Abstract

Solutions of an Inhomogeneous Multiplicatively Advanced Differential Equation using an Analysis of Wavelet Coefficients

by

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In this paper solutions to an Inhomogeneous Multiplicatively Advanced Differential Equation (MADE) of the form $y'(t) - Ay(qt) = f(t)$, for $q > 1$ and certain functions $f(t) \in \mathbb{L}^2(\mathbb{R})$, are provided. Such solutions are obtained with the help of wavelet frames generated by two special functions called ${}_qCos(t)$ and ${}_qSin(t)$. The introduction provides basic definitions and useful tools. Then a specific solution of our main MADE above is obtained, where $f(t)$ is equal to either ${}_qCos(\alpha t + \beta)$ or ${}_qSin(\alpha t + \beta)$. After obtaining the solutions for these specific MADE's, we will prove that the solutions are actually Schwartz wavelets whose series expansion converge uniformly and absolutely with all moments vanishing. The general MADE is then solved by obtaining a series expansion for $f(t)$ in terms of ${}_qCos(t)$ or ${}_qSin(t)$. A series solution, $y(t)$, of the general MADE then follows by linearity. Assumptions on the wavelet coefficients will be obtained sufficient for the series solution $y(t)$ to converge uniformly and absolutely. Finally, pictures of these solutions are provided along with questions about confluence between the solutions of these MADE's and their non-advanced differential equation counterparts.

Solutions of an Inhomogeneous MADE using an Analysis of Wavelet Coefficients

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MADE stands for Multiplicatively Advanced Differential Equation

$\L^p(\mathbb{R})$ is the Banach space of p-integrable functions

$\hat{f}$ is the Fourier transform of f

$\theta(q;x)$ is the Jacobi Theta function

${}_qCos$ is a q-advanced cos function

${}_qSin$ is a q-advanced sin function

1 Start Up Section

We begin the study of the inhomogeneous MADE

$$y'(t) - Ay(qt) = f(t), \tag{1}$$

where $A \in \mathbb{R}\backslash\{0\}$ and $q > 1$

by introducing the basic results and techniques upon which the study of (1) is based.

1.1 Key Functions

We will rely on the following two functions in this thesis. For $q > 1$ the function ${}_qCos(t)$ and ${}_qSin(t)$ are defined as in [PRS3] as follows:

$$\begin{aligned}
{}_qCos(t) &\equiv \frac{1}{C_q}\sum_{k=-\infty}^{\infty}\frac{(-1)^k e^{-q^k|t|}}{q^{k^2}} && (2)\\
C_q &\equiv \sum_{k=-\infty}^{\infty}\frac{(-1)^k}{q^{k^2}} = \theta(q^2; -1/q) > 0 && (3)\\
{}_qSin(t) &\equiv sign(t)\frac{1}{C_q}\sum_{k=-\infty}^{\infty}\frac{(-1)^k e^{-q^k|t|}}{q^{k(k-1)}}. && (4)
\end{aligned}$$

Here, in (3) the theta function is given by (31) below.

To check that

$$[{}_qCos(t)]' = -({}_qSin(t)), \tag{5}$$

one has, for $t \neq 0$, that

$$[_qCos(t)]' = \left[\frac{1}{C_q}\sum_{k=-\infty}^{\infty}\frac{(-1)^k e^{-q^k|t|}}{q^{k^2}}\right]' \tag{6}$$

$$= \frac{1}{C_q}\sum_{k=-\infty}^{\infty}\frac{(-1)^k e^{-q^k|t|}}{q^{k^2}}(-q^k)sign(t) \tag{7}$$

$$= sign(t)\frac{-1}{C_q}\sum_{k=-\infty}^{\infty}\frac{(-1)^k e^{-q^k|t|}}{q^{k^2-k}} \tag{8}$$

$$= -(_qSin(t)). \tag{9}$$

For $t = 0$, the result follows from the Mean Value Theorem.

To check that

$$[_qSin(t)]' = q_qCos(qt) \tag{10}$$

one has, for $t \neq 0$, that

$$[_qSin(t)]' = \left[sign(t)\frac{1}{C_q}\sum_{k=-\infty}^{\infty}\frac{(-1)^k e^{-q^k|t|}}{q^{k(k-1)}}\right]' \tag{11}$$

$$= sign(t)\frac{1}{C_q}\sum_{k=-\infty}^{\infty}\frac{(-1)^k e^{-q^k|t|}}{q^{k(k-1)}}(-q^k)sign(t) \tag{12}$$

$$= -(sign(t))^2\frac{1}{C_q}\sum_{k=-\infty}^{\infty}\frac{(-1)^k e^{-q^k|t|}}{q^{k^2-2k}} \tag{13}$$

$$= -\frac{1}{C_q}\sum_{k=-\infty}^{\infty}\frac{(-1)(-1)^{k-1}e^{-q^{k-1}|qt|}q}{q^{(k-1)^2}}. \tag{14}$$

Now making a substitution of $m = k - 1$ gives,

$$[_qSin(t)]' = -\frac{1}{C_q}\sum_{k=-\infty}^{\infty}\frac{(-1)(-1)^m e^{-q^m|qt|}q}{q^{(m)^2}} \tag{15}$$

$$= q_qCos(qt). \tag{16}$$

For $t = 0$, the result follows by an application of the Mean Value Theorem.

Relying on (5) and (10), we have that the following are true:

$$\int {}_qSin(t)dt = -({}_qCos(t)) + C \tag{17}$$

$$\int {}_qCos(t)dt = {}_qSin\left(\frac{t}{q}\right) + C. \tag{18}$$

From (5) and (10) we can see that ${}_qCos(t)$ and ${}_qSin(t)$ solve some certain Multiplicatively Advanced Differential Equations (MADE's). Specifically, $y(t) = {}_qCos(t)$ solves the homogeneous MADE

$$y''(t) + qy(qt) = 0$$

and $y(t) = {}_qSin(t)$ solves the homogeneous MADE

$$y''(t) + q^2y(qt) = 0.$$

The MADE that we will be studying in this thesis is

$$y'(t) - Ay(qt) = f(t), \quad \text{for} \quad f(t) \in Ł^2(\mathbb{R}). \tag{19}$$

Notice that if $f(t)$ in (19) has 0th moment vanishing, meaning $\int_{-\infty}^{\infty} f(t)dt = 0$ (i.e., $f(t)$ is a basic wavelet), and we assume that our solution $y(t)$ in (19) vanishes at plus minus infinity then, we integrate (19) from negative infinity to positive infinity to give:

$$\int_{-\infty}^{\infty} y'(t)dt - A\int_{-\infty}^{\infty} y(qt)dt = \int_{-\infty}^{\infty} f(t)dt = 0. \tag{20}$$

If $y(t)$ vanishes at plus minus infinity then the term $\displaystyle\int_{-\infty}^{\infty} y'(t)dt$ in (20) is 0 and thus,

$$-A\int_{-\infty}^{\infty} y(qt)dt = 0. \tag{21}$$

Then making a substitution of $u = qt$ in (21) gives,

$$-\frac{A}{q}\int_{-\infty}^{\infty} y(u)du = 0. \tag{22}$$

From (22) we see that if A is invertible, then $\displaystyle\int_{-\infty}^{\infty} y(u)du = 0$, meaning the solution $y(t)$ also has 0th moment vanishing and is a basic wavelet. This means that if we have a forcing term $f(t)$ that is a wavelet, and therefore has 0th moment vanishing, as well as we assume that our solution $y(t)$ vanishes at plus minus infinity, and A is invertible, then it is a property of our MADE (19) that the solution $y(t)$ will also have its 0th moment vanishing.

Remark 1. *We conjecture that if $f(t)$ in* (19)*, is a wavelet in the sense of Definition 4, then the solution $y(t)$ of* (19) *is also a wavelet. We will work to prove this for the solutions* (103)-(104) *of $y'(t)-Ay(qt) = {}_qCos(\alpha t+\beta)$ and* (105)-(106) *of $y'(t)-Ay(qt) = {}_qSin(\alpha t+\beta)$ in section 3.*

1.2 Wavelets

We first begin with the definition of $Ł^p(\mathbb{R})$.

Definition 1. *$Ł^p(\mathbb{R})$ is the Banach space of measurable functions such that, $f \in Ł^p(\mathbb{R})$ if and only if*

$$\int_{-\infty}^{\infty} |f(t)|^p \ dt < \infty.$$

We have that $Ł^p(\mathbb{R})$ is a normed space with norm given by $\|f\|_{Ł^p} = \left(\int_{-\infty}^{\infty} |f(t)|^p \ dt\right)^{\frac{1}{p}}$.

An important case of this is $L^2(\mathbb{R})$ *which is the Hilbert space of functions* $f(t)$ *satisfying*

$$f \in L^2(\mathbb{R}) \quad \textit{if and only if} \quad \int_{-\infty}^{\infty} |f(t)|^2 \, dt < \infty, \tag{23}$$

with the inner product in $L^2(\mathbb{R})$ *given by the following,*

$$\langle f, \Phi \rangle = \int_{-\infty}^{\infty} f(x) \cdot \overline{\Phi(x)} dx.$$

Next we define $l^2(\mathbb{Z})$.

Definition 2. *Let* $l^2(\mathbb{Z})$ *be the Hilbert space of sequences,* $\langle c_n \rangle$ *, such that* $\sum_{n \in \mathbb{Z}} |c_n|^2 < \infty$*. Similar to* $L^2(\mathbb{R})$ *we have that the norm of* $l^2(\mathbb{Z})$ *is given by the following,*
$\|\langle c_n \rangle\|_{l^2} = \left(\sum_{n \in \mathbb{Z}} |c_n|^2 \right)^{1/2}$ *, and inner product* $\langle c, d \rangle_{l^2} = \sum_n c_n \overline{d_n}$*.*

Fourier series are able to represent periodic functions with a basis of sines and cosines. So if f is a continuous periodic function on the interval $[0, 2\pi]$ then we have that f is recovered by the following,

$$f(x) = \sum_{n=0}^{\infty} c_n sin(nx) + b_n cos(nx), \quad \text{where} \quad c_n = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \cdot sin(nx) dx \tag{24}$$

and

$$b_n = \begin{cases} \dfrac{1}{\sqrt{\pi}} \displaystyle\int_0^{2\pi} f(x) \cdot cos(nx) dx \quad \text{for} \quad n > 0 \\ \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^{2\pi} f(x) dx \quad \text{for} \quad n = 0 \end{cases} .$$

Then we have that

$$\left\{ \frac{sin(nx)}{\sqrt{\pi}} \right\} \bigcup \left\{ \frac{cos(nx)}{\sqrt{\pi}} \right\} \bigcup \left\{ \frac{1}{\sqrt{2\pi}} \right\}$$

forms an orthonormal basis, since $\int_0^{2\pi} cos(mx) \cdot sin(nx) dx = 0$ for all m and n, $\int_0^{2\pi} sin(mx) \cdot sin(nx) dx = 0$ for $m \neq n$, $\int_0^{2\pi} cos(mx) \cdot cos(nx) dx = 0$ for $m \neq n$, and

$\frac{1}{\pi}\int_0^{2\pi} cos^2(mx)dx = 1 = \frac{1}{\pi}\int_0^{2\pi} sin^2(mx)dx$ for $m \neq 0$. This is also expressed in terms of inner products as $\langle cos(nx), sin(mx)\rangle = 0, \ \ \forall\, n, m \in \mathbb{Z}$ and $\langle cos(mx), cos(nx)\rangle = \langle sin(mx), sin(nx)\rangle = \pi\delta_{n,m}, \ \ \forall\, m \in \mathbb{Z}$, where $\delta_{n,m}$ is the Kronecker delta.

We next define the Fourier transform of a function.

Definition 3. *For a function $f(x) \in Ł^2(\mathbb{R})$ such that $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$, the Fourier transform of $f(x)$ is defined to be*

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{-itx}dx = \mathcal{F}[f(x)](t).$$

In order to recover the original function from the Fourier transform we have that the inverse Fourier transform:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(t)\cdot e^{itx}dt = \mathcal{F}^{-1}[f(t)](x).$$

Also note that for $f, g \in Ł^2(\mathbb{R})$ one has that, $\|f\| = \|\hat{f}\|$ and $\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$; this is known as Plancherel's theorem.

We will define a wavelet as in [Daubechies] and [Christensen].

Definition 4. *A function $f(t)$ is a wavelet if and only if the following three requirements are met:*

$$i) \quad f(t) \in Ł^1(\mathbb{R}) \bigcap Ł^2(\mathbb{R}) \bigcap Ł^\infty(\mathbb{R}) \tag{25}$$

$$ii) \quad \int_{-\infty}^{\infty} f(t)dt = 0 \quad \textit{(0th moment vanishes)} \tag{26}$$

$$iii) \quad \int_{-\infty}^{\infty} \frac{\left|\hat{f}(t)\right|^2}{|t|}dt < \infty \quad \textit{(Admissibility Condition).} \tag{27}$$

The concept of a mother wavelet will be introduced in section 1.4. Certain wavelets will be able to recover any function $f \in Ł^2(\mathbb{R})$, similar to Fourier series recovery of $f(t)$

in (24). The two such wavelets that we will be using are ${}_qCos(t)$ and ${}_qSin(t)$. It is shown in [PRS3] that ${}_qCos(t)$ and ${}_qSin(t)$ are wavelets, meaning they satisfy, (25), (26), and (27). In addition, in [PRS3] it is shown that

$$\widehat{{}_qCos}(x) = \tilde{C}_q \cdot \frac{1}{\theta(q^2; x^2)} \tag{28}$$

$$\text{and} \qquad \widehat{{}_qSin}(x) = \tilde{C}_q \cdot \frac{-ix}{\theta(q^2; x^2)}. \tag{29}$$

Here $\tilde{C}_q$ is the constant

$$\tilde{C}_q = \frac{2\left(\mu_{q^2}\right)^3}{C_q\sqrt{2\pi}}, \tag{30}$$

with C_q as defined in (3).

$\theta(q, x)$ in (28) and (29), is the Jacobi theta function

$$\theta(q, x) \equiv \sum_{n=-\infty}^{\infty} \frac{x^n}{q^{n(n-1)/2}} = \mu_q \prod_{n=0}^{\infty} \left(1 + \frac{x}{q^n}\right)\left(1 + \frac{1}{xq^{n+1}}\right), \tag{31}$$

where $q > 1$, and μ_q, in (30) and (31), is the constant

$$\mu_q \equiv \prod_{n=0}^{\infty} \left(1 - \frac{1}{q^{n+1}}\right). \tag{32}$$

Essential properties of $\theta(q; x)$, upon which we rely, are

$$\theta(q; q^n x) = q^{n(n+1)/2} x^n \; \theta(q; x) \;\; \forall\, n \in \mathbb{Z}, \quad \text{and} \quad \theta\left(q; \frac{1}{qx}\right) = \theta(q; x). \tag{33}$$

1.3 Frames

We define the notion of a frame for $L^2(\mathbb{R})$.

Definition 5. *Let* $\{\phi_j\}_{j\in J}$ *be a countable collection of functions in* $L^2(\mathbb{R})$*, then* $\{\phi_j\}_{j\in J}$

is a frame if and only if there exists an $A, B \in \mathbb{R}$ *with* $0 < A \leq B < \infty$ *such that*

$$A\|f\|^2 \leq \sum_{j\in J} |\langle f, \phi_j\rangle|^2 \leq B\|f\|^2,$$

for all $f \in Ł^2(\mathbb{R})$.

Let $f \in Ł^2(\mathbb{R})$ and $\{\phi_j\}_{j\in J}$ be a frame for $Ł^2(\mathbb{R})$ we define $F : Ł^2(R) \to l^2(\mathbb{Z})$ depending on $\{\phi_j\}_{j\in J}$, where F is given by $F(f) = \{\langle f, \phi_j\rangle\}_{j\in J}$. Also define $F^* : l^2(\mathbb{Z}) \to Ł^2(\mathbb{R})$ by $F^*(\{c_j\}) = \sum_{j\in J} c_j\phi_j$ which converges in $Ł^2(\mathbb{R})$. Then we have $F^*F : Ł^2(\mathbb{R}) \to Ł^2(\mathbb{R})$ and $F^*F(f) = F^*(F(f)) = F^*\left(\{\langle f, \phi_j\rangle\}_{j\in J}\right) = \sum_{j\in J}\langle f, \phi_j\rangle\phi_j$. If we let $\{\phi_j\}_{j\in J}$ be an orthonormal basis, then we may write $f = \sum_{k\in J} \alpha_k\phi_k$. Then if we take $F^*(F(f)) = \sum_{j\in J}\langle f, \phi_j\rangle\phi_j = \sum_{j\in J}\langle\sum_{k\in J}\alpha_k\phi_k, \phi_j\rangle\phi_j = \sum_{j\in J}\sum_{k\in J}\alpha_k\delta_{j,k}\phi_j$. Now since $\delta_{j,k} = 0$ when $j \neq k$ and $\delta_{j,k} = 1$ when $j = k$, we have that $\sum_{k\in J}\alpha_k\delta_{j,k} = \alpha_j$, thus $F^*(F(f)) = \sum_{j\in J}\sum_{k\in J}\alpha_k\delta_{j,k}\phi_j = \sum_{j\in J}\alpha_j\phi_j = f$. However, for a general frame, unlike an orthonormal basis, we do not necessarily recover f with $\sum_{j\in J}\langle f, \phi_j\rangle\phi_j$. In order to resolve this issue we introduce the dual frame, as in [Daubechies] and [Christensen].

Definition 6. *Let* $\tilde{\phi}_j = (F^*F)^{-1}\phi_j$ *then* $\left\{\tilde{\phi}_j\right\}$ *is called the dual frame to* $\{\phi_j\}$. *This gives us* $\tilde{F} : Ł^2(\mathbb{R}) \to l^2(\mathbb{Z})$ *by* $\tilde{F}(f) = \langle f, \tilde{\phi}_j\rangle_{j\in J}$ *and* $\tilde{F}^* : l^2(\mathbb{Z}) \to Ł^2(\mathbb{R})$ *by* $\tilde{F}^*\left(\{c_j\}_{j\in J}\right) = \sum_{j\in J} c_j\tilde{\phi}_j$.

Then, as shown in [Daubechies], [Christensen], and [David Edwards], for a frame $\{\phi_j\}$ we can write any function $f \in Ł^2(\mathbb{R})$ as

$$f = \sum_{j\in J}\langle f, \phi_j\rangle\tilde{\phi}_j = \sum_{j\in J}\left\langle f, \tilde{\phi}_j\right\rangle\phi_j. \tag{34}$$

From (34) one observes that a frame is a generating set for $Ł^2(\mathbb{R})$ (also known as a spanning set for $Ł^2(\mathbb{R})$). The functions ${}_qCos(t)$ and ${}_qSin(t)$ have been shown in [PRS3] to be in $Ł^1(\mathbb{R})\bigcap Ł^2(\mathbb{R})\bigcap Ł^\infty(\mathbb{R})$ and produce wavelet frames for generating $Ł^2(\mathbb{R})$. For ${}_qCos(t)$ and ${}_qSin(t)$, respectively, the frames are:

$$\left\{\Phi_{N,M} = q^{\frac{N}{2}}\, {}_qCos(q^N t - Mb) \,\middle|\, N, M \in \mathbb{Z}\right\} \tag{35}$$

$$\text{or} \qquad \left\{\Phi_{N,M} = q^{\frac{N}{2}}\, {}_qSin(q^N t - Mb) \,\middle|\, N, M \in \mathbb{Z}\right\}. \tag{36}$$

1.4 Mother Wavelets

A mother wavelet $\Phi(t)$ is a wavelet in the sense of Definition 4 that can generate a family of wavelets. We will denote mother wavelets as $\Phi(t)$ and take the set of functions generated by $\Phi(t)$ to be $\Phi_{N,M}(t) = a_0^{\frac{N}{2}}\Phi(a_0^N t - Mb_0)$ for $N, M \in \mathbb{Z}$.

A wavelet $f(t)$ is a mother wavelet for a frame generating $Ł^2(\mathbb{R})$ of form

$$S(\Phi; a_0, b_0) = \left\{a_0^{n/2}\Phi(a_0^n t + mb_0)/\|\Phi\| \,\middle|\, n, m \in \mathbb{Z}\right\}$$

if $S(\Phi; a_0, b_0)$ generates $Ł^2(\mathbb{R})$, where $a_0 > 1$ is the scale factor, $b_0 > 0$ is the translation parameter, and $\|\Phi\| = \|\Phi\|_2$ is the norm of Φ in $Ł^2(\mathbb{R})$. One defines the diagonal term $G_0[\Phi](x)$ by

$$G_0[\Phi](x) \equiv \frac{1}{\|\Phi\|^2}\sum_{n=-\infty}^{\infty}\left|\hat{\Phi}(a_0^n x)\right|^2$$

and the off-diagonal term $G_1[\Phi](x)$ by

$$G_1[\Phi](x) \equiv \frac{1}{\|\Phi\|^2}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}/\{0\}}\left|\hat{\Phi}(a_0^j x)\cdot\hat{\Phi}(a_0^j x + 2\pi k/b^0)\right|, \tag{37}$$

which together give the frame condition

$$
\begin{aligned}
0 &< \inf_{1\le|x|\le a_0} \{G_0[\Phi](x) - G_1[\Phi](x)\} && (38)\\
&\le \sup_{1\le|x|\le a_0} \{G_0[\Phi](x) - G_1[\Phi](x)\} < \infty && (39)
\end{aligned}
$$

sufficient for $S(\Phi; a_0, b_0)$ to be a frame. It has been shown in [Christensen] that the condition (38)-(39) is implied by the condition (40)-(41) immediately below:

$$
\begin{aligned}
0 < \inf_{1\le|x|\le a_0} \{G_0[\Phi](x)\} \quad \text{and} \quad \exists\, C > 0 && (40)\\
\text{with} \quad \left|\hat{\Phi}(x)\right| \le \frac{C|x|}{(1+x^2)^{3/2}}. && (41)
\end{aligned}
$$

Remark 2. *The functions ${}_qCos(t)$ and ${}_qSin(t)$ have been shown to be mother wavelets generating a frame for $L^2(\mathbb{R})$ in [PRS3]. Also for a mother wavelet Φ it has been shown in [Christensen] that the set*

$$
S(\Phi; a_0, b_0) \equiv \left\{ a^{n/2}\Phi(a_0^n t + mb_0)/\|\Phi\| \middle| n, m \in \mathbb{Z} \right\},
$$

generates a frame for $L^2(\mathbb{R})$, if the conditions (40)-(41) *hold.*

2 Useful Estimates

Here we introduce known estimates that we will use in this thesis.

2.1 Bounds for ${}_qCos$ and ${}_qSin$

The first estimate that we will use is proven in [PRS3] and are bounds for ${}_qCos$ and ${}_qSin$. They are the following:

$$|{}_qCos(u)| \leq 1 \ \ \forall \ u \in \mathbb{R} \tag{42}$$

$$|{}_qSin(u)| \leq q^{1/2} \ \ \forall \ u \in \mathbb{R}. \tag{43}$$

Furthermore in [PRS3] it is shown that for $|u| \geq 1$ there is an upper decay bound on ${}_qCos(u)$ of the form

$$|{}_qCos(u)| \leq B_q |u|^{-2/(2/e+\ln(q))} |u|^{-\ln(|u|)\ln(q)/(2/e+\ln(q))^2}. \tag{44}$$

Similarly in [PRS3] it is shown that for $|u| \geq 1$ there is an upper decay bound on ${}_qSin(u)$ of the form

$$|{}_qSin(u)| \leq B_q e^{-1} |u|^{-1} |u|^{-\ln(|u|)\ln(q)/(2+\ln(q))^2}, \tag{45}$$

where B_q in both (44) and (45) is given by

$$B_q = (C_q)^{-1} \left[1 + \sqrt{\frac{\pi}{4\ln(q)}} \right], \tag{46}$$

and C_q is as defined in (3).

2.2 Bounds for a Gaussian

Another useful estimate is the following proposition that is used to to bound $\sum_{n\in\mathbb{Z}} G(n)$ for G a Gaussian.

