 3.5 is not applicable as the underlying minimal operator,  $T_{min,\alpha,\beta}$ , is not strictly positive. This is easily seen by considering the Jacobi polynomials and the boundary conditions they satisfy. The  $n$ th Jacobi polynomial is defined as (see [53, Eq. 18.5.7])

$$P_n^{\alpha,\beta}(x) := \frac{(\alpha+1)_n}{n!} F(-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2), \quad (4.54)$$

$$n \in \mathbb{N}_0, \quad -\alpha \notin \mathbb{N}, \quad -n-\alpha-\beta-1 \notin \mathbb{N},$$

and can be defined by continuity for all parameters  $\alpha, \beta \in \mathbb{R}$ . We note that  $P_n^{\alpha,\beta}(x)$  is a polynomial of degree at most  $n$ , and has strictly smaller degree if and only if  $-n-\alpha-\beta \in \{1, \dots, n\}$  (cf. [66, p. 64]). It satisfies the differential equation

$$\tau_{\alpha,\beta} P_n^{\alpha,\beta}(x) = \lambda_n^{\alpha,\beta} P_n^{\alpha,\beta}(x), \quad (4.55)$$

where

$$\lambda_n^{\alpha,\beta} = n(n+1+\alpha+\beta), \quad n \in \mathbb{N}_0. \quad (4.56)$$

One verifies that the Jacobi polynomials are solutions of the Jacobi operator eigenvalue equation  $\tau_{\alpha,\beta} y = \lambda_n^{\alpha,\beta} y$  with Neumann boundary conditions in the regular case where  $\alpha, \beta \in (-1, 0)$ , and the Friedrichs boundary conditions in the present case under consideration where  $\alpha, \beta \in [0, 1)$ .

In particular, this implies that  $0 \in \sigma(T_{F,\alpha,\beta})$ ,  $\alpha, \beta \in [0, 1)$ , where  $T_{F,\alpha,\beta}$  denotes the Friedrichs extension of  $T_{min,\alpha,\beta}$ , and hence  $T_{min,\alpha,\beta} \geq 0$  is nonnegative, but not strictly positive when  $\alpha, \beta \in [0, 1)$ .  $\diamond$