Proposition 1. *Let $G(x)$ be a Gaussian of form $G(x) = \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$. Then the following bound holds*

$$\sum_{n\in\mathbb{Z}} G(n) = \sum_{n\in\mathbb{Z}} \exp\left[\frac{-\frac{1}{2}(n-\mu)^2}{\sigma^2}\right] < \left[1 + \sqrt{2\pi}\cdot\sigma\right]. \tag{47}$$

Proof. The proof for this Proposition is completed in [PRS3] □

2.3 Rudin's Theorem

We will also employ the following Theorem (Rudin's Theorem) that is used to rigorously ensure the exchange of derivatives with infinite sums.

Theorem 1. *(Rudin) Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$, such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$, to a function f, and*

$$f'(x) = \lim_{n\to\infty} f_n'(x) \qquad (a \le x \le b). \tag{48}$$

Proof. This is THM 7.17 on pg 152-153 in [Rudin], and it is proven there. □

2.4 Lebesgue Dominated Convergence Theorem

We will use the following Theorem to pass a limit through an integral.

Theorem 2. *Let $\{f_n\}$ be a sequence of complex valued measurable functions on a measure space (S,Σ,μ). Suppose that the sequence converges point-wise to a function f and* *is dominated by some integrable function g in the sense that*

$$|f_n(x)| \leq |g(x)|$$

for all numbers n in the index set of the sequence and all points $x \in S$. Then f is integrable in the Lebesgue sense and

$$\lim_{n\to\infty} \int_S |f_n - f| d\mu \ = 0$$

which also implies

$$\lim_{n\to\infty} \int_S f_n d\mu = \int_S f \ d\mu.$$

2.5 Schwartz Spaces

Here is the definition of a Schwartz space along with some consequences of a function being Schwartz.

Definition 7. *Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The Schwartz space or space of rapidly decreasing functions on $\mathbb{R}$ is the function space*

$$\mathcal{S}(\mathbb{R}, \mathbb{C}) = \left\{ f \in \mathcal{C}^\infty \,\middle|\, \forall\, \alpha, \beta \in \mathbb{N}_0 \,, |f|_{\alpha,\beta} < \infty \right\}, \tag{49}$$

where $\mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$ is the function space of smooth functions from $\mathbb{R}$ into $\mathbb{C}$, and $|f|_{\alpha,\beta} = \sup_{x\in\mathbb{R}} \left| x^\alpha f^{(\beta)}(x) \right|$, where $\alpha, \beta \in \mathbb{N}_0$.

A necessary and sufficient condition for a function $y(t)$ to be Schwartz is the following:

$$y(t) \ \text{ is Schwartz } \iff \quad \forall\, p, k \in \mathbb{N}, \ \exists\, C_{p,k} \in \mathbb{R} \ \text{ s.t. } \ 0 < |t^p y^{(k)}(t)| \leq C_{p,k} < \infty. \tag{50}$$

Two properties of a function $y(t)$ being Schwartz that we will use are the following:

$$\text{if} \quad y(t) \;\text{ is Schwartz, then }\; y(t) \in \L^p(\mathbb{R}^n), \quad \forall\, n \in \mathbb{N} \;\text{ and }\; 1 \leq p \leq \infty, \tag{51}$$

$$\text{if} \quad f(t) \;\text{ is Schwartz, then }\; \hat{f}(x) \;\text{ is also Schwartz.} \tag{52}$$

3 The First Inhomogeneous Multiplicatively Advanced Differential Equation

The main functional differential equation that we want to solve is:

$$y'(t) - Ay(qt) = f(t) \tag{53}$$

for certain $f(t)$ in $Ł^2(\mathbb{R})$. To do this we first look at the case where the $f(t)$ in (53) is one of our ${}_qCos$ or ${}_qSin$ functions.

So first we want to solve the two functional differential equations:

$$y'(t) - Ay(qt) = {}_qCos(\alpha t + \beta) \tag{54}$$

$$y'(t) - Ay(qt) = {}_qSin(\alpha t + \beta). \tag{55}$$

So we want to solve $y'(t) - Ay(qt) = f(t)$, where $f(t) = {}_qCos(\alpha + \beta)$ or ${}_qSin(\alpha + \beta)$. Integrating (54) and (55) over $(-\infty, t]$ and assuming the solution $y(t)$ vanishes at $-\infty$ gives:

$$\int_{-\infty}^{t} y'(u)du - A\int_{-\infty}^{t} y(qu)du = F(t), \tag{56}$$

where, from (18) and (17),

$$F(t) = \begin{cases} \displaystyle\int_{-\infty}^{t} {}_qCos(\alpha t + \beta)dt = \left(\frac{1}{\alpha}\right) {}_qSin\left(\frac{\alpha t + \beta}{q}\right) & \text{if } f(t) = {}_qCos(\alpha t + \beta) \\ \displaystyle\int_{-\infty}^{t} {}_qSin(\alpha t + \beta)dt = \left(\frac{-1}{\alpha}\right) {}_qCos(\alpha t + \beta) & \text{if } f(t) = {}_qSin(\alpha t + \beta) \end{cases}. \tag{57}$$

Now for the term $\int_{-\infty}^{t} y(qu)du$ in (56) we make a change of variables letting $v = qu$.

Then the term $A\int_{-\infty}^{t} y(qu)du$ in (56) becomes $\frac{A}{q}\int_{-\infty}^{qt} y(v)dv = \tilde{K}[y(v)](t)$,

where the linear operator $\tilde{K}$ is defined by:

$$\tilde{K}[y(v)](t) = \frac{A}{q}\int_{-\infty}^{qt} y(v)dv. \tag{58}$$

Then our integral equation (56) becomes:

$$y(t) - \tilde{K}[y(v)](t) = F(t)$$

$$\tilde{I}[y(t)] - \tilde{K}[y(v)](t) = F(t), \tag{59}$$

where $\tilde{I}$ is the identity operator. Assuming invertibility of $\left(\tilde{I} - \tilde{K}\right)$ in (59) gives a formal solution to (54) or (55), namely:

$$y(t) = \left(\tilde{I} - \tilde{K}\right)^{-1}[F(u)](t)$$
$$y(t) = \sum_{n=0}^{\infty} \tilde{K}^n[F(u)](t). \tag{60}$$

Now in order to calculate $\tilde{K}^n[F(u)](t)$ in (60), we first compute $\tilde{K}[{}_qCos(\alpha u + \beta)](t)$.

$$\tilde{K}[{}_qCos(\alpha u + \beta)](t) = \frac{A}{q}\int_{-\infty}^{qt} {}_qCos(\alpha u + \beta)du$$
$$= \frac{A}{q}\left[{}_qSin\left(\frac{\alpha u + \beta}{q}\right)\left(\frac{1}{\alpha}\right)\right]\Bigg|_{-\infty}^{qt} = \left(\frac{A}{q\alpha}\right){}_qSin\left(\frac{\alpha qt + \beta}{q}\right), \tag{61}$$

where (18) was used to obtain the left side of (61).

Next, to calculate $\tilde{K}^2[{}_qCos(\alpha u+\beta)](t)$

$$\begin{aligned}
\tilde{K}^2[{}_qCos(\alpha u+\beta)](t) &= \tilde{K}\left[\left(\frac{A}{q\alpha}\right)\,{}_qSin\left(\frac{\alpha qt+\beta}{q}\right)\right] \\
&= \left(\frac{A^2}{q^2\alpha}\right)\int_{-\infty}^{qt}{}_qSin\left(\frac{\alpha qu+\beta}{q}\right)du \\
&= \left(\frac{A^2}{q^2\alpha}\right)\left[-{}_qCos\left(\frac{\alpha qu+\beta}{q}\right)\left(\frac{1}{\alpha}\right)\right]\Bigg|_{-\infty}^{qt} \qquad (62) \\
&= (-1)\left(\frac{A^2}{q^2\alpha^2}\right)\,{}_qCos\left(\frac{q^2\alpha t+\beta}{q}\right),
\end{aligned}$$

where (17) was used to obtain (62).

Now we need to do this inductively for $\tilde{K}^n[{}_qCos(\alpha t+\beta)]$ and we will start by continuing to take powers of $\tilde{K}^n$ to get a general formula.

For $n=3$ we have:

$$\begin{aligned}
\tilde{K}^3[{}_qCos(\alpha t+\beta)] &= \tilde{K}\left[(-1)\left(\frac{A^2}{q^2\alpha^2}\right)\,{}_qCos\left(\frac{q^2\alpha u+\beta}{q}\right)\right](t) \\
&= (-1)\left(\frac{A^3}{q^3\alpha^2}\right)\int_{-\infty}^{qt}{}_qCos\left(\frac{q^2\alpha u+\beta}{q}\right)du \\
&= (-1)\left(\frac{A^3}{q^3\alpha^2}\right)\left[{}_qSin\left(\frac{q^2\alpha u+\beta}{q^2}\right)\left(\frac{1}{q\alpha}\right)\right]\Bigg|_{-\infty}^{qt} \qquad (63) \\
&= (-1)\left(\frac{A^3}{q^4\alpha^3}\right)\,{}_qSin\left(\frac{q^3\alpha t+\beta}{q^2}\right),
\end{aligned}$$

where (18) was used to obtain (63).

Then for $n = 4$ we have:

$$\begin{aligned}
\tilde{K}^4[{}_qCos(\alpha u + \beta)](t) &= \tilde{K}\left[(-1)\left(\frac{A^3}{q^4\alpha^3}\right)\ {}_qSin\left(\frac{q^3\alpha u + \beta}{q^2}\right)\right](t) \\
&= (-1)\left(\frac{A^4}{q^5\alpha^3}\right)\int_{-\infty}^{qt} {}_qSin\left(\frac{q^3\alpha u + \beta}{q^2}\right) du \\
&= (-1)\left(\frac{A^4}{q^5\alpha^3}\right)\left[-{}_qCos\left(\frac{q^3\alpha u + \beta}{q^2}\right)\left(\frac{1}{q\alpha}\right)\right]\Bigg|_{-\infty}^{qt} \qquad (64) \\
&= (-1)^2\left(\frac{A^4}{q^6\alpha^4}\right)\left({}_qCos\left(\frac{q^4\alpha t + \beta}{q^2}\right)\right),
\end{aligned}$$

where (17) was used to obtain (64).

To compute $\tilde{K}^n[{}_qCos(\alpha u + \beta](t)$ in general, we have the following Proposition.

Proposition 2. *Applying powers of* $\tilde{K}$ *to* ${}_qCos[\alpha u+\beta](t)$ *gives, for even and odd powers, respectively,*

$$\begin{aligned}
\tilde{K}^{2n}[{}_qCos(\alpha u + \beta](t) &= (-1)^n\left(\frac{A}{\alpha}\right)^{2n}\left(\frac{1}{q^{n(n+1)}}\right)\ {}_qCos\left(\frac{q^{2n}\alpha t + \beta}{q^n}\right) \qquad (65) \\
\tilde{K}^{2n+1}[{}_qCos(\alpha u + \beta)](t) &= (-1)^n\left(\frac{A}{\alpha}\right)^{2n+1}\left(\frac{1}{q^{(n+1)^2}}\right)\ {}_qSin\left(\frac{q^{2n+1}\alpha t + \beta}{q^{n+1}}\right). \qquad (66)
\end{aligned}$$

Proof. We will prove this Proposition with induction on n. Assume (65) as the inductive hypothesis. We need to calculate $\tilde{K}^{2(n+1)}[{}_qCos(\alpha u + \beta)](t)$.

First we calculate $\tilde{K}^{2n+1}[{}_qCos(\alpha u+\beta)](t)$.

$$\begin{aligned}
&\tilde{K}^{2n+1}[{}_qCos(\alpha u+\beta)](t) = \tilde{K}[\tilde{K}^{2n}[{}_qCos(\alpha u+\beta)](t)] \\
= \;& (-1)^n \left(\frac{A}{\alpha}\right)^{2n} \left(\frac{1}{q^{n(n+1)}}\right) \tilde{K}\left[{}_qCos\left(\frac{q^{2n}\alpha u+\beta}{q^n}\right)\right](t) \qquad (67)\\
= \;& (-1)^n \left(\frac{A}{\alpha}\right)^{2n} \left(\frac{1}{q^{n(n+1)}}\right) \left[\frac{A}{q}\int_{-\infty}^{qt} {}_qCos\left(\frac{q^{2n}\alpha u+\beta}{q^n}\right)du\right] \\
= \;& (-1)^n \left(\frac{A}{\alpha}\right)^{2n} \left(\frac{1}{q^{n(n+1)}}\right) \left[\frac{A}{q}\left[{}_qSin\left(\frac{q^{2n}\alpha u+\beta}{q^{n+1}}\right)\left(\frac{1}{q^n\alpha}\right)\right]\Bigg|_{-\infty}^{qt}\right] \qquad (68)\\
= \;& (-1)^n \left(\frac{A}{\alpha}\right)^{2n} \left(\frac{1}{q^{n(n+1)}}\right) \left[\left(\frac{A}{q^{n+1}\alpha}\right) {}_qSin\left(\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right)\right] \\
= \;& (-1)^n \left(\frac{A}{\alpha}\right)^{2n+1} \left(\frac{1}{q^{(n+1)^2}}\right) {}_qSin\left(\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right). \qquad (69)
\end{aligned}$$

Here (67) follows from the inductive hypothesis (65), and (68) follows from (18). Then from (69) we have that:

$$\begin{aligned}
&\tilde{K}^{2(n+1)}[{}_qCos(\alpha u+\beta](t) = \tilde{K}\left[\tilde{K}^{2n+1}[{}_qCos(\alpha x+\beta)](u)\right](t) \\
= \;& (-1)^n \left(\frac{A}{\alpha}\right)^{2n+1} \left(\frac{1}{q^{(n+1)^2}}\right) \tilde{K}\left[{}_qSin\left(\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right)\right] \\
= \;& (-1)^n \left(\frac{A}{\alpha}\right)^{2n+1} \left(\frac{1}{q^{(n+1)^2}}\right) \left[\frac{A}{q}\int_{-\infty}^{qt} {}_qSin\left(\frac{q^{2n+1}\alpha u+\beta}{q^{n+1}}\right)du\right] \\
= \;& (-1)^n \left(\frac{A}{\alpha}\right)^{2n+1} \left(\frac{1}{q^{(n+1)^2}}\right) \left[\frac{A}{q}\left[-{}_qCos\left(\frac{q^{2n+1}\alpha u+\beta}{q^{n+1}}\right)\right]\left(\frac{1}{q^n\alpha}\right)\Bigg|_{-\infty}^{qt}\right] \\
= \;& (-1)^{n+1} \left(\frac{A}{\alpha}\right)^{2(n+1)} \left(\frac{1}{q^{(n+1)(n+2)}}\right) \left({}_qCos\left(\frac{q^{2(n+1)}\alpha t+\beta}{q^{n+1}}\right)\right). \qquad (70)
\end{aligned}$$

From (70), we have shown (65) holds by induction. Now that (65) is established for all $n \in \mathbb{N}$, from (69) we obtain that (66) holds for all $n \in \mathbb{N}$. This proves Proposition 2. □

Now in order to calculate $\tilde{K}^n[{}_qSin(\alpha u+\beta)](t)$ we start by finding the expression $\tilde{K}[{}_qSin(\alpha u+\beta)](t)$.

$$
\begin{aligned}
\tilde{K}[{}_qSin(\alpha u+\beta)](t) &= \frac{A}{q}\int_{-\infty}^{qt} {}_qSin(\alpha u+\beta)du \\
&= \frac{A}{q}\left[-{}_qCos(\alpha u+\beta)\left(\frac{1}{\alpha}\right)\right]\Bigg|_{-\infty}^{qt} \qquad (71)\\
&= (-1)\left(\frac{A}{q\alpha}\right){}_qCos(\alpha qt+\beta), \qquad (72)
\end{aligned}
$$

where (17) was used to obtain (71).

Next we compute $\tilde{K}^2[{}_qSin(\alpha u+\beta)](t)$ using (72).

$$
\begin{aligned}
\tilde{K}^2[{}_qSin(\alpha u+\beta)](t) &= \tilde{K}\left[(-1)\left(\frac{A}{q\alpha}\right){}_qCos(\alpha qt+\beta)\right] \\
&= (-1)\left(\frac{A^2}{q^2\alpha}\right)\int_{-\infty}^{qt} {}_qCos(\alpha qu+\beta)du \\
&= (-1)\left(\frac{A^2}{q^2\alpha}\right)\left[{}_qSin\left(\frac{\alpha qu+\beta}{q}\right)\left(\frac{1}{\alpha q}\right)\right]\Bigg|_{-\infty}^{qt} \qquad (73)\\
&= (-1)\left(\frac{A^2}{\alpha^2q^3}\right){}_qSin\left(\frac{q^2\alpha t+\beta}{q}\right), \qquad (74)
\end{aligned}
$$

where (18) was used to obtain (73).

We need to do this inductively for $\tilde{K}^n[{}_qSin(\alpha u+\beta)](t)$. For $n=3$ and relying on (74), we have:

$$
\begin{aligned}
\tilde{K}^3[{}_qSin(\alpha u+\beta)](t) &= \tilde{K}\left[(-1)\left(\frac{A^2}{\alpha^2q^3}\right){}_qSin\left(\frac{q^2\alpha t+\beta}{q}\right)\right] \\
&= \frac{A}{q}\int_{-\infty}^{qt}(-1)\left(\frac{A^2}{q^3\alpha^2}\right)\left({}_qSin\left(\frac{q^2\alpha u+\beta}{q}\right)\right)du \\
&= (-1)\left(\frac{A^3}{q^4\alpha^2}\right)\left[-{}_qCos\left(\frac{q^2\alpha u+\beta}{q}\right)\left(\frac{1}{\alpha q}\right)\right]\Bigg|_{-\infty}^{qt} \qquad (75)\\
&= (-1)^2\left(\frac{A^3}{q^5\alpha^3}\right){}_qCos\left(\frac{q^3\alpha t+\beta}{q}\right), \qquad (76)
\end{aligned}
$$

where (17) was used to obtain (75). Then for $n = 4$ and relying on (76), we have:

$$\begin{aligned}
\tilde{K}^4[{}_qSin(\alpha u+\beta)](t) &= \tilde{K}\left[(-1)^2\left(\frac{A^3}{q^5\alpha^3}\right){}_qCos\left(\frac{q^3\alpha t+\beta}{q}\right)\right] \\
&= \frac{A}{q}\int_{-\infty}^{qt}(-1)^2\left(\frac{A^3}{q^5\alpha^3}\right)\left({}_qCos\left(\frac{q^3\alpha u+\beta}{q}\right)\right)du \\
&= (-1)^2\left(\frac{A^4}{q^6\alpha^3}\right)\left[{}_qSin\left(\frac{q^3\alpha u+\beta}{q^2}\right)\left(\frac{1}{q^2\alpha}\right)\right]\Bigg|_{-\infty}^{qt} \qquad (77) \\
&= (-1)^2\left(\frac{A^4}{q^8\alpha^4}\right){}_qSin\left(\frac{q^4\alpha t+\beta}{q^2}\right), \qquad (78)
\end{aligned}$$

where (18) was used to obtain (77).

Now in order to compute $\tilde{K}^n[{}_qSin(\alpha u+\beta)](t)$ in general, we have the following proposition.

Proposition 3. *Applying powers of $\tilde{K}$ to ${}_qSin[\alpha u+\beta](t)$ gives, for even and odd powers, respectively,*

$$\begin{aligned}
\tilde{K}^{2n}[{}_qSin(\alpha u+\beta](t) &= (-1)^n\left(\frac{A}{\alpha}\right)^{2n}\left(\frac{1}{q^{n(n+2)}}\right){}_qSin\left(\frac{q^{2n}\alpha t+\beta}{q^n}\right) \qquad (79) \\
\tilde{K}^{2n+1}[{}_qSin(\alpha u+\beta)](t) &= (-1)^{n+1}\left(\frac{A}{\alpha}\right)^{2n+1}\left(\frac{1}{q^{n^2+3n+1}}\right){}_qCos\left(\frac{q^{2n+1}\alpha t+\beta}{q^n}\right). \qquad (80)
\end{aligned}$$

Proof. We will prove Proposition 3 with induction on n. Assume (79) as the inductive hypothesis. In order to calculate $\tilde{K}^{2(n+1)}[{}_qSin(\alpha u+\beta)](t)$ we will first calculate $\tilde{K}^{2n+1}[{}_qSin(\alpha u+\beta)](t)$.

$$
\begin{aligned}
&\tilde{K}^{2n+1}[{}_qSin(\alpha u+\beta)](t) = \tilde{K}[\tilde{K}_q^{2n} Sin(\alpha u+\beta)](t) \\
&= (-1)^n \left(\frac{A}{\alpha}\right)^{2n} \left(\frac{1}{q^{n(n+2)}}\right) \tilde{K}\left[{}_qSin\left(\frac{q^{2n}\alpha u+\beta}{q^n}\right)\right](t) \quad (81)\\
&= (-1)^n \left(\frac{A}{\alpha}\right)^{2n} \left(\frac{1}{q^{n(n+2)}}\right) \left[\frac{A}{q}\int_{-\infty}^{qt} {}_qSin\left(\frac{q^{2n}\alpha u+\beta}{q^n}\right) du\right]\\
&= (-1)^n \left(\frac{A}{\alpha}\right)^{2n} \left(\frac{1}{q^{n(n+2)}}\right) \left[\frac{A}{q}\left[-{}_qCos\left(\frac{q^{2n}\alpha u+\beta}{q^n}\right)\right]\left(\frac{1}{q^n\alpha}\right)\Big|_{-\infty}^{qt}\right] \quad (82)\\
&= (-1)^{n+1} \left(\frac{A}{\alpha}\right)^{2n+1} \left(\frac{1}{q^{n^2+3n+1}}\right) {}_qCos\left(\frac{q^{2n+1}\alpha t+\beta}{q^n}\right). \quad (83)
\end{aligned}
$$

Here (81) follows from the inductive hypothesis (79) and (82) follows from (17). Next we calculate $\tilde{K}^{2(n+1)}[{}_qSin(\alpha u+\beta)](t)$. From (83), one has:

$$
\begin{aligned}
&\tilde{K}^{2(n+1)}[{}_qSin(\alpha u+\beta)](t) = \tilde{K}\left[\tilde{K}^{2n+1}[{}_qSin(\alpha x+\beta)](u)\right](t)\\
&= (-1)^{n+1} \left(\frac{A}{\alpha}\right)^{2n+1} \left(\frac{1}{q^{n^2+3n+1}}\right) \tilde{K}\left[{}_qCos\left(\frac{q^{2n+1}\alpha u+\beta}{q^n}\right)\right](t)\\
&= (-1)^{n+1} \left(\frac{A}{\alpha}\right)^{2n+1} \left(\frac{1}{q^{n^2+3n+1}}\right) \left[\frac{A}{q}\int_{-\infty}^{qt} {}_qCos\left(\frac{q^{2n+1}\alpha u+\beta}{q^n}\right) du\right]\\
&= (-1)^{n+1} \left(\frac{A}{\alpha}\right)^{2n+1} \left(\frac{1}{q^{n^2+3n+1}}\right) \left[\frac{A}{q}\,{}_qSin\left(\frac{q^{2n+1}\alpha u+\beta}{q^{n+1}}\right)\left(\frac{1}{q^{n+1}\alpha}\right)\Big|_{-\infty}^{qt}\right]\\
&= (-1)^{n+1} \left(\frac{A^{2n+2}}{\alpha^{2n+2}}\right) \left(\frac{1}{q^{n(n+2)}q^{2n+3}}\right) {}_qSin\left(\frac{q^{2n+2}\alpha t+\beta}{q^{n+1}}\right)\\
&= (-1)^{n+1} \left(\frac{A}{\alpha}\right)^{2(n+1)} \left(\frac{1}{q^{(n+1)(n+3)}}\right) {}_qSin\left(\frac{q^{2(n+1)}\alpha t+\beta}{q^{n+1}}\right). \quad (84)
\end{aligned}
$$

From (84) we have proven (79) by induction. Now that (79) holds for all $n \in \mathbb{N}$, from (83) we obtain that (80) holds for all $n \in \mathbb{N}$. This proves Proposition 3. □

Now we solve (54) in the case where $f(t) = {}_qCos(\alpha t+\beta)$.

As shown in (57) this means that $F(t) = \left(\frac{1}{\alpha}\right) {}_qSin\left(\frac{\alpha t+\beta}{q}\right)$.

Then recall from (60) that:

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} \tilde{K}^n[F(u)](t) \\ &= \sum_{n=0}^{\infty} \tilde{K}^n \left[\left(\frac{1}{\alpha}\right) {}_qSin\left(\frac{\alpha u+\beta}{q}\right)\right](t) \\ &= \sum_{n=0}^{\infty} \tilde{K}^{2n} \left[\left(\frac{1}{\alpha}\right) {}_qSin\left(\frac{\alpha u+\beta}{q}\right)\right](t) \qquad (85) \\ &\quad + \sum_{n=0}^{\infty} \tilde{K}^{2n+1} \left[\left(\frac{1}{\alpha}\right) {}_qSin\left(\frac{\alpha u+\beta}{q}\right)\right](t). \qquad (86) \end{aligned}$$

Using Proposition 3 and applying (79) to (85) and (80) to (86) gives the formal solution to (54) as:

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty}(-1)^n \left(\frac{A^{2n}}{\alpha^{2n+1}}\right)\left(\frac{1}{q^{n^2}}\right){}_qSin\left(\frac{q^{2n}\alpha t+\beta}{q^{n+1}}\right) \qquad (87) \\ &\quad + \sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{A^{2n+1}}{\alpha^{2n+2}}\right)\left(\frac{1}{q^{n(n+1)}}\right){}_qCos\left(\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right). \qquad (88) \end{aligned}$$

Now we solve (55) in the case where $f(t) = {}_qSin(\alpha t+\beta)$.

As shown in (57) this means that $F(t) = \left(\frac{-1}{\alpha}\right) {}_qCos(\alpha t+\beta)$.

Then recall from (60) that a formal solution to (55) is given by:

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} \tilde{K}^n [F(u)](t) \\ &= \sum_{n=0}^{\infty} \tilde{K}^n \left[\left(\frac{-1}{\alpha} \right) {}_qCos\,(\alpha t + \beta) \right] (t) \\ &= \sum_{n=0}^{\infty} \tilde{K}^{2n} \left[\left(\frac{-1}{\alpha} \right) {}_qCos\,(\alpha t + \beta) \right] (t) \qquad (89) \\ &\quad + \sum_{n=0}^{\infty} \tilde{K}^{2n+1} \left[\left(\frac{-1}{\alpha} \right) {}_qCos\,(\alpha t + \beta) \right] (t). \qquad (90) \end{aligned}$$

Using Proposition 2 and applying (65) to (89) and (66) to (90) gives the formal solution to (55) is of the form:

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{A^{2n}}{\alpha^{2n+1}} \right) \left(\frac{1}{q^{n(n+1)}} \right) {}_qCos \left(\frac{q^{2n}\alpha t + \beta}{q^n} \right) \qquad (91) \\ &\quad + \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{A^{2n+1}}{\alpha^{2n+2}} \right) \left(\frac{1}{q^{(n+1)^2}} \right) {}_qSin \left(\frac{q^{2n+1}\alpha t + \beta}{q^{n+1}} \right). \qquad (92) \end{aligned}$$

Now to check our solutions:

First we check the solution (87)-(88); this is when $f(t) = {}_qCos(\alpha t + \beta)$.

So we will formally check that it solves (54) or that in particular,

$$y'(t) = Ay(qt) + {}_qCos(\alpha t + \beta).$$

Recall from (87) and (88) that

$$\begin{aligned} & y(t) \\ &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{A^{2n}}{\alpha^{2n+1}} \right) \left(\frac{1}{q^{n^2}} \right) {}_qSin \left(\frac{q^{2n}\alpha t + \beta}{q^{n+1}} \right) \qquad (93) \\ &\quad + \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{A^{2n+1}}{\alpha^{2n+2}} \right) \left(\frac{1}{q^{n(n+1)}} \right) {}_qCos \left(\frac{q^{2n+1}\alpha t + \beta}{q^{n+1}} \right). \qquad (94) \end{aligned}$$

Then we calculate, under the assumption that differentiation can be exchanged with the infinite sums in (93)-(94), that:

$$y'(t) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{A^{2n}}{\alpha^{2n+1}}\right)\left(\frac{1}{q^{n^2}}\right)\left[{}_qSin\left(\frac{q^{2n}\alpha t+\beta}{q^{n+1}}\right)\right]' \quad (95)$$

$$+\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{A^{2n+1}}{\alpha^{2n+2}}\right)\left(\frac{1}{q^{n(n+1)}}\right)\left[{}_qCos\left(\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right)\right]' \quad (96)$$

$$= \sum_{n=0}^{\infty}(-1)^n\left(\frac{A^{2n}}{\alpha^{2n+1}}\right)\left(\frac{1}{q^{n^2}}\right)q\,{}_qCos\left(\frac{q^{2n+1}\alpha t+q\beta}{q^{n+1}}\right)(q^{n-1}\alpha)$$

$$+\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{A^{2n+1}}{\alpha^{2n+2}}\right)\left(\frac{1}{q^{n(n+1)}}\right)\left[-{}_qSin\left(\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right)\right](q^n\alpha)$$

$$= {}_qCos(\alpha t+\beta)$$

$$+\sum_{n=1}^{\infty}(-1)^n\left(\frac{A^{2n}}{\alpha^{2n}}\right)\left(\frac{1}{q^{n(n-1)}}\right){}_qCos\left(\frac{q^{2n}\alpha t+\beta}{q^n}\right) \quad (97)$$

$$+\sum_{n=0}^{\infty}(-1)^n\left(\frac{A^{2n+1}}{\alpha^{2n+1}}\right)\left(\frac{1}{q^{n^2}}\right){}_qSin\left(\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right).$$

Now making a change of variables by letting $n = m+1$ in (97) gives:

$$y'(t) = {}_qCos(\alpha t+\beta)$$

$$+\sum_{m=0}^{\infty}(-1)^{m+1}\left(\frac{A^{2(m+1)}}{\alpha^{2(m+1)}}\right)\left(\frac{1}{q^{m(m+1)}}\right){}_qCos\left(\frac{q^{2(m+1)}\alpha t+\beta}{q^{m+1}}\right)$$

$$+\sum_{n=0}^{\infty}(-1)^n\left(\frac{A^{2n+1}}{\alpha^{2n+1}}\right)\left(\frac{1}{q^{n^2}}\right){}_qSin\left(\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right)$$

$$= {}_qCos(\alpha t+\beta)$$

$$+A\sum_{m=0}^{\infty}(-1)^{m+1}\left(\frac{A^{2m+1}}{\alpha^{2m+2}}\right)\left(\frac{1}{q^{m(m+1)}}\right){}_qCos\left(\frac{q^{2m+1}\alpha(qt)+\beta}{q^{m+1}}\right)$$

$$+A\sum_{n=0}^{\infty}(-1)^n\left(\frac{A^{2n}}{\alpha^{2n+1}}\right)\left(\frac{1}{q^{n^2}}\right){}_qSin\left(\frac{q^{2n}\alpha(qt)+\beta}{q^{n+1}}\right)$$

$$= {}_qCos(\alpha t+\beta)+Ay(qt).$$

This completes the check that (87)-(88) is a formal solution to (54).

Now we will formally check that the solution (91)-(92), does indeed solve (55). Namely, we will check that

$$y'(t) = Ay(qt) + {}_qSin(\alpha t + \beta).$$

Recall that from (91) and (92)

$$\begin{aligned} & y(t) \\ = & \sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{A^{2n}}{\alpha^{2n+1}}\right)\left(\frac{1}{q^{n(n+1}}\right){}_qCos\left(\frac{q^{2n}\alpha t+\beta}{q^n}\right) \qquad (98) \\ & +\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{A^{2n+1}}{\alpha^{2n+2}}\right)\left(\frac{1}{q^{(n+1)^2}}\right){}_qSin\left(\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right). \qquad (99) \end{aligned}$$

Then we calculate again under the assumption that we may pass the derivatives through the infinite sums in (98)-(99).

$$\begin{aligned} & y'(t) \\ = & \sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{A^{2n}}{\alpha^{2n+1}}\right)\left(\frac{1}{q^{n(n+1)}}\right)\left[{}_qCos\left(\frac{q^{2n}\alpha t+\beta}{q^n}\right)\right]' \qquad (100) \\ & +\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{A^{2n+1}}{\alpha^{2n+2}}\right)\left(\frac{1}{q^{(n+1)^2}}\right)\left[{}_qSin\left(\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right)\right]' \qquad (101) \\ = & \sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{A^{2n}}{\alpha^{2n+1}}\right)\left(\frac{1}{q^{n(n+1)}}\right)\left[-{}_qSin\left(\frac{q^{2n}\alpha t+\beta}{q^n}\right)\right](q^n\alpha) \\ & +\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{A^{2n+1}}{\alpha^{2n+2}}\right)\left(\frac{1}{q^{(n+1)^2}}\right)q\left[{}_qCos\left(\frac{q^{2n+2}\alpha t+q\beta}{q^{n+1}}\right)\right](q^n\alpha) \\ = & {}_qSin(\alpha t+\beta) \\ & +\sum_{n=1}^{\infty}(-1)^{n}\left(\frac{A^{2n}}{\alpha^{2n}}\right)\left(\frac{1}{q^{n^2}}\right)\left[{}_qSin\left(\frac{q^{2n}\alpha t+\beta}{q^n}\right)\right] \qquad (102) \\ & +\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{A^{2n+1}}{\alpha^{2n+1}}\right)\left(\frac{1}{q^{n(n+1)}}\right)\left[{}_qCos\left(\frac{q^{2n+1}\alpha t+\beta}{q^n}\right)\right]. \end{aligned}$$

Now making a change of variables in (102) by letting $n = m+1$ gives:

$$
\begin{aligned}
y'(t) &= {}_qSin(\alpha t+\beta) \\
&\quad + \sum_{m=0}^{\infty}(-1)^{m+1}\left(\frac{A^{2(m+1)}}{\alpha^{2(m+1)}}\right)\left(\frac{1}{q^{(m+1)^2}}\right)\left[{}_qSin\left(\frac{q^{2(m+1)}\alpha t+\beta}{q^{m+1}}\right)\right] \\
&\quad + \sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{A^{2n+1}}{\alpha^{2n+1}}\right)\left(\frac{1}{q^{n(n+1)}}\right)\left[{}_qCos\left(\frac{q^{2n+1}\alpha t+\beta}{q^{n}}\right)\right] \\
&= {}_qSin(\alpha t+\beta) \\
&\quad + A\sum_{m=0}^{\infty}(-1)^{m+1}\left(\frac{A^{2m+1}}{\alpha^{2m+2}}\right)\left(\frac{1}{q^{(m+1)^2}}\right)\left[{}_qSin\left(\frac{q^{2m+1}\alpha(qt)+\beta}{q^{m+1}}\right)\right] \\
&\quad + A\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{A^{2n}}{\alpha^{2n+1}}\right)\left(\frac{1}{q^{n(n+1)}}\right)\left[{}_qCos\left(\frac{q^{2n}\alpha(qt)+\beta}{q^{n}}\right)\right] \\
&= {}_qSin(\alpha t+\beta) + Ay(qt).
\end{aligned}
$$

This completes the formal check for both our results.

We have arrived at the following theorem.

Theorem 3. *A solution to the MADE*

$$y'(t) - Ay(qt) = f(t) \quad , \ \textit{where} \ \ f(t) = {}_qCos(\alpha t+\beta)$$

is of the form:

$$y(t) = \sum_{n=0}^{\infty}(-1)^{n}\left(\frac{A^{2n}}{\alpha^{2n+1}}\right)\left(\frac{1}{q^{n^2}}\right){}_qSin\left(\frac{q^{2n}\alpha t+\beta}{q^{n+1}}\right) \tag{103}$$

$$+\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{A^{2n+1}}{\alpha^{2n+2}}\right)\left(\frac{1}{q^{n(n+1)}}\right){}_qCos\left(\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right). \tag{104}$$

Similarly a solution to the MADE

$$y'(t) - Ay(qt) = f(t) \quad , \ \textit{where} \ \ f(t) = {}_qSin(\alpha t+\beta)$$

is of the form:

$$y(t) = \sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{A^{2n}}{\alpha^{2n+1}}\right)\left(\frac{1}{q^{n(n+1)}}\right){}_qCos\left(\frac{q^{2n}\alpha t+\beta}{q^{n}}\right) \tag{105}$$

$$+\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{A^{2n+1}}{\alpha^{2n+2}}\right)\left(\frac{1}{q^{(n+1)^2}}\right){}_qSin\left(\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right). \tag{106}$$

Proof. We will prove that (103)-(104) and (105)-(106) both converge absolutely and uniformly in order to show that our checks are correct in assuming the exchanging of derivatives with infinite sums. We will bound them using a completion of squares to obtain a Gaussian bound via (47) in Proposition 1. First from the solution (103)-(104) we can see that $|y(t)|$ is:

$$|y(t)| = \left|\sum_{n=0}^{\infty}(-1)^{n}\left(\frac{A^{2n}}{\alpha^{2n+1}}\right)\left(\frac{1}{q^{n^2}}\right){}_qSin\left(\frac{q^{2n}\alpha t+\beta}{q^{n+1}}\right)\right. \tag{107}$$

$$\left.+\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{A^{2n+1}}{\alpha^{2n+2}}\right)\left(\frac{1}{q^{n(n+1)}}\right){}_qCos\left(\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right)\right|. \tag{108}$$

Now using the triangle inequality and the bounds (43) and (42) on (107) and (108) respectively gives:

$$
\begin{aligned}
|y(t)| \;&\leq\; \sum_{n=0}^{\infty}\left(\frac{A^{2n}}{|\alpha|^{2n+1}}\right)\left(\frac{1}{q^{n^2}}\right)\,(q^{1/2}) \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{|A|^{2n+1}}{\alpha^{2n+2}}\right)\left(\frac{1}{q^{n(n+1)}}\right) \\
&= \frac{q^{1/2}}{|\alpha|}\sum_{n=0}^{\infty}\left(\frac{|A|^{2n}}{\alpha^{2n}}\right)\left(\frac{1}{q^{n^2}}\right) \\
&\quad + \frac{|A|}{\alpha^2}\sum_{n=0}^{\infty}\left(\frac{|A|^{2n}}{\alpha^{2n}}\right)\left(\frac{1}{q^{n(n+1)}}\right) \\
&= \frac{q^{1/2}}{|\alpha|}\sum_{n=0}^{\infty}\left(\frac{|A|^{2}}{\alpha^{2}}\right)^n\left(\frac{1}{q^{n^2}}\right) \\
&\quad + \frac{|A|}{\alpha^2}\sum_{n=0}^{\infty}\left(\frac{|A|^{2}}{\alpha^{2}}\right)^n\left(\frac{1}{q^{n^2+n}}\right) \\
&= \frac{q^{1/2}}{|\alpha|}\sum_{n=0}^{\infty}\exp\left[n\ln(|A|^2\alpha^{-2})\right]\exp\left[-n^2\ln(q)\right] \\
&\quad + \frac{|A|}{|\alpha|^2}\sum_{n=0}^{\infty}\exp\left[n\ln(|A|^2\alpha^{-2})\right]\exp[-n\ln(q)]\exp[-n^2\ln(q)] \\
&= \frac{q^{1/2}}{|\alpha|}\sum_{n=0}^{\infty}\exp\left[-\ln(q)\left(n^2-n\left(\frac{\ln(|A|^2\alpha^{-2})}{\ln(q)}\right)\right)\right] \qquad (109) \\
&\quad + \frac{|A|}{\alpha^2}\sum_{n=0}^{\infty}\exp\left[-\ln(q)\left(n^2-n\left(\frac{\ln(|A|^2\alpha^{-2})-\ln(q)}{\ln(q)}\right)\right)\right]. \qquad (110)
\end{aligned}
$$

Now we will complete the square on n in (109) and (110). This gives that:

$$
\begin{aligned}
&|y(t)| \\
&\leq \frac{q^{1/2}}{|\alpha|}\sum_{n=0}^{\infty}\exp\left[-\ln(q)\left(n-\frac{\ln(|A|^2\alpha^{-2})}{2\ln(q)}\right)^2\right] \\
&\qquad\qquad \cdot\exp\left[\frac{\ln^2(|A|^2\alpha^{-2})}{4\ln(q)}\right] \\
&\quad +\frac{|A|}{\alpha^2}\sum_{n=0}^{\infty}\exp\left[-\ln(q)\left(n-\frac{\ln(|A|^2\alpha^{-2})-\ln(q)}{2\ln(q)}\right)^2\right] \\
&\qquad\qquad \cdot\exp\left[\frac{\left[\ln(|A|^2\alpha^{-2})-\ln(q)\right]^2}{4\ln(q)}\right]
\end{aligned}
$$

$$
= \frac{q^{1/2}}{|\alpha|}\cdot e^{k_1}\sum_{n=0}^{\infty}\exp\left[\frac{-\frac{1}{2}\left[n-\left(\frac{\ln(|A|^2\alpha^{-2})}{2\ln(q)}\right)\right]^2}{\left(\frac{1}{\sqrt{2\ln(q)}}\right)^2}\right] \tag{111}
$$

$$
+\frac{|A|}{\alpha^2}\cdot e^{k_2}\sum_{n=0}^{\infty}\exp\left[\frac{-\frac{1}{2}\left[n-\left(\frac{\ln(|A|^2\alpha^{-2})-\ln(q)}{2\ln(q)}\right)\right]^2}{\left(\frac{1}{\sqrt{2\ln(q)}}\right)^2}\right] \tag{112}
$$

$$
\leq \frac{q^{1/2}}{|\alpha|}\cdot e^{k_1}\left[1+\sqrt{2\pi}\left(\frac{1}{\sqrt{2\ln(q)}}\right)\right] \tag{113}
$$

$$
+\frac{|A|}{\alpha^2}\cdot e^{k_2}\left[1+\sqrt{2\pi}\left(\frac{1}{\sqrt{2\ln(q)}}\right)\right] \tag{114}
$$

$$
\begin{aligned}
&= \frac{q^{1/2}}{|\alpha|}\cdot e^{k_1}\left[1+\sqrt{\frac{\pi}{\ln(q)}}\right] \\
&\quad +\frac{|A|}{\alpha^2}\cdot e^{k_2}\left[1+\sqrt{\frac{\pi}{\ln(q)}}\right] < \infty
\end{aligned}
$$

where in (113) and (114) we used Proposition 1 on the Gaussian's in (111) and (112) respectively, and the k_1, introduced in (111) is equal to $\dfrac{\ln^2(|A|^2\alpha^{-2})}{4\ln(q)}$, and the k_2, introduced in (112), is equal to $\dfrac{\left[\ln(|A|^2\alpha^{-2})-\ln(q)\right]^2}{4\ln(q)}$. Now that we have shown $y(t)$ converges absolutely and uniformly for all $t \in \mathbb{R}$, we look to similarly bound $y'(t)$ so that we may apply Theorem 1. Recall that $y'(t) = Ay(qt) + f(t)$ and since we have just shown that

$y(t)$ is bounded, clearly $Ay(qt)$ is bounded as well, with the summation converging uniformly and absolutely on all of $\mathbb{R}$. For this solution recall that $f(t) = {}_qCos(\alpha t + \beta)$, which is a single finite term and thus it is clear from (42) that $y'(t)$ is also bounded for all $t \in \mathbb{R}$. Then applying Theorem 1 tells us that we may exchange the derivatives with the infinite sums in our check (95)-(96) of $y(t)$. This makes our solution, (103)-(104) rigorous.

We will apply the same techniques to the solution (105)-(106) to ensure that it converges absolutely and uniformly. From (105)-(106) we can see that the absolute value of our solution is:

$$|y(t)| = \left| \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{A^{2n}}{\alpha^{2n+1}} \right) \left(\frac{1}{q^{n(n+1)}} \right) {}_qCos \left(\frac{q^{2n}\alpha t + \beta}{q^n} \right) \right. \tag{115}$$

$$\left. + \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{A^{2n+1}}{\alpha^{2n+2}} \right) \left(\frac{1}{q^{(n+1)^2}} \right) {}_qSin \left(\frac{q^{2n+1}\alpha t + \beta}{q^{n+1}} \right) \right|. \tag{116}$$

Now using the triangle inequality and the bounds (42) and (43) on (115) and (116) respectively gives:

$$
\begin{aligned}
|y(t)| \leq & \sum_{n=0}^{\infty} \left(\frac{|A|^{2n}}{|\alpha|^{2n+1}} \right) \left(\frac{1}{q^{n(n+1)}} \right) \\
& + \sum_{n=0}^{\infty} \left(\frac{|A|^{2n+1}}{\alpha^{2n+2}} \right) \left(\frac{1}{q^{(n+1)^2}} \right) (q^{1/2}) \\
= & \frac{1}{|\alpha|} \sum_{n=0}^{\infty} \left(\frac{|A|^{2n}}{\alpha^{2n}} \right) \left(\frac{1}{q^{n(n+1)}} \right) \\
& + \frac{|A|}{\alpha^2} \sum_{n=0}^{\infty} \left(\frac{|A|^{2n}}{\alpha^{2n}} \right) \left(\frac{1}{q^{n^2+2n+1}} \right) (q^{1/2}) \\
= & \frac{1}{|\alpha|} \sum_{n=0}^{\infty} \left(\frac{|A|^2}{\alpha^2} \right)^n \left(\frac{1}{q^{n^2+n}} \right) \\
& + \frac{|A|}{q^{1/2}\alpha^2} \sum_{n=0}^{\infty} \left(\frac{|A|^2}{\alpha^2} \right)^n \left(\frac{1}{q^{n^2+2n}} \right) \\
= & \frac{1}{|\alpha|} \sum_{n=0}^{\infty} \exp\left[n \ln(|A|^2 \alpha^{-2}) \right] \exp[-n \ln(q)] \exp[-n^2 \ln(q)] \\
& + \frac{|A|}{q^{1/2}\alpha^2} \sum_{n=0}^{\infty} \exp\left[n \ln(|A|^2 \alpha^{-2}) \right] \exp[-2n \ln(q)] \exp[-n^2 \ln(q)] \\
= & \frac{1}{|\alpha|} \sum_{n=0}^{\infty} \exp\left[-\ln(q) \left(n^2 - n \left(\frac{\ln(|A|^2 \alpha^{-2}) - \ln(q)}{\ln(q)} \right) \right) \right] \qquad (117) \\
& + \frac{|A|}{q^{1/2}\alpha^2} \sum_{n=0}^{\infty} \exp\left[-\ln(q) \left(n^2 - n \left(\frac{\ln(|A|^2 \alpha^{-2}) - 2\ln(q)}{\ln(q)} \right) \right) \right]. \qquad (118)
\end{aligned}
$$

Now we will complete the square on n in (117) and (118). This gives that:

$$
\begin{aligned}
&|y(t)| \\
&\leq \frac{1}{|\alpha|}\sum_{n=0}^{\infty}\exp\left[-\ln(q)\left(n-\frac{\ln(|A|^2\alpha^{-2})-\ln(q)}{2\ln(q)}\right)^2\right] \\
&\qquad\cdot\exp\left[\frac{[\ln(|A|^2\alpha^{-2})-\ln(q)]^2}{4\ln(q)}\right] \\
&\quad+\frac{|A|}{q^{1/2}\alpha^2}\sum_{n=0}^{\infty}\exp\left[-\ln(q)\left(n-\frac{\ln(|A|^2\alpha^{-2})-2\ln(q)}{2\ln(q)}\right)^2\right] \\
&\qquad\cdot\exp\left[\frac{[\ln(|A|^2\alpha^{-2})-2\ln(q)]^2}{4\ln(q)}\right] \\
&= \frac{1}{|\alpha|}\cdot e^{k_3}\sum_{n=0}^{\infty}\exp\left[\frac{-\frac{1}{2}\left[n-\left(\frac{\ln(|A|^2\alpha^{-2})-\ln(q)}{2\ln(q)}\right)\right]^2}{\left(\frac{1}{\sqrt{2\ln(q)}}\right)^2}\right] \quad (119)\\
&\quad+\frac{|A|}{q^{1/2}\alpha^2}\cdot e^{k_4}\sum_{n=0}^{\infty}\exp\left[\frac{-\frac{1}{2}\left[n-\left(\frac{\ln(|A|^2\alpha^{-2})-2\ln(q)}{2\ln(q)}\right)\right]^2}{\left(\frac{1}{\sqrt{2\ln(q)}}\right)^2}\right] \quad (120)\\
&\leq \frac{1}{|\alpha|}\cdot e^{k_3}\left[1+\sqrt{2\pi}\left(\frac{1}{\sqrt{2\ln(q)}}\right)\right] \quad (121)\\
&\quad+\frac{|A|}{q^{1/2}\alpha^2}\cdot e^{k_4}\left[1+\sqrt{2\pi}\left(\frac{1}{\sqrt{2\ln(q)}}\right)\right] \quad (122)\\
&= \frac{1}{|\alpha|}\cdot e^{k_3}\left[1+\sqrt{\frac{\pi}{\ln(q)}}\right] \\
&\quad+\frac{|A|}{q^{1/2}\alpha^2}\cdot e^{k_4}\left[1+\sqrt{\frac{\pi}{\ln(q)}}\right]<\infty
\end{aligned}
$$

where in (121) and (122) we used Proposition 1 on the Gaussian's in (119) and (120) respectively, and the k_3, introduced in (119) is equal to $\frac{[\ln(|A|^2\alpha^{-2})-\ln(q)]^2}{4\ln(q)}$, and the k_4, introduced in (120), is equal to $\frac{[\ln(|A|^2\alpha^{-2})-2\ln(q)]^2}{4\ln(q)}$. Now that we have shown $y(t)$ converges absolutely and uniformly for all $t\in\mathbb{R}$, we look to similarly bound $y'(t)$ so that we may apply Theorem 1. Recall that $y'(t)=Ay(qt)+f(t)$ and since we have just shown that $y(t)$ is bounded and converges uniformly and absolutely on $\mathbb{R}$, clearly so does $Ay(qt)$. For this solution recall that $f(t) = {}_qSin(\alpha t + \beta)$, which is a single finite term and is bounded by (43). Thus it is clear that $y'(t)$ is also bounded and converges uniformly and absolutely on $\mathbb{R}$. Then applying Theorem 1 tells us that we may exchange the derivatives with the infinite sums in our check (100)-(101) of $y(t)$. This makes our solution, (105)-(106) rigorous as well. This completes the proof of Theorem 1 as we have shown both our solutions to converge uniformly and absolutely on the entire real line. □

Now we introduce a proposition for the Fourier transform of our solution $y(t)$ (103)-(104).