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## REFERENCES

- [1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1972.
- [2] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space, Volume II*, Pitman, Boston, 1981.
- [3] A. Alonso and B. Simon, *The Birman-Krein-Vishik theory of selfadjoint extensions of semi-bounded operators*, *J. Operator Th.* **4**, 251–270 (1980); Addenda: **6**, 407 (1981).
- [4] T. Ando and K. Nishio, *Positive selfadjoint extensions of positive symmetric operators*, *Tohoku Math. J. (2)*, **22**, 65–75 (1970).
- [5] A. Yu. Anan’eva and V. S. Budyka, *On the spectral theory of the Bessel operator on a finite interval and the half-line*, *Diff. Eq.* **52**, 1517–1522 (2016).
- [6] A. Yu. Ananieva and V. S. Budyka, *To the spectral theory of the Bessel operator on finite interval and half-line*, *J. Math. Sci.* **211**, 624–645 (2015).
- [7] Yu. M. Arlinskii, S. Hassi, Z. Sebestyén, and H. S. V. de Snoo, *On the class of extremal extensions of a nonnegative operator*, in *Recent Advances in Operator Theory and Related Topics*, L. Kérchy, C. Foias, I. Gohberg, and H. Langer (eds.), *Operator Theory: Advances and Applications*, Vol. 127, Birkhäuser, Basel, 2001, pp. 41–81.
- [8] Yu. M. Arlinskii and E. R. Tsekanovskii, *The von Neumann problem for nonnegative symmetric operators*, *Integr. Equ. Oper. Theory* **51**, 319–356 (2005).
- [9] Yu. Arlinskii and E. Tsekanovskii, *M. Krein’s research on semibounded operators, its contemporary developments, and applications*, in *Modern Analysis and Applications. The Mark Krein Centenary Conference*, Vol. 1, V. Adamyan, Y. M. Berezansky, I. Gohberg, M. L. Gorbachuk, V. Gorbachuk, A. N. Kochubei, H. Langer, and G. Popov (eds.), *Operator Theory: Advances and Applications*, Vol. 190, Birkhäuser, Basel, 2009, pp. 65–112.
- [10] M. S. Ashbaugh, F. Gesztesy, M. Mitrea, and G. Teschl, *Spectral theory for perturbed Krein Laplacians in nonsmooth domains*, *Adv. Math.* **223**, 1372–1467 (2010).
- [11] M. S. Ashbaugh, F. Gesztesy, M. Mitrea, R. Shterenberg, and G. Teschl, *The Krein–von Neumann extension and its connection to an abstract buckling problem*, *Math. Nachr.* **283**, 165–179 (2010).
- [12] M. S. Ashbaugh, F. Gesztesy, M. Mitrea, R. Shterenberg, and G. Teschl, *A survey on the Krein–von Neumann extension, the corresponding abstract buckling problem, and Weyl-type spectral asymptotics for perturbed Krein Laplacians in non smooth domains*, in *Mathematical Physics, Spectral Theory and Stochastic Analysis*, M. Demuth and W. Kirsch (eds.), *Operator Theory: Advances and Applications*, Vol. 232, Birkhäuser, Springer, Basel, 2013, pp. 1–106.
- [13] M. S. Ashbaugh, F. Gesztesy, A. Laptev, M. Mitrea, and S. Sukhtaiev, *A bound for the eigenvalue counting function for Krein–von Neumann and Friedrichs extensions*, *Adv. Math.* **304**, 1108–1155 (2017).
- [14] B. P. Belinskiy, D. B. Hinton, and R. Nichols, *Singular Sturm–Liouville operators with extreme properties that generate black holes*, *Stud. Appl. Math.* **147**, 180–208 (2021).
- [15] J. Behrndt, S. Hassi, and H. de Snoo, *Boundary Value Problems, Weyl Functions, and Differential Operators*, *Monographs in Math.*, Vol. 108, Birkhäuser, Springer, 2020.
- [16] M. Sh. Birman, *On the theory of self-adjoint extensions of positive definite operators*, *Mat. Sbornik* **38**, 431–450 (1956). (Russian.)
- [17] L. Bruneau, J. Dereziński, and V. Georgescu, *Homogeneous Schrödinger operators on half-line*, *Ann. H. Poincaré* **12**, 547–590 (2011).
- [18] M. Bush, D. Frymark, and C. Liaw, *Singular boundary conditions for Sturm–Liouville operators via perturbation theory*, preprint, 2020.
- [19] S. Clark, F. Gesztesy, and R. Nichols, *Principal solutions revisited*, in *Stochastic and Infinite Dimensional Analysis*, C. C. Bernido, M. V. Carpio-Bernido, M. Grothaus, T. Kuna, M. J.