Proposition 4. *The Fourier transform of* $y(t)$ *in* (103)-(104) *is given by*

$$\hat{y}(x) = \left[\frac{1}{\theta(q^2; x^2/\alpha^2)}\right]\left[\frac{-\tilde{C}_q i}{x\alpha}\sum_{n=0}^{\infty}(-1)^n\left(\frac{A^{2n}}{\alpha^{4n}}\right)\left(\frac{x^{2n}}{q^{2n^2-n}}\right)\exp\left[\frac{i\beta}{q^{2n}\alpha}\right] \right. \tag{123}$$

$$\left. + \frac{-\tilde{C}_q A}{\alpha^3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{A^{2n}}{\alpha^{4n}}\right)\left(\frac{x^{2n}}{q^{2n^2+n}}\right)\exp\left[\frac{i\beta}{q^{2n+1}\alpha}\right]\right]. \tag{124}$$

Proof. We start by taking the Fourier transform of (103)-(104). Note that we may exchange the integration with the infinite sums in (103)-(104) by the Lebesgue Dominated Convergence of $y^{(m)}(t)$ by $g_m(t)$, for $m = 0$, in (214) below.

$$\hat{y}(x)$$

$$= \sum_{n=0}^{\infty}(-1)^n\left(\frac{A^{2n}}{\alpha^{2n+1}}\right)\left(\frac{1}{q^{n^2}}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-itx}\,{}_qSin\left(\frac{q^{2n}\alpha t+\beta}{q^{n+1}}\right)dt \tag{125}$$

$$+ \sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{A^{2n+1}}{\alpha^{2n+2}}\right)\left(\frac{1}{q^{n(n+1)}}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-itx}\,{}_qCos\left(\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right)dt \tag{126}$$

Making a u substitution of $u = q^{n-1}\alpha t + \beta q^{-n-1}$ in (125) and the u substitution of

$u = q^n \alpha t + \beta q^{-n-1}$ in (126) gives:

$$\begin{aligned} \hat{y}(x) & \\ &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{A^{2n}}{\alpha^{2n+2}}\right) \left(\frac{1}{q^{n^2+n-1}}\right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\left(\frac{u-\beta q^{-n-1}}{q^{n-1}\alpha}\right)x} {}_qSin\,(u) du \quad (127) \\ &+ \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{A^{2n+1}}{\alpha^{2n+3}}\right)\left(\frac{1}{q^{n^2+2n}}\right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\left(\frac{u-\beta q^{-n-1}}{q^{n}\alpha}\right)x} {}_qCos\,(u) du\,. \quad (128) \end{aligned}$$

Now applying (29) and (28) to (127) and (128) respectively gives:

$$\begin{aligned} \hat{y}(x) & \\ &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{A^{2n}}{\alpha^{2n+2}}\right) \left(\frac{1}{q^{n^2+n-1}}\right) e^{\frac{i\beta}{q^{2n}\alpha}} \left[\frac{\tilde{C}_q \cdot (-i) \left(\frac{x}{q^{n-1}\alpha}\right)}{\theta\left(q^2; \frac{x^2}{q^{2n-2}\alpha^2}\right)}\right] \quad (129) \\ &\quad + \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{A^{2n+1}}{\alpha^{2n+3}}\right) \left(\frac{1}{q^{n^2+2n}}\right) e^{\frac{i\beta}{q^{2n+1}\alpha}} \left[\frac{\tilde{C}_q}{\theta\left(q^2; \frac{x^2}{q^{2n}\alpha^2}\right)}\right]. \quad (130) \end{aligned}$$

Now using (33) on both (129) and (130) gives:

$$\begin{aligned}
&\hat{y}(x) \\
&= \frac{-\tilde{C}_q i x q^2}{\alpha^3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{A^{2n}}{\alpha^{2n}}\right) \left(\frac{1}{q^{n^2+2n}}\right) e^{\frac{i\beta}{q^{2n}\alpha}} && (131)\\
&\quad \cdot \left[\frac{1}{q^{(-n+1)(-n+2)} \left(\frac{x^2}{\alpha^2}\right)^{-n+1}}\right] \left[\frac{1}{\theta\left(q^2; x^2/\alpha^2\right)}\right] && (132)\\
&\quad + \frac{-\tilde{C}_q A}{\alpha^3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{A^{2n}}{\alpha^{2n}}\right) \left(\frac{1}{q^{n^2+2n}}\right) e^{\frac{i\beta}{q^{2n+1}\alpha}} && (133)\\
&\quad \cdot \left[\frac{1}{q^{(-n)(-n+1)} \left(\frac{x^2}{\alpha^2}\right)^{-n}}\right] \left[\frac{1}{\theta\left(q^2; x^2/\alpha^2\right)}\right] && (134)\\
&= \frac{-\tilde{C}_q i}{x\alpha} \sum_{n=0}^{\infty} (-1)^n \left(\frac{A^{2n}}{\alpha^{4n}}\right) \left(\frac{x^{2n}}{q^{2n^2-n}}\right) e^{\frac{i\beta}{q^{2n}\alpha}} \left[\frac{1}{\theta\left(q^2; x^2/\alpha^2\right)}\right] && (135)\\
&\quad + \frac{-\tilde{C}_q A}{\alpha^3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{A^{2n}}{\alpha^{4n}}\right) \left(\frac{x^{2n}}{q^{2n^2+n}}\right) e^{\frac{i\beta}{q^{2n+1}\alpha}} \left[\frac{1}{\theta\left(q^2; x^2/\alpha^2\right)}\right] && (136)\\
&= \left[\frac{1}{\theta(q^2; x^2/\alpha^2)}\right] \left[\frac{-\tilde{C}_q i}{x\alpha} \sum_{n=0}^{\infty} (-1)^n \left(\frac{A^{2n}}{\alpha^{4n}}\right) \left(\frac{x^{2n}}{q^{2n^2-n}}\right) \exp\left[\frac{i\beta}{q^{2n}\alpha}\right] \right.\\
&\quad \left. + \frac{-\tilde{C}_q A}{\alpha^3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{A^{2n}}{\alpha^{4n}}\right) \left(\frac{x^{2n}}{q^{2n^2+n}}\right) \exp\left[\frac{i\beta}{q^{2n+1}\alpha}\right]\right],
\end{aligned}$$

where (132) and (134) follow from (33). This gives (123)-(124) as needed. □

Next we have a proposition for the $2k$ and $2k+1$ derivatives of the solution y (103)-(104).

Proposition 5. *For even and odd order derivatives of $y(t)$, respectively, we have that:*

$$\begin{aligned} & y^{(2k)}(t) \\ = & \sum_{n=0}^{\infty}(-1)^{n+k}\left(\frac{A^{2n}}{\alpha^{2n+1}}\right)\left(\frac{q^{2kn}q^{k(k-1)}\alpha^{2k}}{q^{n^2}}\right){}_qSin\left(q^k\frac{q^{2n}\alpha t+\beta}{q^{n+1}}\right) && (137)\\ & +\sum_{n=0}^{\infty}(-1)^{n+1+k}\left(\frac{A^{2n+1}}{\alpha^{2n+2}}\right)\left(\frac{q^{2kn}q^{k^2}\alpha^{2k}}{q^{n(n+1)}}\right){}_qCos\left(q^k\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right) && (138)\end{aligned}$$

$$\begin{aligned} & y^{(2k+1)}(t) \\ = & \sum_{n=0}^{\infty}(-1)^{n+k}\left(\frac{A^{2n}}{\alpha^{2n+1}}\right)\left(\frac{q^{2kn}q^{k(k-1)}q^{n+k}\alpha^{2k+1}}{q^{n^2}}\right){}_qCos\left(q^{k+1}\frac{q^{2n}\alpha t+\beta}{q^{n+1}}\right) && (139)\\ & +\sum_{n=0}^{\infty}(-1)^{n+k}\left(\frac{A^{2n+1}}{\alpha^{2n+2}}\right)\left(\frac{q^{2kn}q^{k^2}q^{n+k}\alpha^{2k+1}}{q^{n(n+1)}}\right){}_qSin\left(q^k\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right). && (140)\end{aligned}$$

Proof. We will prove this proposition using induction on k. Also note that we may pass the derivatives through these infinite sums since $g_k(t)$ dominates each $y^{(k)}(t)$'s series expansion in (214) below. First note that for the $k = 0$ case (137)-(138) holds from (103)-(104). Assume (137)-(138) as the inductive hypothesis. Then in order to calculate $y^{(2(k+1))}(t) = y^{(2k+2)}(t)$ we first calculate $y^{(2k+1)}(t)$ using the inductive hypothesis.

$$
\begin{aligned}
& y^{(2k+1)}(t) = [y^{(2k)}(t)]' = \\
= & \sum_{n=0}^{\infty}(-1)^{n+k}\left(\frac{A^{2n}}{\alpha^{2n+1}}\right)\left(\frac{q^{2kn}q^{k(k-1)}\alpha^{2k}}{q^{n^2}}\right)\left[{}_qSin\left(q^k\frac{q^{2n}\alpha t+\beta}{q^{n+1}}\right)\right]' && (141)\\
& +\sum_{n=0}^{\infty}(-1)^{n+1+k}\left(\frac{A^{2n+1}}{\alpha^{2n+2}}\right)\left(\frac{q^{2kn}q^{k^2}\alpha^{2k}}{q^{n(n+1)}}\right)\left[{}_qCos\left(q^k\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right)\right]' && (142)\\
= & \sum_{n=0}^{\infty}(-1)^{n+k}\left(\frac{A^{2n}}{\alpha^{2n+1}}\right)\left(\frac{q^{2kn}q^{k(k-1)}\alpha^{2k}}{q^{n^2}}\right)\\
& \qquad\cdot\left[q\,{}_qCos\left(q^{k+1}\frac{q^{2n}\alpha t+\beta}{q^{n+1}}\right)\left(q^{n+k-1}\alpha\right)\right]\\
& +\sum_{n=0}^{\infty}(-1)^{n+1+k}\left(\frac{A^{2n+1}}{\alpha^{2n+2}}\right)\left(\frac{q^{2kn}q^{k^2}\alpha^{2k}}{q^{n(n+1)}}\right)\\
& \qquad\cdot\left[-\,{}_qSin\left(q^k\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right)\left(q^{n+k}\alpha\right)\right]\\
= & \sum_{n=0}^{\infty}(-1)^{n+k}\left(\frac{A^{2n}}{\alpha^{2n+1}}\right)\left(\frac{q^{2kn}q^{k(k-1)}q^{n+k}\alpha^{2k+1}}{q^{n^2}}\right){}_qCos\left(q^{k+1}\frac{q^{2n}\alpha t+\beta}{q^{n+1}}\right) && (143)\\
& +\sum_{n=0}^{\infty}(-1)^{n+k}\left(\frac{A^{2n+1}}{\alpha^{2n+2}}\right)\left(\frac{q^{2kn}q^{k^2}q^{n+k}\alpha^{2k+1}}{q^{n(n+1)}}\right){}_qSin\left(q^k\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right). && (144)
\end{aligned}
$$

Next we calculate $y^{(2(k+1))}(t) = y^{(2k+2)}(t)$ from $y^{(2k+1)}(t)$.

$$
\begin{aligned}
& y^{(2(k+1))}(t) = y^{(2k+2)}(t) = [y^{(2k+1)}(t)]' \\
= & \sum_{n=0}^{\infty}(-1)^{n+k}\left(\frac{A^{2n}}{\alpha^{2n+1}}\right)\left(\frac{q^{2kn}q^{k(k-1)}q^{n+k}\alpha^{2k+1}}{q^{n^2}}\right)\left[{}_qCos\left(q^{k+1}\frac{q^{2n}\alpha t+\beta}{q^{n+1}}\right)\right]' \quad (145)\\
& +\sum_{n=0}^{\infty}(-1)^{n+k}\left(\frac{A^{2n+1}}{\alpha^{2n+2}}\right)\left(\frac{q^{2kn}q^{k^2}q^{n+k}\alpha^{2k+1}}{q^{n(n+1)}}\right)\left[{}_qSin\left(q^{k}\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right)\right]' \quad (146)\\
= & \sum_{n=0}^{\infty}(-1)^{n+k}\left(\frac{A^{2n}}{\alpha^{2n+1}}\right)\left(\frac{q^{2kn}q^{k(k-1)}q^{n+k}\alpha^{2k+1}}{q^{n^2}}\right)\\
& \quad\cdot\left[-{}_qSin\left(q^{k+1}\frac{q^{2n}\alpha t+\beta}{q^{n+1}}\right)\left(q^{n+k}\alpha\right)\right]\\
& +\sum_{n=0}^{\infty}(-1)^{n+k}\left(\frac{A^{2n+1}}{\alpha^{2n+2}}\right)\left(\frac{q^{2kn}q^{k^2}q^{n+k}\alpha^{2k+1}}{q^{n(n+1)}}\right)\\
& \quad\cdot\left[q\,{}_qCos\left(q^{k+1}\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right)\left(q^{n+k}\alpha\right)\right]\\
= & \sum_{n=0}^{\infty}(-1)^{n+k+1}\left(\frac{A^{2n}}{\alpha^{2n+1}}\right)\left(\frac{q^{2kn+2n}q^{k(k+1)}\alpha^{2(k+1)}}{q^{n^2}}\right) \quad (147)\\
& \quad\cdot{}_qSin\left(q^{k+1}\frac{q^{2n}\alpha t+\beta}{q^{n+1}}\right) \quad (148)\\
& +\sum_{n=0}^{\infty}(-1)^{n+1+k+1}\left(\frac{A^{2n+1}}{\alpha^{2n+2}}\right)\left(\frac{q^{2n(k+1)}q^{(k+1)^2}\alpha^{2(k+1)}}{q^{n(n+1)}}\right) \quad (149)\\
& \quad\cdot{}_qCos\left(q^{k+1}\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right). \quad (150)
\end{aligned}
$$

From (147)-(150), we have proven (137)-(138) by induction. Now that (137)-(138) is established for all $k \in \mathbb{N}$, from (143)-(144) we obtain that (139)-(140) holds for all $k \in \mathbb{N}$. Note that we are allowed to exchange the derivatives with the infinite sums in (145)-(146) and (141)-(142) from (214) given below. □

Proposition 6. *The solution in* (103)-(104) *and the solution* (105)-(106) *in Theorem 3 are Schwartz.*

Proof. Recall from (50) that a necessary and sufficient condition for $y(t)$ to be Schwartz is that

$$\forall\, p,k \in \mathbb{N}_0 \;\; \exists\, C_{p,k} \in \mathbb{R} \;\text{ s.t. }\; 0 < |t^p y^{(k)}(t)| \leq C_{p,k} < \infty. \tag{151}$$

In order to bound every $|y^{(k)}(t)|$, we bound the even order derivatives, $|y^{(2k)}(t)|$, and the odd order derivatives, $|y^{(2k+1)}(t)|$. Examine Proposition 5, apply the triangle inequality to $|y^{(2k)}(t)|$ and $|y^{(2k+1)}(t)|$, and factor out common factors to obtain:

$$\begin{aligned}
&|y^{(2k)}(t)| \\
\leq\;& q^{k(k-1)}|\alpha|^{2k-1}\sum_{n=0}^{\infty}\left(\frac{A^{2n}}{\alpha^{2n}}\right)\left(\frac{q^{2kn}}{q^{n^2}}\right)\left|{}_qSin\left(q^k\frac{q^{2n}\alpha t+\beta}{q^{n+1}}\right)\right| && (152)\\
&+Aq^{k^2}|\alpha|^{2k-2}\sum_{n=0}^{\infty}\left(\frac{A^{2n}}{\alpha^{2n}}\right)\left(\frac{q^{2kn}}{q^{n(n+1)}}\right)\left|{}_qCos\left(q^k\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right)\right| && (153)
\end{aligned}$$

$$\begin{aligned}
&|y^{(2k+1)}(t)| \\
\leq\;& q^{k^2}\alpha^{2k}\sum_{n=0}^{\infty}\left(\frac{A^{2n}}{\alpha^{2n}}\right)\left(\frac{q^{2kn}q^n}{q^{n^2}}\right)\left|{}_qCos\left(q^{k+1}\frac{q^{2n}\alpha t+\beta}{q^{n+1}}\right)\right| && (154)\\
&+q^{k(k+1)}|\alpha|^{2k-1}\sum_{n=0}^{\infty}\left(\frac{A^{2n}}{\alpha^{2n}}\right)\left(\frac{q^{2kn}q^n}{q^{n(n+1)}}\right)\left|{}_qSin\left(q^k\frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right)\right| && (155)
\end{aligned}$$

Now we will apply the decay estimate, (44) for ${}_qCos(u)$ to (153) and (154), and the decay estimate, (45) for ${}_qSin(u)$ to (152) and (155). Recall that in order to apply the decay estimates (44) and (45) to ${}_qCos(u)$ and ${}_qSin(u)$ we must have $|u| \geq 1$. So we look at the argument of the ${}_qCos(t)$ and ${}_qSin(t)$ in (152), (153), (154), and (155), and work to get a standard assumption on t sufficient for every argument to be greater than or equal to 1 in absolute value. We will have four such arguments to handle.

For the argument of ${}_qSin$ in (152), we assume

$$\left|q^k \frac{q^{2n}\alpha t+\beta}{q^{n+1}}\right| = \left|q^{n+k-1}\alpha t+\beta q^{k-n-1}\right| \tag{156}$$

$$\geq \left|q^{n+k-1}\alpha t\right| - \left|\beta q^{k-n-1}\right| \tag{157}$$

$$\geq \left|q^{-1}\alpha t\right| - \left|\beta q^{k}\right| \geq 1. \tag{158}$$

Second, for the argument of ${}_qCos$ in (153), we assume

$$\left|q^k \frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right| = \left|q^{n+k}\alpha t+\beta q^{k-n-1}\right| \tag{159}$$

$$\geq \left|q^{n+k}\alpha t\right| - \left|\beta q^{k-n-1}\right| \tag{160}$$

$$\geq |\alpha t| - \left|\beta q^{k}\right| \geq 1. \tag{161}$$

Third, the argument of ${}_qCos$ in (154), we assume

$$\left|q^{k+1} \frac{q^{2n}\alpha t+\beta}{q^{n+1}}\right| = \left|q^{n+k}\alpha t+\beta q^{k-n-1}\right| \tag{162}$$

$$\geq \left|q^{n+k}\alpha t\right| - \left|\beta q^{k-n-1}\right| \tag{163}$$

$$\geq |\alpha t| - \left|\beta q^{k}\right| \geq 1. \tag{164}$$

Fourth, the argument of ${}_qSin$ in (155), we assume

$$\left|q^k \frac{q^{2n+1}\alpha t+\beta}{q^{n+1}}\right| = \left|q^{n+k}\alpha t+\beta q^{k-n-1}\right| \tag{165}$$

$$\geq \left|q^{n+k}\alpha t\right| - \left|\beta q^{k-n-1}\right| \tag{166}$$

$$\geq |\alpha t| - \left|\beta q^{k}\right| \geq 1. \tag{167}$$

Since $|q^{-1}\alpha t| - \left|\beta q^k\right| \geq 1$ implies that $|\alpha t| - \left|\beta q^k\right| \geq 1$, we have that

$$\left|q^{-1}\alpha t\right| - \left|\beta q^k\right| \geq 1 \quad \Longleftrightarrow \quad |t| \geq \frac{1+|\beta| q^k}{q^{-1}|\alpha|} \equiv C(\alpha,\beta,q,k) \tag{168}$$

is the standard assumption that we will use when we apply our decay estimates. Here $C(\alpha, \beta, q, k)$ is given by (168). Now using our assumption, $|t| \geq C(\alpha, \beta, q, k)$, with the decay estimate, (44) on (153) and (154), and the decay estimate (45) on (152) and (155) we get that:

$$
\begin{aligned}
& |y^{(2k)}(t)| \\
\leq \quad & q^{k(k-1)}|\alpha|^{2k-1} \sum_{n=0}^{\infty} \left(\frac{A^{2n}}{\alpha^{2n}}\right) \left(\frac{q^{2kn}}{q^{n^2}}\right) B_q e^{-1} \left||q^{-1}\alpha t| - |\beta q^k|\right|^{-1} && (169) \\
& \quad \cdot \left||q^{-1}\alpha t| - |\beta q^k|\right|^{-\ln\left(\left||q^{-1}\alpha t|-|\beta q^k|\right|\right)\cdot \ln(q)/[2+\ln(q)]^2} && (170) \\
& + A q^{k^2} \alpha^{2k-2} \sum_{n=0}^{\infty} \left(\frac{A^{2n}}{\alpha^{2n}}\right) \left(\frac{q^{2kn}}{q^{n(n+1)}}\right) B_q \left||q^{-1}\alpha t| - |\beta q^k|\right|^{-2/[2e^{-1}+\ln(q)]} && (171) \\
& \quad \cdot \left||q^{-1}\alpha t| - |\beta q^k|\right|^{-\ln\left(\left||q^{-1}\alpha t|-|\beta q^k|\right|\right)\cdot \ln(q)/[2e^{-1}+\ln(q)]^2} && (172)
\end{aligned}
$$

and

$$
\begin{aligned}
& |y^{(2k+1)}(t)| \\
\leq \quad & q^{k^2} \alpha^{2k} \sum_{n=0}^{\infty} \left(\frac{A^{2n}}{\alpha^{2n}}\right) \left(\frac{q^{2kn} q^n}{q^{n^2}}\right) B_q \left||q^{-1}\alpha t| - |\beta q^k|\right|^{-2/[2e^{-1}+\ln(q)]} && (173) \\
& \quad \cdot \left||q^{-1}\alpha t| - |\beta q^k|\right|^{-\ln\left(\left||q^{-1}\alpha t|-|\beta q^k|\right|\right)\cdot \ln(q)/[2e^{-1}+\ln(q)]^2} && (174) \\
& + q^{k(k+1)}|\alpha|^{2k-1} \sum_{n=0}^{\infty} \left(\frac{A^{2n}}{\alpha^{2n}}\right) \left(\frac{q^{2kn} q^n}{q^{n(n+1)}}\right) B_q e^{-1} \left||q^{-1}\alpha t| - |\beta q^k|\right|^{-1} && (175) \\
& \quad \cdot \left||q^{-1}\alpha t| - |\beta q^k|\right|^{-\ln\left(\left||q^{-1}\alpha t|-|\beta q^k|\right|\right)\cdot \ln(q)/[2+\ln(q)]^2}. && (176)
\end{aligned}
$$

Now we will bound the sums in (169), (171), (173), and (175) using a completion of squares into a Gaussian Bound.