- Oliveira, and J. L. da Silva (eds.), Trends in Mathematics, Birkhäuser, Springer, 2016, pp. 85–117.
- [20] S. Clark, F. Gesztesy, R. Nichols, and M. Zinchenko, *Boundary data maps and Krein’s resolvent formula for Sturm–Liouville operators on a finite interval*, Operators and Matrices **8**, 1–71 (2014).
- [21] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, Krieger Publ., Malabar, FL, 1985.
- [22] V. A. Derkach and M. M. Malamud, *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*, J. Funct. Anal. **95**, 1–95 (1991).
- [23] V. A. Derkach and M. M. Malamud, *The extension theory of Hermitian operators and the moment problem*, J. Math. Sci. **73**, 141–242 (1995).
- [24] N. Dunford and J. T. Schwartz, *Linear Operators. Part II: Spectral Theory*, Wiley, Interscience, New York, 1988.
- [25] W. N. Everitt, *A catalogue of Sturm–Liouville differential equations*, in *Sturm–Liouville Theory: Past and Present*, W. O. Amrein, A. M. Hinz, D. B. Pearson (eds.), Birkhäuser, Basel, 2005, pp. 271–331.
- [26] W. N. Everitt, K. H. Kwon, L. L. Littlejohn, R. Wellman, and G. J. Yoon, *Jacobi Stirling numbers, Jacobi polynomials, and the left-definite analysis of the classical Jacobi differential expression*, J. Comput. Appl. Math. **208**, 29–56 (2007).
- [27] W. G. Faris, *Self-Adjoint Operators*, Lecture Notes in Mathematics, Vol. 433, Springer, Berlin, 1975.
- [28] D. Frymark, *Boundary triples and Weyl  $m$ -functions for powers of the Jacobi differential operator*, J. Diff Eq. **269**, 7931–7974 (2020).
- [29] F. Gesztesy, N. Kalton, K. Makarov, and E. Tsekanovskii, *Some applications of operator-valued Herglotz functions*, in *Operator Theory, System Theory and Related Topics*. The Moshe Livsic Anniversary Volume, D. Alpay and V. Vinnikov (eds.), Operator Theory: Advances and Applications, Vol. 123, Birkhäuser, Basel, 2001, pp. 271–321.
- [30] F. Gesztesy, L. L. Littlejohn, and R. Nichols, *On self-adjoint boundary conditions for singular Sturm–Liouville operators bounded from below*, J. Diff. Eq. **269**, 6448–6491 (2020).
- [31] F. Gesztesy, L. L. Littlejohn, M. Piorkowski, and J. Stanfill, *The Jacobi operator and its Weyl–Titchmarsh–Kodaira  $m$ -functions*, preprint 2020.
- [32] F. Gesztesy, R. Nichols, and J. Stanfill, *A Survey of Some Norm Inequalities*, Complex Anal. Operator Th., **15**, No. 23 (2021).
- [33] F. Gesztesy and M. Zinchenko, *Sturm–Liouville Operators, Their Spectral Theory, and Some Applications*, book in preparation.
- [34] G. Grubb, *Spectral asymptotics for the “soft” selfadjoint extension of a symmetric elliptic differential operator*, J. Operator Th. **10**, 9–20 (1983).
- [35] G. Grubb, *Distributions and Operators*, Graduate Texts in Mathematics, Vol. 252, Springer, New York, 2009.
- [36] U. Grunewald, *Jacobische Differentialoperatoren*, Math. Nachr. **63**, 239–253 (1974).
- [37] P. Hartman, *Ordinary Differential Equations*. SIAM, Philadelphia, 2002.
- [38] P. Hartman and A. Wintner, *On the assignment of asymptotic values for the solutions of linear differential equations of second order*, Amer. J. Math. **77**, 475–483 (1955).
- [39] S. Hassi, M. Malamud, and H. de Snoo, *On Krein’s extension theory of nonnegative operators*, Math. Nachr. **274–275**, 40–73 (2004).
- [40] S. Hassi, A. Sandovici, H. de Snoo, and H. Winkler, *A general factorization approach to the extension theory of nonnegative operators and relations*, J. Operator Th. **58**, 351–386 (2007).
- [41] K. Jörgens and F. Rellich, *Eigenwerttheorie Gewöhnlicher Differentialgleichungen*, Springer-Verlag, Berlin, 1976.
- [42] H. Kalf, *A characterization of the Friedrichs extension of Sturm–Liouville operators*, J. London Math. Soc. (2) **17**, 511–521 (1978).
- [43] E. Kamke, *Differentialgleichungen. Lösungsmethoden und Lösungen. Gewöhnliche Differentialgleichungen*, 7th ed., Akademische Verlagsgesellschaft, Leipzig, 1961.
- [44] T. Kato, *Perturbation Theory for Linear Operators*, corr. printing of the 2nd ed., Springer, Berlin, 1980.
- [45] T. Koornwinder, A. Kostenko and G. Teschl, *Jacobi polynomials, Bernstein-type inequalities and dispersion estimates for the discrete Laguerre operator*, Adv. Math. **333**, 796–821 (2018).