First the sum in (169), namely

$$\begin{aligned}
& \sum_{n=0}^{\infty}\left(\frac{A^{2n}}{\alpha^{2n}}\right)\left(\frac{q^{2kn}}{q^{n^2}}\right) && (177)\\
= & \sum_{n=0}^{\infty}\left(\frac{A^2 q^{2k}}{\alpha^2}\right)^n\left(\frac{1}{q^{n^2}}\right)\\
= & \sum_{n=0}^{\infty}\exp\left[n\ln(A^2q^{2k}\alpha^{-2})\right]\exp\left[-n^2\ln(q)\right]\\
= & \sum_{n=0}^{\infty}\exp\left[-\ln(q)\left(n^2-n\frac{\ln(A^2q^{2k}\alpha^{-2})}{\ln(q)}\right)\right]\\
= & \sum_{n=0}^{\infty}\exp\left[-\ln(q)\left(n^2-n\frac{\ln(A^2q^{2k}\alpha^{-2})}{\ln(q)}+\left(\frac{\ln(A^2q^{2k}\alpha^{-2})}{2\ln(q)}\right)^2-\left(\frac{\ln(A^2q^{2k}\alpha^{-2})}{2\ln(q)}\right)^2\right)\right]\\
= & \exp\left[\frac{\ln^2(A^2q^{2k}\alpha^{-2})}{4\ln(q)}\right]\sum_{n=0}^{\infty}\exp\left[-\ln(q)\left(n-\frac{\ln(A^2q^{2k}\alpha^{-2})}{2\ln(q)}\right)^2\right]\\
= & \exp\left[\frac{\ln^2(A^2q^{2k}\alpha^{-2})}{4\ln(q)}\right]\sum_{n=0}^{\infty}\exp\left[\frac{-\frac{1}{2}\left(n-\frac{\ln(A^2q^{2k}\alpha^{-2})}{2\ln(q)}\right)^2}{\left(\frac{1}{\sqrt{2\ln(q)}}\right)^2}\right] && (178)\\
\leq & \exp\left[\frac{\ln^2(A^2q^{2k}\alpha^{-2})}{4\ln(q)}\right]\left[1+\sqrt{2\pi}\frac{1}{\sqrt{2\ln(q)}}\right] && (179)\\
= & \exp\left[\frac{\ln^2(A^2q^{2k}\alpha^{-2})}{4\ln(q)}\right]\left[1+\sqrt{\frac{\pi}{\ln(q)}}\right]. && (180)
\end{aligned}$$

Here the bound (179) follows from using (47) on the Gaussian in (178).

Now we will do the same for the sum in (171), namely

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\frac{A^{2n}}{\alpha^{2n}}\right)\left(\frac{q^{2kn}}{q^{n^2+n}}\right) && (181)\\
=\;& \sum_{n=0}^{\infty}\left(\frac{A^2 q^{2k-1}}{\alpha^2}\right)^n\left(\frac{1}{q^{n^2}}\right)\\
=\;& \sum_{n=0}^{\infty}\exp\left[n\ln(A^2q^{2k-1}\alpha^{-2})\right]\exp\left[-n^2\ln(q)\right]\\
=\;& \sum_{n=0}^{\infty}\exp\left[-\ln(q)\left(n^2-n\frac{\ln(A^2q^{2k-1}\alpha^{-2})}{\ln(q)}\right)\right]\\
=\;& \exp\left[\frac{\ln^2(A^2q^{2k-1}\alpha^{-2})}{4\ln(q)}\right]\sum_{n=0}^{\infty}\exp\left[-\ln(q)\left(n-\frac{\ln(A^2q^{2k-1}\alpha^{-2})}{2\ln(q)}\right)^2\right]\\
=\;& \exp\left[\frac{\ln^2(A^2q^{2k-1}\alpha^{-2})}{4\ln(q)}\right]\sum_{n=0}^{\infty}\exp\left[\frac{-\frac{1}{2}\left(n-\frac{\ln(A^2q^{2k-1}\alpha^{-2})}{2\ln(q)}\right)^2}{\left(\frac{1}{\sqrt{2\ln(q)}}\right)^2}\right] && (182)\\
\leq\;& \exp\left[\frac{\ln^2(A^2q^{2k-1}\alpha^{-2})}{4\ln(q)}\right]\left[1+\sqrt{2\pi}\frac{1}{\sqrt{2\ln(q)}}\right] && (183)\\
=\;& \exp\left[\frac{\ln^2(A^2q^{2k-1}\alpha^{-2})}{4\ln(q)}\right]\left[1+\sqrt{\frac{\pi}{\ln(q)}}\right]. && (184)
\end{aligned}$$

Here the bound (183) follows from applying (47) on the Gaussian in (182).

Similarly we do the same for the sum in (173), namely

$$\begin{aligned}
& \sum_{n=0}^{\infty}\left(\frac{A^{2n}}{\alpha^{2n}}\right)\left(\frac{q^{2kn}q^{n}}{q^{n^2}}\right) && (185)\\
= & \sum_{n=0}^{\infty}\left(\frac{A^{2}q^{2k+1}}{\alpha^{2}}\right)^{n}\left(\frac{1}{q^{n^2}}\right)\\
= & \sum_{n=0}^{\infty}\exp\left[n\ln(A^2q^{2k+1}\alpha^{-2})\right]\exp\left[-n^2\ln(q)\right]\\
= & \sum_{n=0}^{\infty}\exp\left[-\ln(q)\left(n^2-n\frac{\ln(A^2q^{2k+1}\alpha^{-2})}{\ln(q)}\right)\right]\\
= & \exp\left[\frac{\ln^2(A^2q^{2k+1}\alpha^{-2})}{4\ln(q)}\right]\sum_{n=0}^{\infty}\exp\left[-\ln(q)\left(n-\frac{\ln(A^2q^{2k}\alpha^{-2})}{2\ln(q)}\right)^2\right]\\
= & \exp\left[\frac{\ln^2(A^2q^{2k+1}\alpha^{-2})}{4\ln(q)}\right]\sum_{n=0}^{\infty}\exp\left[\frac{-\frac{1}{2}\left(n-\frac{\ln(A^2q^{2k+1}\alpha^{-2})}{2\ln(q)}\right)^2}{\left(\frac{1}{\sqrt{2\ln(q)}}\right)^2}\right] && (186)\\
\leq & \exp\left[\frac{\ln^2(A^2q^{2k+1}\alpha^{-2})}{4\ln(q)}\right]\left[1+\sqrt{2\pi}\frac{1}{\sqrt{2\ln(q)}}\right] && (187)\\
= & \exp\left[\frac{\ln^2(A^2q^{2k+1}\alpha^{-2})}{4\ln(q)}\right]\left[1+\sqrt{\frac{\pi}{\ln(q)}}\right]. && (188)
\end{aligned}$$

Here the bound (187) follows from applying (47) to the Gaussian in (186).

We will proceed similarly for the sum in (175), namely

$$\begin{aligned}
& \sum_{n=0}^{\infty}\left(\frac{A^{2n}}{\alpha^{2n}}\right)\left(\frac{q^{2kn}q^{n}}{q^{n(n+1)}}\right) && (189)\\
= & \sum_{n=0}^{\infty}\left(\frac{A^{2n}}{\alpha^{2n}}\right)\left(\frac{q^{2kn}}{q^{n^2}}\right) && (190)\\
\leq & \exp\left[\frac{\ln^2(A^2q^{2k}\alpha^{-2})}{4\ln(q)}\right]\left[1+\sqrt{\frac{\pi}{\ln(q)}}\right] && (191)
\end{aligned}$$

The sum in (175) is equal to (190) which is exact same as the sum in (177) so the bound (191) is the same as the bound for (177), namely (180).

Now that all of the sums in (169), (171), (173), and (175) are bounded by (180), (184), (188), and (191) respectively, we look back to (169)-(176) and say that:

$$
\begin{aligned}
& |y^{(2k)}(t)| \\
\leq\ & q^{k(k-1)}|\alpha|^{2k-1} \exp\left[\frac{\ln^2(A^2 q^{2k}\alpha^{-2})}{4\ln(q)}\right]\left[1+\sqrt{\frac{\pi}{\ln(q)}}\right] && (192)\\
& \quad \cdot B_q e^{-1}\left||q^{-1}\alpha t|-|\beta q^k|\right|^{-1} && (193)\\
& \quad \cdot \left||q^{-1}\alpha t|-|\beta q^k|\right|^{-\ln\left(\left||q^{-1}\alpha t|-|\beta q^k|\right|\right)\cdot\ln(q)/[2+\ln(q)]^2} && (194)\\
& + A q^{k^2}\alpha^{2k-2} \exp\left[\frac{\ln^2(A^2 q^{2k-1}\alpha^{-2})}{4\ln(q)}\right]\left[1+\sqrt{\frac{\pi}{\ln(q)}}\right] && (195)\\
& \quad \cdot B_q \left||q^{-1}\alpha t|-|\beta q^k|\right|^{-2/[2e^{-1}+\ln(q)]} && (196)\\
& \quad \cdot \left||q^{-1}\alpha t|-|\beta q^k|\right|^{-\ln\left(\left||q^{-1}\alpha t|-|\beta q^k|\right|\right)\cdot\ln(q)/[2e^{-1}+\ln(q)]^2} && (197)\\
& \equiv T_{2k}(t) \quad \text{for} \quad |t| \geq C(\alpha,\beta,q,k) && (198)
\end{aligned}
$$

$$
\begin{aligned}
& |y^{(2k+1)}(t)| \\
\leq\ & q^{k^2}\alpha^{2k} \exp\left[\frac{\ln^2(A^2 q^{2k+1}\alpha^{-2})}{4\ln(q)}\right]\left[1+\sqrt{\frac{\pi}{\ln(q)}}\right] && (199)\\
& \quad \cdot B_q \left||q^{-1}\alpha t|-|\beta q^k|\right|^{-2/[2e^{-1}+\ln(q)]} && (200)\\
& \quad \cdot \left||q^{-1}\alpha t|-|\beta q^k|\right|^{-\ln\left(\left||q^{-1}\alpha t|-|\beta q^k|\right|\right)\cdot\ln(q)/[2e^{-1}+\ln(q)]^2} && (201)\\
& + q^{k(k+1)}|\alpha|^{2k-1} \exp\left[\frac{\ln^2(A^2 q^{2k}\alpha^{-2})}{4\ln(q)}\right]\left[1+\sqrt{\frac{\pi}{\ln(q)}}\right] && (202)\\
& \quad \cdot B_q e^{-1}\left||q^{-1}\alpha t|-|\beta q^k|\right|^{-1} && (203)\\
& \quad \cdot \left||q^{-1}\alpha t|-|\beta q^k|\right|^{-\ln\left(\left||q^{-1}\alpha t|-|\beta q^k|\right|\right)\cdot\ln(q)/[2+\ln(q)]^2} && (204)\\
& \equiv T_{2k+1}(t) \quad \text{for} \quad |t| \geq C(\alpha,\beta,q,k), && (205)
\end{aligned}
$$

where B_q is given in (46).

We defined the function $T_{2k}(t)$ in (198) by (192)-(197), and the function $T_{2k+1}(t)$ in (205) by (199)-(204). Recall that the bounds above are for $|t| \geq C(\alpha, \beta, q, k)$, where $C(\alpha, \beta, q, k)$ is given by (168). For the case where $|t| < C(\alpha, \beta, q, k)$, we use the bounds (42) and (43) on (152)-(155) to obtain the bounds

$$
\begin{aligned}
& |y^{(2k)}(t)| \\
\leq \quad & q^{1/2} q^{k(k-1)} |\alpha|^{2k-1} \exp\left[\frac{\ln^2(A^2 q^{2k} \alpha^{-2})}{4\ln(q)}\right] \left[1 + \sqrt{\frac{\pi}{\ln(q)}}\right] && (206) \\
& + A q^{k^2} |\alpha|^{2k-2} \exp\left[\frac{\ln^2(A^2 q^{2k-1} \alpha^{-2})}{4\ln(q)}\right] \left[1 + \sqrt{\frac{\pi}{\ln(q)}}\right] && (207) \\
& \equiv M_{2k} \quad \text{for} \quad |t| < C(\alpha, \beta, q, k) && (208)
\end{aligned}
$$

and

$$
\begin{aligned}
& |y^{(2k+1)}(t)| \\
\leq \quad & q^{k^2} \alpha^{2k} \exp\left[\frac{\ln^2(A^2 q^{2k+1} \alpha^{-2})}{4\ln(q)}\right] \left[1 + \sqrt{\frac{\pi}{\ln(q)}}\right] && (209) \\
& + q^{1/2} q^{k(k+1)} |\alpha|^{2k-1} \exp\left[\frac{\ln^2(A^2 q^{2k} \alpha^{-2})}{4\ln(q)}\right] \left[1 + \sqrt{\frac{\pi}{\ln(q)}}\right] && (210) \\
& \equiv M_{2k+1} \quad \text{for} \quad |t| < C(\alpha, \beta, q, k). && (211)
\end{aligned}
$$

We defined the constant M_{2k} in (208) by (206)-(207) and the constant M_{2k+1} in (211) by (209)-(210).

Now we will define two functions $g_{2k}(t)$ and $g_{2k+1}(t)$ as follows:

$$
g_{2k}(t) = \begin{cases} T_{2k}(t)\ , & \text{if} \quad |t| \geq C(\alpha, \beta, q, k) \\ M_{2k}\ , & \text{if} \quad |t| < C(\alpha, \beta, q, k) \end{cases} \tag{212}
$$

and

$$
g_{2k+1}(t) = \begin{cases} T_{2k+1}(t)\ , & \text{if} \quad |t| \geq C(\alpha, \beta, q, k) \\ M_{2k+1}\ , & \text{if} \quad |t| < C(\alpha, \beta, q, k), \end{cases} \tag{213}
$$

where $C(\alpha, \beta, q, k)$ is given by (168).

Then by construction we have for $m = 2k$ or $m = 2k + 1$ that

$$|y^{(m)}(t)| \leq |g_m(t)| < \infty, \quad \forall\ m \in \mathbb{N} \text{ and } \ \forall\ t \in \mathbb{R} \tag{214}$$

and $g_m(t)$ is integrable on $\mathbb{R}$ since it is a constant on the interval $(-C(\alpha, \beta, q, \lfloor m/2 \rfloor), C(\alpha, \beta, q, \lfloor m/2 \rfloor))$ and it is both continuous and decaying to 0 rapidly as $|t| \to \infty$ (see below) on the intervals $(-\infty, -C(\alpha, \beta, q, \lfloor m/2 \rfloor)]$ and $[C(\alpha, \beta, q, \lfloor m/2 \rfloor), \infty)$. Note that although $g_m(t)$ need not be continuous at the points $C(\alpha, \beta, q, \lfloor m/2 \rfloor)$ or $-C(\alpha, \beta, q, \lfloor m/2 \rfloor)$ this is a set of measure 0, and we may still apply Theorem 2 (Lebesgue Dominated Convergence).

Now notice that the Schwartz bound, (50), on $|t^p y^{(m)}(t)|$ clearly holds for $-C(\alpha, \beta, q, \lfloor m/2 \rfloor) < t < C(\alpha, \beta, q, \lfloor m/2 \rfloor)$ since if $t \in (-C(\alpha, \beta, q, \lfloor m/2 \rfloor), C(\alpha, \beta, q, \lfloor m/2 \rfloor))$, then $|t^p| \leq |C(\alpha, \beta, q, \lfloor m/2 \rfloor)|^p \ , \quad \forall\ p \in \mathbb{N}$.
Then recall from (214) we have that $|y^{(m)}(t)| \leq |g_m(t)|$, and therefore $|t^p y^{(m)}(t)| < |C(\alpha, \beta, q, \lfloor m/2 \rfloor)^p M_m| < \infty$ holds for $-C(\alpha, \beta, q, \lfloor m/2 \rfloor) < t < C(\alpha, \beta, q, \lfloor m/2 \rfloor)$.
To get the desired result, namely that $|t^p y^{(m)}(t)|$ is bounded, when $t \geq C(\alpha, \beta, q, \lfloor m/2 \rfloor)$ or $t \leq -C(\alpha, \beta, q, \lfloor m/2 \rfloor)$, we rely on the fact that $\forall\ A > 0$ and $B \in \mathbb{R}$

$$\lim_{|t| \to \infty} \frac{\ln |At - B|}{\ln |t|} = 1. \tag{215}$$

From (215) we may write that,

$$\begin{aligned}
& \forall\ 1 > \epsilon > 0\ ,\ \exists\ \ t_0 > 1 \text{ s.t. } \forall\ |t| > t_0 && (216)\\
\text{s.t.}\quad & C_\epsilon = 1-\epsilon < \frac{\ln|At-B|}{\ln|t|} < 1+\epsilon = B_\epsilon && (217)\\
\Longrightarrow\quad & C_\epsilon \ln|t| < \ln|At-B| < B_\epsilon \ln|t| && (218)\\
\Longrightarrow\quad & C_\epsilon^2 \ln^2|t| < \ln^2|At-B| < B_\epsilon^2 \ln^2|t| && (219)\\
\Longrightarrow\quad & Ke^{-C_1\ln|At-B|-C_2\ln^2|At-B|} < Ke^{-C_1C_\epsilon\ln|t|-C_2C_\epsilon^2\ln^2|t|} && (220)\\
\Longrightarrow\quad & |t|^pKe^{-C_1\ln|At-B|-C_2\ln^2|At-B|} < Ke^{\left[p-C_1C_\epsilon-C_2C_\epsilon^2\ln|t|\right]\ln|t|} && (221)
\end{aligned}$$

for $p \in \mathbb{N}\cup\{0\}$ fixed, and for all C_1,$C_2 > 0$.

Then, recall from (214) that

$$|y^{(m)}(t)| \leq |g_m(t)|. \tag{222}$$

Now setting K, C_1, C_2, A, B to be as needed to match (221) with (192)-(205), then applying this to (222) we obtain that for $|t| \geq C(\alpha,\beta,q,\lfloor m/2\rfloor)$

$$\begin{aligned}
|t^py^{(m)}(t)| &< Ke^{\left[p-C_1C_\epsilon-C_2C_\epsilon^2\ln|t|\right]\ln|t|} && (223)\\
\Longrightarrow \lim_{|t|\to\pm\infty}|t^py^{(m)}(t)| &< \lim_{|t|\to\pm\infty}Ke^{\left[p-C_1C_\epsilon-C_2C_\epsilon^2\ln|t|\right]\ln|t|} = 0 && (224)\\
\Longrightarrow \lim_{|t|\to\pm\infty}|t^py^{(m)}(t)| &= 0 && (225)
\end{aligned}$$

Now that it is established that $\lim_{|t|\to\pm\infty}|t^py^{(m)}(t)| = 0$, we note that it is clear that

$$\begin{aligned}
|t^py^{(m)}(t)| \quad & \text{is continuous on the intervals} && (226)\\
[C(\alpha,\beta,q,\lfloor m/2\rfloor),\infty) \ \text{ and } \ & (-\infty,-C(\alpha,\beta,q,\lfloor m/2\rfloor),\infty)], && (227)
\end{aligned}$$

since $|t^p|$ and $|y^{(m)}(t)|$ are continuous on both of the intervals $[C(\alpha,\beta,q,\lfloor m/2\rfloor),\infty)$ and

$(-\infty, -C(\alpha, \beta, q, \lfloor m/2 \rfloor), \infty)]$. Then using (225) and (226) we obtain that $|t^p y^{(n)}(t)|$ is bounded on the intervals $(-\infty, C(\alpha, \beta, q, \lfloor m/2 \rfloor)]$ and $[C(\alpha, \beta, q, \lfloor m/2 \rfloor), \infty)$, which is equivalent to the statement

$$\begin{aligned}
&\exists\ B(\alpha, \beta, q, \lfloor m/2 \rfloor) \\
&\text{s.t. on the intervals } [C(\alpha, \beta, q, \lfloor m/2 \rfloor), \infty) \text{ and } (-\infty, -C(\alpha, \beta, q, \lfloor m/2 \rfloor)] \\
&|t^p y^{(m)}(t)| < B(\alpha, \beta, q, \lfloor m/2 \rfloor) < \infty. \qquad (228)
\end{aligned}$$

This completes all cases of bounding $|t^p y^{(m)}(t)|$ and gives that since $|t^p y^{(m)}(t)|$ is bounded on the entire real line, we can conclude that $y(t)$ is Schwartz, as desired. A similar proof shows that the solution (105)-(106) is also Schwartz. $\square$

Next we have another theorem.