- [46] M. G. Krein, *The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. I*, Mat. Sbornik **20**, 431–495 (1947). (Russian).
- [47] M. G. Krein, *The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. II*, Mat. Sbornik **21**, 365–404 (1947). (Russian).
- [48] A. Kuijlaars, A. Martinez-Finkelshtein and R. Orive, *Orthogonality of Jacobi polynomials with general parameters*, Electron. Trans. Numer. Anal. **19**, 1–17 (2005).
- [49] W. Leighton and M. Morse, *Singular quadratic functionals*, Trans. Amer. Math. Soc. **40**, 252–286 (1936).
- [50] M. A. Naimark, *Linear Differential Operators. Part II: Linear Differential Operators in Hilbert Space*, Transl. by E. R. Dawson, Engl. translation edited by W. N. Everitt, Ungar Publishing, New York, 1968.
- [51] G. Nenciu, *Applications of the Krein resolvent formula to the theory of self-adjoint extensions of positive symmetric operators*, J. Operator Th. **10**, 209–218 (1983).
- [52] H.-D. Niessen and A. Zettl, *Singular Sturm–Liouville problems: the Friedrichs extension and comparison of eigenvalues*, Proc. London Math. Soc. (3) **64**, 545–578 (1992).
- [53] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), *NIST Handbook of Mathematical Functions*, National Institute of Standards and Technology (NIST), U.S. Dept. of Commerce, and Cambridge Univ. Press, 2010.
- [54] F. W. J. Olver et al., *NIST Handbook of Mathematical Functions*, <http://dlmf.nist.gov/>, Release 1.0.26 of 2020-03-15.
- [55] D. B. Pearson, *Quantum Scattering and Spectral Theory*, Academic Press, London, 1988.
- [56] V. Prokaj and Z. Sebestyén, *On extremal positive operator extensions*, Acta Sci. Math. (Szeged) **62**, 485–491 (1996).
- [57] F. Rellich, *Die zulässigen Randbedingungen bei den singulären Eigenwertproblemen der mathematischen Physik. (Gewöhnliche Differentialgleichungen zweiter Ordnung.)*, Math. Z. **49**, 702–723 (1943/44). (German.)
- [58] F. Rellich, *Halbbeschränkte gewöhnliche Differentialoperatoren zweiter Ordnung*. Math. Ann. **122**, 343–368 (1951). (German.)
- [59] R. Rosenberger, *A new characterization of the Friedrichs extension of semibounded Sturm–Liouville operators*, J. London Math. Soc. (2) **31**, 501–510 (1985).
- [60] G. Teschl, *Mathematical Methods in Quantum Mechanics. With Applications to Schrödinger Operators*, 2nd ed., Graduate Studies in Math., Vol. 157, Amer. Math. Soc., RI, 2014.
- [61] Z. Sebestyén and E. Sikolya, *On Krein–von Neumann and Friedrichs extensions*, Acta Sci. Math. (Szeged) **69**, 323–336 (2003).
- [62] B. Simon, *The classical moment problem as a self-adjoint finite difference operator*, Adv. Math. **137**, 82–203 (1998).
- [63] C. F. Skau, *Positive self-adjoint extensions of operators affiliated with a von Neumann algebra*, Math. Scand. **44**, 171–195 (1979).
- [64] O. G. Storozh, *On the hard and soft extensions of a nonnegative operator*, J. Math. Sci. **79**, 1378–1380 (1996).
- [65] A. V. Štraus, *On extensions of a semibounded operator*, Sov. Math. Dokl. **14**, 1075–1079 (1973).
- [66] G. Szegő, *Orthogonal Polynomials*, 4th Edition, Colloquium Publications, Vol. 23, Amer. Math. Soc., Providence, RI 1975.
- [67] E. R. Tsekanovskii, *Friedrichs and Krein extensions of positive operators and holomorphic contraction semigroups*, Funct. Anal. Appl. **15**, 308–309 (1981).
- [68] M. L. Višik, *On general boundary problems for elliptic differential equations*, Trudy Moskov. Mat. Obsc. **1**, 187–246 (1952) (Russian); Engl. transl. in Amer. Math. Soc. Transl. (2), **24**, 107–172 (1963).
- [69] J. von Neumann, *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren*, Math. Ann. **102**, 49–131 (1929–30).
- [70] J. Weidmann, *Linear Operators in Hilbert Spaces*, Graduate Texts in Mathematics, Vol. 68, Springer, New York, 1980.
- [71] J. Weidmann, *Lineare Operatoren in Hilberträumen. Teil II: Anwendungen*, Teubner, Stuttgart, 2003.
- [72] A. Zettl, *Sturm–Liouville Theory*, Mathematical Surveys and Monographs, Vol. 121, Amer. Math. Soc., Providence, RI, 2005.

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