Theorem 4. *The solution* (103)-(104) *in Theorem 3 is a Schwartz wavelet with all moments vanishing.*

Proof. We have already shown that $y(t)$, (103)-(104), is Schwartz from Proposition 6. Now we must show that it satisfies the three conditions to be a wavelet, namely (25), (26), and (27) in Definition 4. Since $y(t)$ is Schwartz, as shown in Proposition 6, then from (51) this means that $y(t) \in Ł^1(\mathbb{R}) \bigcap Ł^2(\mathbb{R}) \bigcap Ł^\infty(\mathbb{R})$. This is (25) confirmed. For (26), recall from (19), (20), (21), and (22) that if $f(t)$, in our main MADE $y'(t) - Ay(qt) = f(t)$, is a wavelet, and we assume that the solution $y(t)$ vanishes at plus minus infinity, then it is a property of our main MADE that the solution $y(t)$ will have its 0th moment vanishing. For our solution, $y(t)$ recall that $f(t) = {}_qCos(\alpha t + \beta)$, which is clearly a wavelet as stated earlier and proven in [PRS3]. Since we have shown that $y(t)$ is Schwartz, it must vanish at plus minus infinity, meaning that our solution, $y(t)$, (103)-(104), must have its 0th moment vanishing. This is (26) confirmed. To prove (27), we need to show that $\displaystyle\int_{-\infty}^{\infty} \frac{|\hat{y}(x)|^2}{|x|}\, dx < \infty$. Recall from (214) that we have $|y^{(m)}(t)| \leq |g_m(t)| < \infty, \;\; \forall\, m \in \mathbb{N}$ and $\forall\, t \in \mathbb{R}$, where $g_k(t)$ is an integrable function as it constant on $(-C(\alpha, \beta, q, \lfloor m/2 \rfloor), C(\alpha, \beta, q, \lfloor m/2 \rfloor))$ and it is continuous and rapidly decaying on $(-\infty, -C(\alpha, \beta, q, \lfloor m/2 \rfloor)]$ and $[C(\alpha, \beta, q, \lfloor m/2 \rfloor), \infty)$. Thus by Theorem 2, we may pass integrals through our infinite sums in the solution $y(t)$, (103)-(104). This makes the computation of $\hat{y}(x)$ in Proposition 4 rigorous. Then, since $y(t)$ has been established to be Schwartz, from (52) we get that $\hat{y}(x)$ is also Schwartz. Now that $\hat{y}(x)$ is Schwartz it is clear that $\displaystyle\int_{-\infty}^{-1} \frac{|\hat{y}(x)|^2}{|x|}\, dx < \infty$ and $\displaystyle\int_{1}^{\infty} \frac{|\hat{y}(x)|^2}{|x|}\, dx < \infty$, however we need to make sure that $\displaystyle\int_{-1}^{0} \frac{|\hat{y}(x)|^2}{|x|}\, dx$ and $\displaystyle\int_{0}^{1} \frac{|\hat{y}(x)|^2}{|x|}\, dx$ are finite. To do this first recall, from Proposition 4, that

$$\begin{aligned}\hat{y}(x) & \\ = & \left[\frac{1}{\theta(q^2;x^2/\alpha^2)}\right]\left[\frac{-\tilde{C}_q i}{x\alpha}\sum_{n=0}^{\infty}(-1)^n\left(\frac{A^{2n}}{\alpha^{4n}}\right)\left(\frac{x^{2n}}{q^{2n^2-n}}\right)\exp\left[\frac{i\beta}{q^{2n}\alpha}\right]\right. & (229)\\ & \left.+\frac{-\tilde{C}_q A}{\alpha^3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{A^{2n}}{\alpha^{4n}}\right)\left(\frac{x^{2n}}{q^{2n^2+n}}\right)\exp\left[\frac{i\beta}{q^{2n+1}\alpha}\right]\right]. & (230)\end{aligned}$$

Taking the absolute value of (229)-(230) and using the triangle equality we obtain that:

$$\begin{aligned}&|\hat{y}(x)| \\ &\leq \frac{1}{\theta(q^2;x^2/\alpha^2)}\left[\frac{|\tilde{C}_q|}{|x\alpha|}\sum_{n=0}^{\infty}\left(\frac{A^{2n}}{\alpha^{4n}}\right)\left(\frac{x^{2n}}{q^{2n^2-n}}\right)+\frac{|\tilde{C}_q A|}{|\alpha|^3}\sum_{n=0}^{\infty}\left(\frac{A^{2n}}{\alpha^{4n}}\right)\left(\frac{x^{2n}}{q^{2n^2+n}}\right)\right] \quad (231)\end{aligned}$$

Denote $P_1(x)=\sum_{n=0}^{\infty}\left(\frac{A^{2n}}{\alpha^{4n}}\right)\left(\frac{x^{2n}}{q^{2n^2-n}}\right)$ and $P_2(x)=\sum_{n=0}^{\infty}\left(\frac{A^{2n}}{\alpha^{4n}}\right)\left(\frac{x^{2n}}{q^{2n^2+n}}\right)$ so that, from (231), we may write

$$|\hat{y}(x)| \leq \frac{|\tilde{C}_q|}{\theta(q^2;x^2/\alpha^2)}\left[\frac{1}{|x\alpha|}P_1(x)+\frac{|A|}{|\alpha|^3}P_2(x)\right] \tag{232}$$

Then squaring both sides of (232) we obtain that:

$$|\hat{y}(x)|^2 \leq \frac{|\tilde{C}_q|^2}{\theta^2(q^2;x^2/\alpha^2)}\left[\frac{P_1^2(x)}{x^2\alpha^2}+\frac{2|A|P_1(x)P_2(x)}{|x|\alpha^4}+\frac{A^2P_2^2(x)}{\alpha^6}\right]. \tag{233}$$

Now let us study $\theta^2(q^2;x^2/\alpha^2)$. Recall from (31) that, $\theta(q;x)=\sum_{n=-\infty}^{\infty}\frac{x^n}{q^{n(n-1)/2}}$ and therefore $\theta(q^2;x^2/\alpha^2)=\sum_{n=-\infty}^{\infty}\frac{(x^2/\alpha^2)^n}{(q^2)^{n(n-1)/2}}=\sum_{n=-\infty}^{\infty}\frac{x^{2n}}{\alpha^{2n}q^{n(n-1)}}$. Now notice that every term in $\theta(q^2;x^2/\alpha^2)>0$ and therefore we can isolate one term, say $n=-2$, and get that

$$\theta(q^2;x^2/\alpha^2) \geq \frac{x^{2\cdot(-2)}}{\alpha^{2\cdot(-2)}q^{-2(-2-1)}}=\frac{\alpha^4}{x^4q^6}. \tag{234}$$

Since everything in (234) is non-negative, when we square both sides we preserve the inequality; so from (234) we obtain that

$$\theta^2(q^2; x^2/\alpha^2) \geq \frac{\alpha^8}{x^8 q^{12}}. \tag{235}$$

Now taking the reciprocal of (235) gives:

$$\frac{1}{\theta^2(q^2; x^2/\alpha^2)} \leq \frac{x^8 q^{12}}{\alpha^8} \tag{236}$$

Looking back to (233) and using (236) to bound the $\frac{1}{\theta^2(q^2; x^2/\alpha^2)}$ term and multiplying by a $\frac{1}{|x|}$ we get that:

$$\frac{|\hat{y}(x)|^2}{|x|} \leq \frac{|\tilde{C}_q|^2 x^8 q^{12}}{|x|\alpha^8}\left[\frac{P_1^2(x)}{x^2\alpha^2} + \frac{2|A|P_1(x)P_2(x)}{|x|\alpha^4} + \frac{A^2 P_2^2(x)}{\alpha^6}\right] \tag{237}$$

$$= \frac{|\tilde{C}_q|^2 q^{12}}{\alpha^8}\left[\frac{|x^5|P_1^2(x)}{\alpha^2} + \frac{2|A|x^6 P_1(x)P_2(x)}{\alpha^4} + \frac{|x^7|A^2 P_2^2(x)}{\alpha^6}\right]. \tag{238}$$

Now it is clear that since $\lim_{x\to 0} \frac{|\hat{y}(x)|^2}{|x|} = 0$ the integrals $\int_{-1}^{0} \frac{|\hat{y}(x)|^2}{|x|}\, dx$ and $\int_{0}^{1} \frac{|\hat{y}(x)|^2}{|x|}\, dx$ converge. Thus we have that $\int_{-\infty}^{\infty} \frac{|\hat{y}(x)|^2}{|x|}\, dx < \infty$, and that confirms (27), the third and last condition for $y(t)$ to be a wavelet. Therefore our solution $y(t)$, (103)-(104), is a Schwartz wavelet as desired. In order to show that $y(t)$ has all moments vanishing notice that from (5), (10) with (18), (17) one sees analogous to (137)-(138) and (139)-(140) that a $(2k)^{th}$ anti-derivative of $y(t)$ is given by

$$y^{(-2k)}(t)$$

$$= \frac{(-1)^k q^{k(k+1)}}{\alpha^{2k+1}} \sum_{n=0}^{\infty} (-1)^n \frac{A^{2n}}{\alpha^{2n} q^{2kn}} \frac{1}{q^{n^2}} \, {}_qSin\left(q^{-k}\frac{q^{2n}\alpha t + \beta}{q^{n+1}}\right) \tag{239}$$

$$+ \frac{(-1)^{k+1} A q^{k^2}}{\alpha^{2k+2}} \sum_{n=0}^{\infty} (-1)^n \frac{A^{2n}}{\alpha^{2n} q^{2kn}} \frac{1}{q^{n(n+1)}} \, {}_qCos\left(q^{-k}\frac{q^{2n+1}\alpha t + \beta}{q^{n+1}}\right) \tag{240}$$

and a $(2k+1)^{st}$ anti-derivative of $y(t)$ is given by

$$\begin{aligned} & y^{(-[2k+1])}(t) \\ = \quad & \frac{(-1)^k q^{(k+1)^2}}{\alpha^{2k+2}} \sum_{n=0}^{\infty} (-1)^n \frac{A^{2n}}{\alpha^{2n} q^{2kn}} \frac{1}{q^{n(n+1)}} \, {}_qCos\left(q^{-k} \frac{q^{2n}\alpha t + \beta}{q^{n+1}} \right) \qquad (241) \\ & + \frac{(-1)^{k+1} A q^{k(k+1)}}{\alpha^{2k+3}} \sum_{n=0}^{\infty} (-1)^n \frac{A^{2n}}{\alpha^{2n} q^{2kn}} \frac{1}{q^{n(n+2)}} \, {}_qSin\left(q^{-k-1} \frac{q^{2n}\alpha t + \beta}{q^{n+1}} \right) \qquad (242) \end{aligned}$$

Now from (239)-(240) and (241)-(242) we see, via an argument entirely analogous to that given in Proposition 6, that each of the anti-derivatives $y^{(-n)}(t)$ is Schwartz. Thus for $y(t)$ our solution to $y'(t) - Ay(qt) = {}_qCos(\alpha t + \beta)$, we have for each $m \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} & \int_{-\infty}^{\infty} x^m y(t) dt \\ = \quad & x^m y^{(-1)}(t) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} m x^{m-1} y^{(-1)}(t) dt \\ = \quad & 0 - m x^{m-1} y^{(-2)}(t) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} m(m-1) x^{m-2} y^{(-2)}(t) dt \\ & \vdots \\ = \quad & (-1)^k m(m-1)\ldots(m-[k-1]) \int_{-\infty}^{\infty} x^{m-k} y^{(-k)}(t) dt \\ & \vdots \\ = \quad & (-1)^m m! \int_{-\infty}^{\infty} y^{(-m)}(t) dt \\ = \quad & (-1)^m m! \, y^{(-m-1)}(t) \Big|_{-\infty}^{\infty} = 0 \end{aligned}$$

This is the definition of having all moments vanishing and therefore completes the proof that $y(t)$, (103)-(104) is a Schwartz wavelet with all moments vanishing. A similar argument showing the solution $y(t)$, (105)-(106), is a Schwartz wavelet with all moments vanishing holds. □

Now we provide some pictures of our solution in the case of $y'(t) - Ay(qt) = {}_qSin(\alpha t + \beta)$, where $A = 2$, $\alpha = 1$, $\beta = 0$, and q takes on several values that get closer to 1.

In Figure 1 - Figure 4 below the red curve is the forcing $f(t) = {}_qSin(t)$ with q taking on the value given in the figure's caption.

The green curve is a normal $sin(t)$.

The light blue curve is the classical solution to the non-advanced differential equation, $y'(t) - 2y(t) = sin(t)$.

The dark blue curve is our $y(t)$ solution to the MADE, $y'(t) - 2y(qt) = {}_qSin(t)$, with q taking on the value given in the figure's caption.

As seen in Figure 1 - Figure 4 below, as $q \to 1$ it appears that our solution is confluencing towards the classical, non-advanced solution to the differential equation $y'(t) - Ay(t) = sin(\alpha t + \beta)$ for $A = 2$, $\alpha = 1$, and $\beta = 0$.

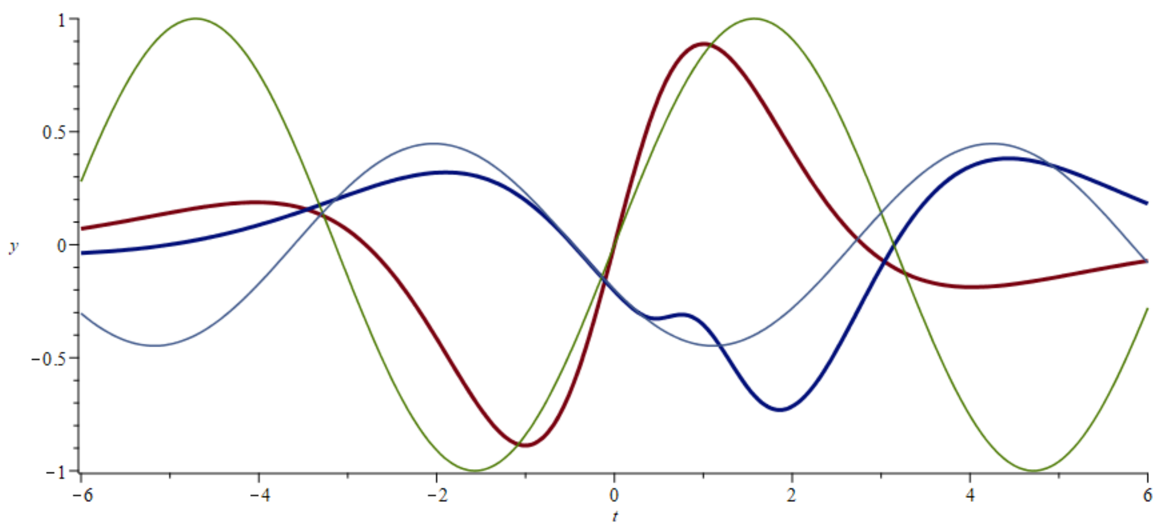

Figure 1: ${}_qSin$ forcing solution with q = 1.5

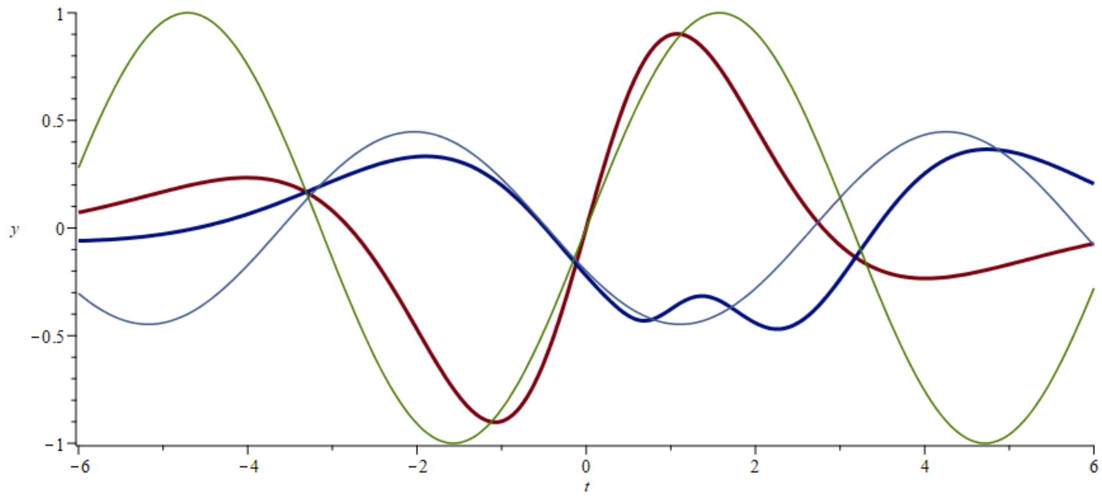

Figure 2: ${}_qSin$ forcing solution with q = 1.4

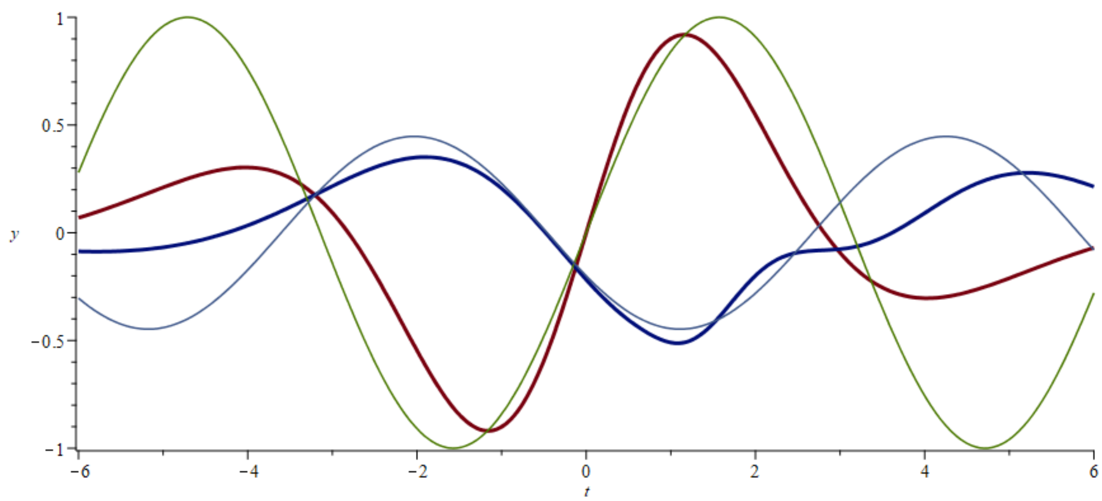

Figure 3: ${}_qSin$ forcing solution with q = 1.3

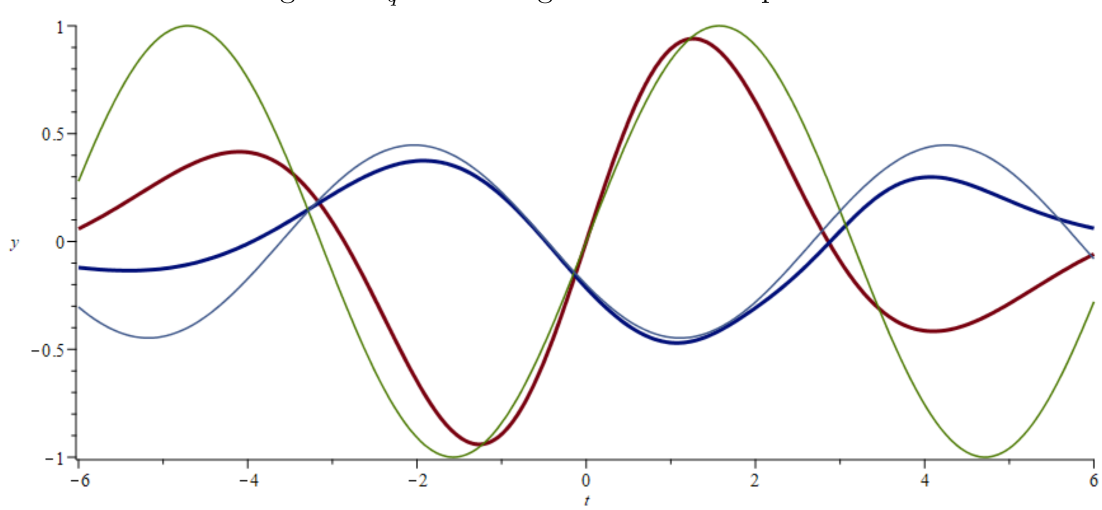

Figure 4: ${}_qSin$ forcing solution with q = 1.2

4 Applying our results to Wavelet frames to get a solution for a general forcing term in Ł^2

Now we apply the above results to wavelets expansions so that we may solve (53) for a larger class of functions $f(t)$. We will do this by expressing our forcing term, $f(t)$, with frames. We rely on two frames, one using ${}_qCos(t)$ and the other using ${}_qSin(t)$ as the mother wavelets. Specifically our two choices of frames $\Phi_{N,M}$ are:

$$\left\{ \Phi_{N,M} = q^{\frac{N}{2}}\, {}_qCos(q^N t - Mb) \,\middle|\, N, M \in \mathbb{Z} \right\} \tag{243}$$

$$\text{or} \qquad \left\{ \Phi_{N,M} = q^{\frac{N}{2}}\, {}_qSin(q^N t - Mb) \,\middle|\, N, M \in \mathbb{Z} \right\}. \tag{244}$$

Now from [PRS3] we have that for any function $f(t) \in Ł^2(\mathbb{R})$, we may write $f(t)$ as:

$$f(t) = \sum_N \sum_M c_{N,M} \Phi_{N,M}. \tag{245}$$

Then from (53), our MADE becomes

$$y'(t) - Ay(qt) = f(t) \tag{246}$$

$$= \sum_N \sum_M c_{N,M} \Phi_{N,M}. \tag{247}$$

Now using the (243) expansion of $\Phi_{N,M}$, (246)-(247) gives that:

$$y'(t) - Ay(qt) = f(t) \tag{248}$$

$$= \sum_N \sum_M c_{N,M} q^{\frac{N}{2}}\, {}_qCos(q^N t - Mb). \tag{249}$$

Now integrating (248)-(249) from $-\infty$ to t and relying on the same $\tilde{K}$ operator as defined in (58) and assuming the exchange of the integral in $\tilde{K}$ with $\sum_N \sum_M$ yields:

$$\begin{aligned} y(t) - \tilde{K}[y](t) &= \sum_N \sum_M c_{N,M}\, q^{\frac{N}{2}} \int_{-\infty}^{t} {}_qCos(q^N t - Mb) \\ &= \sum_N \sum_M c_{N,M} \frac{q^{\frac{N}{2}}}{q^N}\, {}_qSin\left(\frac{q^N t - Mb}{q}\right), \end{aligned} \tag{250}$$

where (18) was used to obtain (250).

Now applying (60) to (250) and assuming the exchange of $\sum_{n=0}^{\infty} \tilde{K}^n$ with $\sum_N \sum_M$ gives that:

$$y(t) = \sum_N \sum_M c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} \sum_{n=0}^{\infty} \tilde{K}^n \left[{}_qSin\left(\frac{q^N u - Mb}{q}\right)\right](t)$$

$$= \sum_N \sum_M c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} \sum_{n=0}^{\infty} \tilde{K}^{2n} \left[{}_qSin\left(\frac{q^N u - Mb}{q}\right)\right](t) \tag{251}$$

$$+ \sum_N \sum_M c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} \sum_{n=0}^{\infty} \tilde{K}^{2n+1} \left[{}_qSin\left(\frac{q^N u - Mb}{q}\right)\right](t). \tag{252}$$

Now using Proposition 3, and applying (79) to (251) and (80) to (252) we get a formal solution to (248)-(249) of the form:

$$y(t)$$

$$= \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^n \left(\frac{A}{q^N}\right)^{2n} \left(\frac{1}{q^{n^2}}\right) {}_qSin\left(\frac{q^{2n} q^N t - Mb}{q^{n+1}}\right) \tag{253}$$

$$+ \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{n+1} \left(\frac{A}{q^N}\right)^{2n+1} \left(\frac{1}{q^{n(n+1)}}\right) {}_qCos\left(\frac{q^{2n} q^{N+1} t - Mb}{q^{n+1}}\right). \tag{254}$$

Next we choose (244) as our frame. Then (246)-(247) becomes:

$$y'(t) - Ay(qt) = f(t) \tag{255}$$

$$= \sum_N \sum_M c_{N,M} q^{\frac{N}{2}}\, {}_qSin(q^N t - Mb). \tag{256}$$

Now integrating (255)-(256) from $-\infty$ to t and relying on the same $\tilde{K}$ operator as defined in (58) and assuming the exchange of the integral in $\tilde{K}$ with $\sum_N \sum_M$ yields:

$$\begin{aligned} y(t) - \tilde{K}[y](t) &= \sum_N \sum_M c_{N,M}\, q^{\frac{N}{2}} \int_{-\infty}^{t} {}_qSin(q^N t - Mb) \\ &= \sum_N \sum_M c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} \left[-{}_qCos(q^N t - Mb) \right], \end{aligned} \tag{257}$$

where (17) was used to obtain (257).

Now using (60) and assuming the exchange of $\sum_{n=0}^{\infty} \tilde{K}^n$ with $\sum_N \sum_M$ we have that:

$$\begin{aligned} y(t) &= \sum_N \sum_M c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} \sum_{n=0}^{\infty} \tilde{K}^n \left[-{}_qCos(q^N u - Mb) \right](t) \\ &= \sum_N \sum_M c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} \sum_{n=0}^{\infty} \tilde{K}^{2n} \left[-{}_qCos(q^N u - Mb) \right](t) && (258) \\ &\quad + \sum_N \sum_M c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} \sum_{n=0}^{\infty} \tilde{K}^{2n+1} \left[-{}_qCos(q^N u - Mb) \right] && (259) \end{aligned}$$

Now using Proposition 2, specifically applying (65) to (258) and (66) to (259) we get that a formal solution to (255)-(256) is of the form:

$$\begin{aligned} &y(t) \\ &= \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{n+1} \left(\frac{A}{q^N} \right)^{2n} \left(\frac{1}{q^{n(n+1)}} \right) {}_qCos\left(\frac{q^{2n} q^N t - Mb}{q^n} \right) && (260) \\ &\quad + \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{n+1} \left(\frac{A}{q^N} \right)^{2n+1} \left(\frac{1}{q^{(n+1)^2}} \right) {}_qSin\left(\frac{q^{2n} q^{N+1} t - Mb}{q^{n+1}} \right). && (261) \end{aligned}$$

We will now check that our solution $y(t)$, in (260)-(261), does formally solve (53) by differentiating $y(t)$. Specifically we will check that $y'(t) = Ay(qt) + f(t)$. So we compute $y'(t)$ formally from (260)-(261) assuming that we may pass the derivatives through the infinite sums. We have:

$$
\begin{aligned}
& y'(t) \\
= & \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{n+1} \left(\frac{A}{q^N}\right)^{2n} \left(\frac{1}{q^{n(n+1)}}\right) && (262)\\
& \cdot \left[{}_qCos\left(\frac{q^{2n}q^N t - Mb}{q^n}\right)\right]' && (263)\\
& + \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{n+1} \left(\frac{A}{q^N}\right)^{2n+1} \left(\frac{1}{q^{(n+1)^2}}\right) && (264)\\
& \cdot \left[{}_qSin\left(\frac{q^{2n}q^{N+1} t - Mb}{q^{n+1}}\right)\right]' && (265)\\
= & \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{n+1} \left(\frac{A}{q^N}\right)^{2n} \left(\frac{1}{q^{n(n+1)}}\right) && (266)\\
& \cdot \left[-{}_qSin\left(\frac{q^{2n}q^N t - Mb}{q^n}\right)(q^n q^N)\right] && (267)\\
& + \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{n+1} \left(\frac{A}{q^N}\right)^{2n+1} \left(\frac{1}{q^{(n+1)^2}}\right) \\
& \cdot \left[(q)_q Cos\left(\frac{q^{2n}q^{N+1} t - Mb}{q^n}\right)(q^n q^N)\right] \\
= & \sum_N \sum_M c_{N,M} q^{\frac{N}{2}} {}_qSin(q^N t - Mb) && (268)\\
& + \sum_N \sum_M \sum_{n=1}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{n} \left(\frac{A}{q^N}\right)^{2n} \left(\frac{1}{q^{n^2}}\right)(q^N) && (269)\\
& \cdot \left[{}_qSin\left(\frac{q^{2n}q^N t - Mb}{q^n}\right)\right] && (270)\\
& + \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{n+1} \left(\frac{A}{q^N}\right)^{2n+1} \left(\frac{1}{q^{n(n+1)}}\right)(q^N) \\
& \cdot \left[{}_qCos\left(\frac{q^{2n}q^{N+1} t - Mb}{q^n}\right)\right].
\end{aligned}
$$

Here we have separated the sum over n in (266)-(267) into the $n = 0$ and $n \geq 1$ cases in (268) and (269)-(270) respectively. Now note that (268) is exactly $f(t)$ and then in (269)-(270) making a change of variables letting $n = m + 1$ gives:

$$
\begin{aligned}
& y'(t) \\
= \; & f(t) \\
& + \sum_N \sum_M \sum_{m=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{m+1} \left(\frac{A}{q^N} \right)^{2(m+1)} \left(\frac{1}{q^{(m+1)^2}} \right) (q^N) \\
& \qquad \cdot \left[{}_qSin \left(\frac{q^{2(m+1)} q^N t - Mb}{q^{m+1}} \right) \right] \\
& + \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{n+1} \left(\frac{A}{q^N} \right)^{2n+1} \left(\frac{1}{q^{(n(n+1)}} \right) (q^N) \\
& \qquad \cdot \left[{}_qCos \left(\frac{q^{2n} q^{N+1} t - Mb}{q^n} \right) \right] \\
= \; & f(t) \\
& + A \sum_N \sum_M \sum_{m=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{m+1} \left(\frac{A}{q^N} \right)^{2m+1} \left(\frac{1}{q^{(m+1)^2}} \right) \\
& \qquad \cdot \left[{}_qSin \left(\frac{q^{2m} q^{N+1} (qt) - Mb}{q^{m+1}} \right) \right] \\
& + A \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{n+1} \left(\frac{A}{q^N} \right)^{2n} \left(\frac{1}{q^{(n(n+1)}} \right) \\
& \qquad \cdot \left[{}_qCos \left(\frac{q^{2n} q^N (qt) - Mb}{q^n} \right) \right] \\
= \; & f(t) + Ay(qt)
\end{aligned}
$$

as desired.

Now to check that (253)-(254) is a formal solution to (53) we will again compute $y'(t)$ formally from (253)-(254). We will assume that we are allowed to exchange the derivatives with the infinite sums in (253)-(254) while we compute $y'(t)$ formally from

(253)-(254). We have:

$$\begin{aligned}
& y'(t) \\
= & \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^n \left(\frac{A}{q^N}\right)^{2n} \left(\frac{1}{q^{n^2}}\right) && (271) \\
& \cdot \left[{}_qSin\left(\frac{q^{2n}q^N t - Mb}{q^{n+1}}\right)\right]' && (272) \\
& + \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{n+1} \left(\frac{A}{q^N}\right)^{2n+1} \left(\frac{1}{q^{n(n+1)}}\right) && (273) \\
& \cdot \left[{}_qCos\left(\frac{q^{2n}q^{N+1} t - Mb}{q^{n+1}}\right)\right]' && (274) \\
= & \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^n \left(\frac{A}{q^N}\right)^{2n} \left(\frac{1}{q^{n^2}}\right) && (275) \\
& \cdot \left[q_qCos\left(\frac{q^{2n}q^N t - Mb}{q^{n}}\right)\left(q^{n-1}q^N\right)\right] && (276) \\
& + \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{n+1} \left(\frac{A}{q^N}\right)^{2n+1} \left(\frac{1}{q^{n(n+1)}}\right) \\
& \cdot \left[-{}_qSin\left(\frac{q^{2n}q^{N+1} t - Mb}{q^{n+1}}\right)\left(q^{n}q^N\right)\right] \\
= & \sum_N \sum_M c_{N,M} q^{\frac{N}{2}} {}_qCos(q^N t - Mb) && (277) \\
& + \sum_N \sum_M \sum_{n=1}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^n \left(\frac{A}{q^N}\right)^{2n} \left(\frac{1}{q^{n(n-1)}}\right)\left(q^N\right) && (278) \\
& \cdot \left[{}_qCos\left(\frac{q^{2n}q^N t - Mb}{q^{n}}\right)\right] && (279) \\
& + \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^n \left(\frac{A}{q^N}\right)^{2n+1} \left(\frac{1}{q^{n^2}}\right)\left(q^N\right) \\
& \cdot \left[{}_qSin\left(\frac{q^{2n}q^{N+1} t - Mb}{q^{n+1}}\right)\right].
\end{aligned}$$

Here we have separated the sum over n in (275)-(276) into the $n = 0$ and $n \geq 1$ cases in (277) and (278)-(279) respectively. Now note that (277) is exactly $f(t)$ and then in (278)-(279) making a change of variables letting $n = m + 1$ gives:

$$
\begin{aligned}
& y'(t) \\
= \; & f(t) \\
& + \sum_N \sum_M \sum_{m=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{m+1} \left(\frac{A}{q^N} \right)^{2(m+1)} \left(\frac{1}{q^{m(m+1)}} \right) (q^N) \\
& \qquad \cdot \left[{}_qCos \left(\frac{q^{2(m+1)} q^N t - Mb}{q^{m+1}} \right) \right] \\
& + \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^n \left(\frac{A}{q^N} \right)^{2n+1} \left(\frac{1}{q^{n^2}} \right) (q^N) \\
& \qquad \cdot \left[{}_qSin \left(\frac{q^{2n} q^{N+1} t - Mb}{q^{n+1}} \right) \right] \\
= \; & f(t) \\
& + A \sum_N \sum_M \sum_{m=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{m+1} \left(\frac{A}{q^N} \right)^{2m+1} \left(\frac{1}{q^{m(m+1)}} \right) \\
& \qquad \cdot \left[{}_qCos \left(\frac{q^{2m} q^{N+1} (qt) - Mb}{q^{m+1}} \right) \right] \\
& + A \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{n+1} \left(\frac{A}{q^N} \right)^{2n} \left(\frac{1}{q^{n^2}} \right) \\
& \qquad \cdot \left[{}_qSin \left(\frac{q^{2n} q^N (qt) - Mb}{q^{n+1}} \right) \right] \\
= \; & f(t) + Ay(qt)
\end{aligned}
$$

as desired.

This completes both of the checks that (260)-(261) and (253)-(254) are both formal solutions to (53).

Now we will introduce some restrictions on the coefficients of the $c_{N,M}$ to ensure that all of our sums have absolute and uniform convergence, thus ensuring that the derivatives can pass through the sums in (262)-(265) and (271)-(274).

Now to ensure that our solution $y(t)$ absolutely and uniformly converges, we will start to bound $|y(t)|$, via bounding the sum of the absolute values of the summands in the formal solutions (253)-(254) and (260)-(261). This will imply both absolute and uniform converge of both (253)-(254) and (260)-(261). First we look at the solution (260)-(261) which is from the case where $\phi_{N,M} = q^{N/2}\,{}_qSin(q^N t - Mb)$.
From (260)-(261) we can see that the absolute value of our solution is:

$$\begin{aligned} &|y(t)| \\ =\ & \Bigg|\sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{n+1} \left(\frac{A}{q^N}\right)^{2n} \left(\frac{1}{q^{n(n+1)}}\right) {}_qCos\left(\frac{q^{2n}q^N t - Mb}{q^n}\right) \quad (280) \\ &+ \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{n+1} \left(\frac{A}{q^N}\right)^{2n+1} \left(\frac{1}{q^{(n+1)^2}}\right) {}_qSin\left(\frac{q^{2n}q^{N+1} t - Mb}{q^{n+1}}\right)\Bigg| \quad (281) \end{aligned}$$

Then applying the triangle inequality and the bounds (42) and (43) respectively to (280)-(281) gives that,

$$\begin{aligned} &|y(t)| \\ \leq\ & \sum_N \sum_M \sum_{n=0}^{\infty} \frac{|c_{N,M}|}{q^{N/2}} \left(\frac{|A|^{2n}}{q^{2Nn}}\right) \left(\frac{1}{q^{n(n+1)}}\right) \\ +\ & \sum_N \sum_M \sum_{n=0}^{\infty} \frac{|c_{N,M}|}{q^{N/2}} \left(\frac{|A|}{q^N}\right) \left(\frac{|A|^{2n}}{q^{2Nn}}\right) \left(\frac{1}{q^{n^2+2n+1}}\right) (q^{1/2}). \\ =\ & \sum_N \sum_M \frac{|c_{N,M}|}{q^{N/2}} \sum_{n=0}^{\infty} \frac{|A|^{2n}}{q^{2Nn+n}} \left(\frac{1}{q^{n^2}}\right) \quad (282) \\ +\ & \frac{|A|}{q^{1/2}} \sum_N \sum_M \frac{|c_{N,M}|}{q^{3N/2}} \sum_{n=0}^{\infty} \frac{|A|^{2n}}{q^{2Nn+2n}} \left(\frac{1}{q^{n^2}}\right). \quad (283) \end{aligned}$$

Now we take each sum only over n in (282)-(283) and work to bound it using a completion of squares in (285)-(286) below. First the sum in (282):

$$\sum_{n=0}^{\infty} \frac{|A|^{2n}}{q^{2Nn+n}} \left(\frac{1}{q^{n^2}}\right) \tag{284}$$

$$= \sum_{n=0}^{\infty} \exp\left[n \cdot \ln\left(\frac{|A|^2}{q^{2N+1}}\right)\right] \exp\left[-n^2 \cdot \ln(q)\right]$$

$$= \sum_{n=0}^{\infty} \exp\left[-\ln(q)\left(n^2 - \frac{n}{\ln(q)} \cdot \ln\left[\frac{|A|^2}{q^{2N+1}}\right]\right)\right]$$

$$= \sum_{n=0}^{\infty} \exp\left[-\ln(q)\left(n^2 - \frac{n}{\ln(q)} \cdot \ln\left[\frac{|A|^2}{q^{2N+1}}\right] + \left(\frac{\ln\left(|A|^2 q^{-2N-1}\right)}{2\ln(q)}\right)^2\right)\right] \tag{285}$$

$$\cdot \exp\left[\ln(q)\left(\frac{\ln\left(|A|^2 q^{-2N-1}\right)}{2\ln(q)}\right)^2\right] \tag{286}$$

$$= \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)}\right] \sum_{n=0}^{\infty} \exp\left[-\ln(q)\left(\left[n - \frac{\ln\left(|A|^2 q^{-2N-1}\right)}{2\ln(q)}\right]^2\right)\right]$$

$$= \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)}\right] \sum_{n=0}^{\infty} \exp\left[\frac{-\left(n - \frac{\ln\left(|A|^2 q^{-2N-1}\right)}{2\ln(q)}\right)^2}{\left(\frac{1}{\ln(q)}\right)}\right]$$

$$= \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)}\right] \sum_{n=0}^{\infty} \exp\left[\frac{-\frac{1}{2}\left(n - \frac{\ln\left(|A|^2 q^{-2N-1}\right)}{2\ln(q)}\right)^2}{\left(\frac{1}{\sqrt{2\ln(q)}}\right)^2}\right] \tag{287}$$

$$\leq \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)}\right]\left[1 + \sqrt{2\pi}\left(\frac{1}{\sqrt{2\ln(q)}}\right)\right] \tag{288}$$

$$= \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)}\right]\left[1 + \sqrt{\frac{\pi}{\ln(q)}}\right]. \tag{289}$$

Note that the expression (287) is a Gaussian with mean $\frac{\ln(|A|^2 q^{-2N-1})}{2\ln(q)}$ and standard deviation $\frac{1}{\sqrt{2\ln(q)}}$. Furthermore the bound in (288) follows from Proposition 1. Note that the mixed term q^{2Nn} in (284) has been handled via the completion of squares in (285)-(286) and the resulting Gaussian bound in (288). Also notice that the dependence on n is gone in (289).

Now we will do the same for the sum over n in (283), again using a completion of squares in (291)-(292) below.

$$\sum_{n=0}^{\infty} \frac{|A|^{2n}}{q^{2Nn+2n}} \left(\frac{1}{q^{n^2}} \right) \tag{290}$$

$$= \sum_{n=0}^{\infty} \exp\left[n \cdot \ln\left(\frac{|A|^2}{q^{2N+2}} \right) \right] \exp\left[-n^2 \cdot \ln(q) \right]$$

$$= \sum_{n=0}^{\infty} \exp\left[-\ln(q) \left(n^2 - \frac{n}{\ln(q)} \cdot \ln\left[\frac{|A|^2}{q^{2N+2}} \right] \right) \right]$$

$$= \sum_{n=0}^{\infty} \exp\left[-\ln(q) \left(n^2 - \frac{n}{\ln(q)} \cdot \ln\left[\frac{|A|^2}{q^{2N+2}} \right] + \left(\frac{\ln\left(|A|^2 q^{-2N-2}\right)}{2\ln(q)} \right)^2 \right) \right] \tag{291}$$

$$\cdot \exp\left[\ln(q) \left(\frac{\ln\left(|A|^2 q^{-2N-2}\right)}{2\ln(q)} \right)^2 \right] \tag{292}$$

$$= \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-2}\right)}{4\ln(q)} \right] \sum_{n=0}^{\infty} \exp\left[-\ln(q) \left(\left[n - \frac{\ln\left(|A|^2 q^{-2N-2}\right)}{2\ln(q)} \right]^2 \right) \right]$$

$$= \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-2}\right)}{4\ln(q)} \right] \sum_{n=0}^{\infty} \exp\left[\frac{-\left(n - \frac{\ln\left(|A|^2 q^{-2N-2}\right)}{2\ln(q)} \right)^2}{\left(\frac{1}{\ln(q)} \right)} \right]$$

$$= \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-2}\right)}{4\ln(q)} \right] \sum_{n=0}^{\infty} \exp\left[\frac{-\frac{1}{2}\left(n - \frac{\ln\left(|A|^2 q^{-2N-2}\right)}{2\ln(q)} \right)^2}{\left(\frac{1}{\sqrt{2\ln(q)}} \right)^2} \right] \tag{293}$$

$$\leq \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-2}\right)}{4\ln(q)} \right] \left[1 + \sqrt{2\pi} \left(\frac{1}{\sqrt{2\ln(q)}} \right) \right] \tag{294}$$

$$= \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-2}\right)}{4\ln(q)} \right] \left[1 + \sqrt{\frac{\pi}{\ln(q)}} \right]. \tag{295}$$

Again notice that the expression in (293) is a Gaussian with mean $\frac{\ln(|A|^2 q^{-2N-2})}{2\ln(q)}$ and standard deviation $\frac{1}{\sqrt{2\ln(q)}}$. Then the bound in (294) follows from Proposition 1. Once more the dependence on n is gone in (295) and the mixed term q^{2Nn} in (290) has been absorbed by the Gaussian bound (294). Now that we have bounded these terms we can look back to (282)-(283) and say that:

$$\begin{aligned} &|y(t)| \\ \leq\ & \left[1+\sqrt{\frac{\pi}{\ln(q)}}\right] \sum_N \sum_M \frac{|c_{N,M}|}{q^{N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)}\right] \qquad (296) \\ +\ & \frac{|A|}{q^{1/2}} \left[1+\sqrt{\frac{\pi}{\ln(q)}}\right] \sum_N \sum_M \frac{|c_{N,M}|}{q^{3N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-2}\right)}{4\ln(q)}\right]. \qquad (297) \end{aligned}$$

Now we will apply the same completion of squares process on the summations over the N's in (296)-(297). This will allow us to obtain conditions on our $c_{N,M}$'s that will be sufficient for (296)-(297) to converge absolutely and uniformly.

Starting with the sums in (296) we have:

$$\begin{aligned} & \sum_N \sum_M \frac{|c_{N,M}|}{q^{N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)}\right] \\ =& \sum_N \sum_M |c_{N,M}| \\ & \quad \cdot \exp\left[-\frac{N}{2}\cdot\ln(q)\right] \exp\left[\frac{\left[-2N\ln(q)+\ln(|A|^2q^{-1})\right]^2}{4\ln(q)}\right] \\ =& \sum_N \sum_M |c_{N,M}| \\ & \quad \cdot \exp\left[\frac{4N^2\ln^2(q) - 4N\ln(|A|^2q^{-1})\ln(q) + \ln^2(|A|^2q^{-1}) - 2N\ln^2(q)}{4\ln(q)}\right] \\ =& \sum_N \sum_M |c_{N,M}| \qquad (298) \\ & \quad \cdot \exp\left[N^2\ln(q) - \frac{N\ln(|A|^2q^{-1})\ln(q)}{\ln(q)} + \frac{\ln^2(|A|^2q^{-1})}{4\ln(q)} - \frac{N\ln(q)}{2}\right]. \qquad (299) \end{aligned}$$

At this point we see that $|c_{N,M}|$ must decay in N faster than the exponential $\exp\left[N^2\ln(q)\right]$ to gain convergence. We will make the assumption that:

$$|c_{N,M}| \leq \exp[-\delta(N-\mu)^2]\cdot g(M) \ \text{, where } \ \delta > \ln(q) \qquad (300)$$

$$\text{and} \qquad \sum_M |g(M)| = B < \infty. \qquad (301)$$

Then continuing from (298)-(299) and utilizing our new conditions for $|c_{N,M}|$ in (300) , we get that:

$$\sum_N \sum_M \frac{|c_{N,M}|}{q^{N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)}\right]$$
$$\leq \sum_M |g(M)| \sum_N \exp\left[-\delta N^2 + 2\delta N\mu - \delta\mu^2\right] \tag{302}$$
$$\cdot \exp\left[N^2\ln(q) - \frac{N\ln(|A|^2q^{-1})\ln(q)}{\ln(q)} + \frac{\ln^2(|A|^2q^{-1})}{4\ln(q)} - \frac{N\ln(q)}{2}\right] \tag{303}$$

Now using the bound B from (301), and combining like powers of N in (302)-(303) gives:

$$\sum_N \sum_M \frac{|c_{N,M}|}{q^{N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)}\right]$$
$$\leq B\sum_N \exp\left[N^2[\ln(q)-\delta] - N\left(\frac{-4\delta\mu\ln(q) + 2\ln(|A|^2q^{-1})\ln(q) + \ln^2(q)}{2\ln(q)}\right)\right] \tag{304}$$
$$\cdot \exp\left[\frac{\ln^2(|A|^2q^{-1}) - 4\delta\mu^2\ln(q)}{4\ln(q)}\right] \tag{305}$$
$$= B\cdot e^{C_1}\sum_N \exp\left[[\ln(q)-\delta]\left(N^2 - N\left(\frac{-4\delta\mu\ln(q) + 2\ln(|A|^2q^{-1})\ln(q) + \ln^2(q)}{2\ln(q)[\ln(q)-\delta]}\right)\right)\right] \tag{306}$$

Where in (306) $C_1 = \dfrac{\ln^2(|A|^2q^{-1}) - 4\delta\mu^2\ln(q)}{4\ln(q)}$. Now to complete the square on N from (306) gives:

$$\sum_N \sum_M \frac{|c_{N,M}|}{q^{N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)}\right]$$
$$\leq B\cdot e^{C_1}$$
$$\cdot \sum_N \exp\left[[\ln(q)-\delta]\left(\left[N - \left(\frac{-4\delta\mu\ln(q) + 2\ln(|A|^2q^{-1})\ln(q) + \ln^2(q)}{4\ln(q)[\ln(q)-\delta]}\right)\right]^2\right)\right] \tag{307}$$
$$\cdot \exp\left[-\frac{\left[-4\delta\mu\ln(q) + 2\ln(|A|^2q^{-1})\ln(q) + \ln^2(q)\right]^2}{16\ln^2(q)[\ln(q)-\delta]}\right] \tag{308}$$
$$= B\cdot e^{C_1}e^{C_2} \tag{309}$$
$$\cdot \sum_N \exp\left[[\ln(q)-\delta]\left(\left[N - \left(\frac{-4\delta\mu\ln(q) + 2\ln(|A|^2q^{-1})\ln(q) + \ln^2(q)}{4\ln(q)[\ln(q)-\delta]}\right)\right]^2\right)\right], \tag{310}$$

where in (309) $C_2 = \dfrac{-\left[-4\delta\mu\ln(q) + 2\ln(|A|^2q^{-1})\ln(q) + \ln^2(q)\right]^2}{16\ln^2(q)[\ln(q)-\delta]}$. Then continuing from (309)-(310) and denoting $C_3 = C_1 + C_2$ we write that

$$\begin{aligned}
&\sum_N\sum_M \frac{|c_{N,M}|}{q^{N/2}} \exp\left[\frac{\ln^2\left(|A|^2q^{-2N-1}\right)}{4\ln(q)}\right] \\
\leq\ & B\cdot e^{C_3}\sum_N \exp\left[\frac{-\frac{1}{2}\left[N-\left(\frac{-4\delta\mu\ln(q)+2\ln(|A|^2q^{-1})\ln(q)+\ln^2(q)}{4\ln(q)[\ln(q)-\delta]}\right)\right]^2}{\left(\frac{1}{\sqrt{2\delta-2\ln(q)}}\right)^2}\right] && (311)\\
\leq\ & Be^{C_3}\left[1+\sqrt{2\pi}\left(\frac{1}{\sqrt{2\delta-2\ln(q)}}\right)\right] && (312)\\
=\ & Be^{C_3}\left[1+\sqrt{\frac{\pi}{\delta-\ln(q)}}\right] < \infty && (313)
\end{aligned}$$

Now that we have bounded (296) by (313) which is finite, we will do the same for (297) in order to completely bound $|y(t)|$ and show absolute and uniform convergence of all the sums in $y(t)$ under the assumption of (300) and (301).

Recall the sums in (297) are of the form:

$$\begin{aligned}
&\sum_N\sum_M \frac{|c_{N,M}|}{q^{3N/2}} \exp\left[\frac{\ln^2\left(|A|^2q^{-2N-2}\right)}{4\ln(q)}\right] \\
=\ & \sum_N\sum_M |c_{N,M}| \exp\left[-\frac{3N}{2}\cdot\ln(q)\right]\exp\left[\frac{[-2N\ln(q)+\ln(|A|^2q^{-2})]^2}{4\ln(q)}\right] \\
=\ & \sum_N\sum_M |c_{N,M}| \\
&\cdot\exp\left[\frac{4N^2\ln^2(q)-4N\ln(|A|^2q^{-2})\ln(q)+\ln^2(|A|^2q^{-2})-6N\ln^2(q)}{4\ln(q)}\right] \\
=\ & \sum_N\sum_M |c_{N,M}| && (314)\\
&\cdot\exp\left[N^2\ln(q) - \frac{N\ln(|A|^2q^{-2})\ln(q)}{\ln(q)} + \frac{\ln^2(|A|^2q^{-2})}{4\ln(q)} - \frac{3N\ln(q)}{2}\right] && (315)
\end{aligned}$$

Again we see that $|c_{N,M}|$ must decay in N faster that an exponential with a leading coefficient of $N^2\ln(q)$ so we will make the same assumption on the $c_{N,M}$ as in (300)-(301), namely that:

$$|c_{N,M}| \leq \exp[-\delta(N-\mu)^2]\cdot g(M) \ \text{, where } \ \delta > \ln(q) \tag{316}$$

$$\text{and} \qquad \sum_M |g(M)| = B < \infty. \tag{317}$$

Then continuing from (314)-(315) and utilizing our condition for $|c_{N,M}|$ in (316), we get:

$$\sum_N\sum_M \frac{|c_{N,M}|}{q^{3N/2}}\exp\left[\frac{\ln^2\left(|A|^2q^{-2N-2}\right)}{4\ln(q)}\right] \tag{318}$$

$$\leq \sum_M |g(M)|\sum_N \exp\left[-\delta N^2+2\delta N\mu-\delta\mu^2\right] \tag{319}$$

$$\cdot\exp\left[N^2\ln(q)-\frac{N\ln(|A|^2q^{-2})\ln(q)}{\ln(q)}+\frac{\ln^2(|A|^2q^{-2})}{4\ln(q)}-\frac{3N\ln(q)}{2}\right] \tag{320}$$

Now using the bound (317) and combining like powers of N in (319)-(320) gives:

$$\sum_N\sum_M \frac{|c_{N,M}|}{q^{3N/2}}\exp\left[\frac{\ln^2\left(|A|^2q^{-2N-2}\right)}{4\ln(q)}\right]$$

$$\leq B\sum_N \exp\left[N^2[\ln(q)-\delta]-N\left(\frac{-4\delta\mu\ln(q)+2\ln(|A|^2q^{-2})\ln(q)+3\ln^2(q)}{2\ln(q)}\right)\right] \tag{321}$$

$$\cdot\exp\left[\frac{\ln^2(|A|^2q^{-2})-4\delta\mu^2\ln(q)}{4\ln(q)}\right] \tag{322}$$

$$= B\cdot e^{C_4}\sum_N \exp\left[[\ln(q)-\delta]\left(N^2-N\left(\frac{-4\delta\mu\ln(q)+2\ln(|A|^2q^{-2})\ln(q)+3\ln^2(q)}{2\ln(q)[\ln(q)-\delta]}\right)\right)\right] \tag{323}$$

Where in (323) $C_4 = \dfrac{\ln^2(|A|^2q^{-2})-4\delta\mu^2\ln(q)}{4\ln(q)}$. Now to complete the square on N in

(323) gives,

$$\begin{aligned}
&\sum_N \sum_M \frac{|c_{N,M}|}{q^{3N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-2}\right)}{4\ln(q)}\right] \\
&\leq B \cdot e^{C_4} \\
&\quad \cdot \sum_N \exp\left[[\ln(q)-\delta]\left(\left[N-\left(\frac{-4\delta\mu\ln(q)+2\ln(|A|^2q^{-2})\ln(q)+3\ln^2(q)}{4\ln(q)[\ln(q)-\delta]}\right)\right]^2\right)\right] \\
&\qquad \cdot \exp\left[-[\ln(q)-\delta]\left(\frac{-4\delta\mu\ln(q)+2\ln(|A|^2q^{-2})\ln(q)+3\ln^2(q)}{4\ln(q)[\ln(q)-\delta]}\right)^2\right] \\
&= B \cdot e^{C_4} e^{C_5} \qquad (324) \\
&\quad \cdot \sum_N \exp\left[[\ln(q)-\delta]\left[N-\left(\frac{-4\delta\mu\ln(q)+2\ln(|A|^2q^{-2})\ln(q)+3\ln^2(q)}{4\ln(q)[\ln(q)-\delta]}\right)\right]^2\right] \qquad (325)
\end{aligned}$$

Where in (324) $C_5 = \dfrac{-\left[-4\delta\mu\ln(q)+2\ln(|A|^2q^{-2})\ln(q)+3\ln^2(q)\right]^2}{16\ln^2(q)[\ln(q)-\delta]}$. Then continuing from (324) and denoting $C_6 = C_4 + C_5$ we get that,

$$\begin{aligned}
&\sum_N \sum_M \frac{|c_{N,M}|}{q^{3N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-2}\right)}{4\ln(q)}\right] \\
\leq\ & B \cdot e^{C_6} \sum_N \exp\left[\frac{-\frac{1}{2}\left[N-\left(\frac{-4\delta\mu\ln(q)+2\ln(|A|^2q^{-2})\ln(q)+3\ln^2(q)}{4\ln(q)[\ln(q)-\delta]}\right)\right]^2}{\left(\frac{1}{\sqrt{2\delta-2\ln(q)}}\right)^2}\right] \qquad (326) \\
\leq\ & Be^{C_6}\left[1+\sqrt{2\pi}\left(\frac{1}{\sqrt{2\delta-2\ln(q)}}\right)\right] \qquad (327) \\
=\ & Be^{C_6}\left[1+\sqrt{\frac{\pi}{\delta-\ln(q)}}\right] < \infty \qquad (328)
\end{aligned}$$

Where (327) was obtain by using Proposition 1 on the Gaussian in (326).

Now that we have bounded the sums in (297) by (328) we can look back to (296) and (297) and using the bounds (313) and (328) respectively we obtain the bound:

$$
\begin{aligned}
& |y(t)| \\
\leq \;& \left[1+\sqrt{\frac{\pi}{\ln(q)}}\right] \sum_N \sum_M \frac{|c_{N,M}|}{q^{N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)}\right] \\
& + \frac{|A|}{q^{1/2}} \left[1+\sqrt{\frac{\pi}{\ln(q)}}\right] \sum_N \sum_M \frac{|c_{N,M}|}{q^{3N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-2}\right)}{4\ln(q)}\right] \\
\leq \;& Be^{C_3}\left(1+\sqrt{\frac{\pi}{\delta-\ln(q)}}\right)\left(1+\sqrt{\frac{\pi}{\ln(q)}}\right) && (329) \\
& + \frac{|A|Be^{C_6}}{q^{1/2}}\left(1+\sqrt{\frac{\pi}{\delta-\ln(q)}}\right)\left(1+\sqrt{\frac{\pi}{\ln(q)}}\right) < \infty. && (330)
\end{aligned}
$$

Now that we have bounded $y(t)$ we need to bound the $y'(t)$ expression in order to use Theorem 1. Recall from (246)-(247) that,

$$
\begin{aligned}
y'(t) &= Ay(qt) + f(t) && (331) \\
&= Ay(qt) + \sum_N \sum_M c_{N,M}\Phi_{N,M} && (332)
\end{aligned}
$$

Clearly since we have already shown in (329)-(330) that $|y(t)| < \infty$ and converges uniformly and absolutely, it follows that $|Ay(qt)| < \infty$, with the series $Ay(qt)$ converging absolutely and uniformly as well. So we will focus on bounding the $f(t)$ term in (331)-(332), namely $\sum_N \sum_M c_{N,M}\Phi_{N,M}$.

Recall that for this version of $y(t)$ that (244) was our choice of $\Phi_{N,M}$ and then using

(316) for our condition on $c_{N,M}$ we can rely on (43) to obtain:

$$\begin{aligned}
& \left|\sum_N \sum_M c_{N,M} \Phi_{N,M}\right| && (333)\\
\leq \; & \sum_N \sum_M \exp\left[-\delta(N-\mu)^2\right] \cdot |g(M)| q^{\frac{N}{2}} \; \left|{}_q Sin(q^N t - Mb)\right| \\
\leq \; & q^{\frac{1}{2}} \sum_M |g(M)| \sum_N \exp\left[-\delta N^2 + 2\delta N\mu - \delta\mu^2\right] \exp\left[\frac{N}{2}\ln(q)\right] \\
\leq \; & Bq^{\frac{1}{2}} \sum_N \exp\left[-\delta\left(N^2 - N\left(\frac{4\delta\mu + \ln(q)}{2\delta}\right) + \mu^2\right)\right] \\
= \; & Bq^{\frac{1}{2}} \sum_N \exp\left[-\delta\left[\left(N - \frac{4\delta\mu + \ln(q)}{4\delta}\right)^2 - \left(\frac{4\delta\mu + \ln(q)}{4\delta}\right)^2 + \mu^2\right]\right] \\
= \; & Bq^{\frac{1}{2}} \cdot e^{C_7} \sum_N \exp\left[\frac{-\frac{1}{2}\left(N - \frac{4\delta\mu + \ln(q)}{4\delta}\right)^2}{\left(\frac{1}{\sqrt{2\delta}}\right)^2}\right] && (334)\\
\leq \; & Bq^{\frac{1}{2}} \cdot e^{C_7} \left[1 + \sqrt{2\pi}\left(\frac{1}{\sqrt{2\delta}}\right)\right] && (335)\\
= \; & Bq^{\frac{1}{2}} \cdot e^{C_7} \left[1 + \sqrt{\frac{\pi}{\delta}}\right] < \infty. && (336)
\end{aligned}$$

Here B is given in (317) and $C_7 = \dfrac{[4\delta\mu + \ln(q)]^2}{16\delta} - \delta\mu^2$, and Proposition 1 was used to obtain (335). Thus we can now say from using (333)-(336) in (331)-(332) that $|y'(t)| < \infty$. Specifically that, $y'(t)$ converges absolutely and uniformly on the whole real line. Thus we have shown that $y(t)$ also converges at each point on the real line. Then by Theorem 1 we are allowed to pass the derivatives through the infinite sums as we did in (262)-(265). This means that the solution for $y(t)$, (260)-(261), converges absolutely and uniformly on the entire real line as needed. This rigorously allows the exchange of derivatives with infinite sums and allows us to conclude that our solution (260)-(261) does indeed satisfy the MADE (53).

The above solution (260)-(261), $y(t)$, was for the case when (244) was chosen as the frame. We will now show that the solution in (253)-(254), $y(t)$, when the frame is $\Phi_{N,M} = q^{\frac{N}{2}}\,{}_qCos(q^N t - Mb)$, also converges absolutely and uniformly on the whole real line. We will start this by bounding $|y(t)|$. From (253)-(254) we can see that the absolute value of $y(t)$ is:

$$
\begin{aligned}
&|y(t)| \\
= \;& \left|\sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^n \left(\frac{A}{q^N}\right)^{2n} \left(\frac{1}{q^{n^2}}\right) {}_qSin\left(\frac{q^{2n} q^N t - Mb}{q^{n+1}}\right)\right. \qquad (337)\\
&\left. + \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{n+1} \left(\frac{A}{q^N}\right)^{2n+1} \left(\frac{1}{q^{n(n+1)}}\right) {}_qCos\left(\frac{q^{2n} q^{N+1} t - Mb}{q^{n+1}}\right)\right|. \qquad (338)
\end{aligned}
$$

Then applying the triangle inequality and the bounds (43) and (42) respectively to (337)-(338) gives that,

$$
\begin{aligned}
&|y(t)| \\
\leq \;& \sum_N \sum_M \sum_{n=0}^{\infty} \frac{|c_{N,M}|}{q^{N/2}} \left(\frac{|A|^{2n}}{q^{2Nn}}\right) \left(\frac{1}{q^{n^2}}\right) (q^{1/2}) \\
+ \;& \sum_N \sum_M \sum_{n=0}^{\infty} \frac{|c_{N,M}|}{q^{N/2}} \left(\frac{|A|}{q^N}\right) \left(\frac{|A|^{2n}}{q^{2Nn}}\right) \left(\frac{1}{q^{n^2+n}}\right). \\
= \;& (q^{1/2}) \sum_N \sum_M \frac{|c_{N,M}|}{q^{N/2}} \sum_{n=0}^{\infty} \frac{|A|^{2n}}{q^{2Nn}} \left(\frac{1}{q^{n^2}}\right) \qquad (339)\\
+ \;& |A| \sum_N \sum_M \frac{|c_{N,M}|}{q^{3N/2}} \sum_{n=0}^{\infty} \frac{|A|^{2n}}{q^{2Nn+n}} \left(\frac{1}{q^{n^2}}\right). \qquad (340)
\end{aligned}
$$

Now we take each sum only over n in (339)-(340) and work to bound it using a completion of squares in (341)-(342) below. First the sum in (339):

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{|A|^{2n}}{q^{2Nn}} \left(\frac{1}{q^{n^2}}\right) \\
&= \sum_{n=0}^{\infty} \exp\left[n \cdot \ln\left(\frac{|A|^2}{q^{2N}}\right)\right] \exp\left[-n^2 \cdot \ln(q)\right] \\
&= \sum_{n=0}^{\infty} \exp\left[-\ln(q)\left(n^2 - \frac{n}{\ln(q)} \cdot \ln\left[\frac{|A|^2}{q^{2N}}\right]\right)\right] \\
&= \sum_{n=0}^{\infty} \exp\left[-\ln(q)\left(n^2 - \frac{n}{\ln(q)} \cdot \ln\left[\frac{|A|^2}{q^{2N}}\right] + \left(\frac{\ln\left(|A|^2 q^{-2N}\right)}{2\ln(q)}\right)^2\right)\right] && (341) \\
&\qquad\qquad \cdot \exp\left[\ln(q)\left(\frac{\ln\left(|A|^2 q^{-2N}\right)}{2\ln(q)}\right)^2\right] && (342) \\
&= \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N}\right)}{4\ln(q)}\right] \sum_{n=0}^{\infty} \exp\left[-\ln(q)\left(\left[n - \frac{\ln\left(|A|^2 q^{-2N}\right)}{2\ln(q)}\right]^2\right)\right] \\
&= \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N}\right)}{4\ln(q)}\right] \sum_{n=0}^{\infty} \exp\left[\frac{-\left(n - \frac{\ln\left(|A|^2 q^{-2N}\right)}{2\ln(q)}\right)^2}{\left(\frac{1}{\ln(q)}\right)}\right] \\
&= \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N}\right)}{4\ln(q)}\right] \sum_{n=0}^{\infty} \exp\left[\frac{-\frac{1}{2}\left(n - \frac{\ln\left(|A|^2 q^{-2N}\right)}{2\ln(q)}\right)^2}{\left(\frac{1}{\sqrt{2\ln(q)}}\right)^2}\right] && (343) \\
&\leq \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N}\right)}{4\ln(q)}\right] \left[1 + \sqrt{2\pi}\left(\frac{1}{\sqrt{2\ln(q)}}\right)\right] && (344) \\
&= \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N}\right)}{4\ln(q)}\right] \left[1 + \sqrt{\frac{\pi}{\ln(q)}}\right]. && (345)
\end{aligned}$$

Note that the expression (343) is a Gaussian with mean $\frac{\ln(|A|^2 q^{-2N})}{2\ln(q)}$ and standard deviation $\frac{1}{\sqrt{2\ln(q)}}$. Furthermore the bound in (344) follows from Proposition 1. Note that the dependence on n is gone in (345).

Now we will do the same for the sum over n in (340), again using a completion of squares in (346)-(347) below.

$$
\begin{aligned}
& \sum_{n=0}^{\infty} \frac{|A|^{2n}}{q^{2Nn+n}} \left(\frac{1}{q^{n^2}} \right) \\
= & \sum_{n=0}^{\infty} \exp\left[n \cdot \ln \left(\frac{|A|^2}{q^{2N+1}} \right) \right] \exp\left[-n^2 \cdot \ln(q) \right] \\
= & \sum_{n=0}^{\infty} \exp\left[-\ln(q) \left(n^2 - \frac{n}{\ln(q)} \cdot \ln\left[\frac{|A|^2}{q^{2N+1}} \right] \right) \right] \\
= & \sum_{n=0}^{\infty} \exp\left[-\ln(q) \left(n^2 - \frac{n}{\ln(q)} \cdot \ln\left[\frac{|A|^2}{q^{2N+1}} \right] + \left(\frac{\ln\left(|A|^2 q^{-2N-1}\right)}{2\ln(q)} \right)^2 \right) \right] && (346) \\
& \cdot \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)} \right] && (347) \\
= & \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)} \right] \sum_{n=0}^{\infty} \exp\left[-\ln(q) \left(\left[n - \frac{\ln\left(|A|^2 q^{-2N-1}\right)}{2\ln(q)} \right]^2 \right) \right] \\
= & \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)} \right] \sum_{n=0}^{\infty} \exp\left[\frac{-\left(n - \frac{\ln\left(|A|^2 q^{-2N-1}\right)}{2\ln(q)} \right)^2}{\left(\frac{1}{\ln(q)} \right)} \right] \\
= & \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)} \right] \sum_{n=0}^{\infty} \exp\left[\frac{-\frac{1}{2}\left(n - \frac{\ln\left(|A|^2 q^{-2N-1}\right)}{2\ln(q)} \right)^2}{\left(\frac{1}{\sqrt{2\ln(q)}} \right)^2} \right] && (348) \\
\leq & \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)} \right] \left[1 + \sqrt{2\pi} \left(\frac{1}{\sqrt{2\ln(q)}} \right) \right] && (349) \\
= & \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)} \right] \left[1 + \sqrt{\frac{\pi}{\ln(q)}} \right]. && (350)
\end{aligned}
$$

Again notice that the expression in (348) is a Gaussian with mean $\dfrac{\ln(|A|^2 q^{-2N-1})}{2\ln(q)}$ and standard deviation $\dfrac{1}{\sqrt{2\ln(q)}}$. Then the bound in (349) follows from Proposition 1. Once more the dependence on n is gone in (350). Now that we have bounded these terms we can look back to (339)-(340) and say that,

$$
\begin{aligned}
& |y(t)| \\
\leq \ & (q^{1/2}) \left[1+\sqrt{\frac{\pi}{\ln(q)}}\right] \sum_N \sum_M \frac{|c_{N,M}|}{q^{N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N}\right)}{4\ln(q)}\right] \qquad (351) \\
+ \ & |A| \left[1+\sqrt{\frac{\pi}{\ln(q)}}\right] \sum_N \sum_M \frac{|c_{N,M}|}{q^{3N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)}\right] \qquad (352)
\end{aligned}
$$

Now we will apply the same completion of squares process on the summations over the N's in (351)-(352). This will allow us to obtain conditions on our $c_{N,M}$'s that will be sufficient for (351)-(352) to converge absolutely and uniformly.

Starting with the sums in (351) we have:

$$
\begin{aligned}
& \sum_N \sum_M \frac{|c_{N,M}|}{q^{N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N}\right)}{4\ln(q)}\right] \\
= & \sum_N \sum_M |c_{N,M}| \\
& \cdot \exp\left[-\frac{N}{2}\cdot\ln(q)\right] \exp\left[\frac{\left[-2N\ln(q)+\ln(|A|^2)\right]^2}{4\ln(q)}\right] \\
= & \sum_N \sum_M |c_{N,M}| \\
& \cdot \exp\left[\frac{4N^2\ln^2(q) - 4N\ln(|A|^2)\ln(q) + \ln^2(|A|^2) - 2N\ln^2(q)}{4\ln(q)}\right] \\
= & \sum_N \sum_M |c_{N,M}| \qquad (353) \\
& \cdot \exp\left[N^2\ln(q) - \frac{N\ln(|A|^2)\ln(q)}{\ln(q)} + \frac{\ln^2(|A|^2)}{4\ln(q)} - \frac{N\ln(q)}{2}\right] \qquad (354)
\end{aligned}
$$

At this point we see that $|c_{N,M}|$ must decay in N faster than the exponential $\exp\left[N^2\ln(q)\right]$ to gain convergence. We will make the assumption that:

$$|c_{N,M}| \leq \exp[-\delta(N-\mu)^2]\cdot g(M) \ \text{, where } \ \delta > \ln(q) \qquad (355)$$

$$\text{and} \qquad \sum_M |g(M)| = B < \infty. \qquad (356)$$

Then continuing from (353)-(354) and utilizing our new conditions for $|c_{N,M}|$ in (355) , we get that:

$$\begin{aligned}
&\sum_N \sum_M \frac{|c_{N,M}|}{q^{N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N}\right)}{4\ln(q)}\right] \\
\leq\ & \sum_M |g(M)| \sum_N \exp\left[-\delta N^2 + 2\delta N\mu - \delta\mu^2\right] && (357)\\
&\cdot \exp\left[N^2 \ln(q) - \frac{N\ln(|A|^2)\ln(q)}{\ln(q)} + \frac{\ln^2(|A|^2)}{4\ln(q)} - \frac{N\ln(q)}{2}\right] && (358)
\end{aligned}$$

Now using the bound B from (356), and combining like powers of N in (357)-(358) gives:

$$\begin{aligned}
&\sum_N \sum_M \frac{|c_{N,M}|}{q^{N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N}\right)}{4\ln(q)}\right] \\
\leq\ & B\sum_N \exp\left[N^2[\ln(q)-\delta] - N\left(\frac{-4\delta\mu\ln(q) + 2\ln(|A|^2)\ln(q) + \ln^2(q)}{2\ln(q)}\right)\right] && (359)\\
&\cdot \exp\left[\frac{\ln^2(|A|^2) - 4\delta\mu^2\ln(q)}{4\ln(q)}\right] && (360)\\
=\ & B\cdot e^{C_8}\sum_N \exp\left[[\ln(q)-\delta]\left(N^2 - N\left(\frac{-4\delta\mu\ln(q) + 2\ln(|A|^2)\ln(q) + \ln^2(q)}{2\ln(q)[\ln(q)-\delta]}\right)\right)\right] && (361)
\end{aligned}$$

Where in (361) $C_8 = \dfrac{\ln^2(|A|^2) - 4\delta\mu^2\ln(q)}{4\ln(q)}$. Now to complete the square on N from (361) gives,

$$\begin{aligned}
&\sum_N \sum_M \frac{|c_{N,M}|}{q^{N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N}\right)}{4\ln(q)}\right] \\
\leq\ & B\cdot e^{C_8} \\
&\cdot \sum_N \exp\left[[\ln(q)-\delta]\left(\left[N - \left(\frac{-4\delta\mu\ln(q) + 2\ln(|A|^2)\ln(q) + \ln^2(q)}{4\ln(q)[\ln(q)-\delta]}\right)\right]^2\right)\right] && (362)\\
&\cdot \exp\left[-\frac{\left[-4\delta\mu\ln(q) + 2\ln(|A|^2)\ln(q) + \ln^2(q)\right]^2}{16\ln^2(q)[\ln(q)-\delta]}\right] && (363)\\
=\ & B\cdot e^{C_8} e^{C_9} && (364)\\
&\cdot \sum_N \exp\left[[\ln(q)-\delta]\left(\left[N - \left(\frac{-4\delta\mu\ln(q) + 2\ln(|A|^2)\ln(q) + \ln^2(q)}{4\ln(q)[\ln(q)-\delta]}\right)\right]^2\right)\right] && (365)
\end{aligned}$$

Where in (364) $C_9 = \dfrac{-\left[-4\delta\mu\ln(q) + 2\ln(|A|^2)\ln(q) + \ln^2(q)\right]^2}{16\ln^2(q)[\ln(q)-\delta]}$. Then continuing from (364)-(365) and denoting $C_{10} = C_8 + C_9$ we write that

$$\sum_N \sum_M \frac{|c_{N,M}|}{q^{N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N}\right)}{4\ln(q)}\right]$$

$$\leq B \cdot e^{C_{10}} \sum_N \exp\left[\frac{-\frac{1}{2}\left[N - \left(\frac{-4\delta\mu\ln(q)+2\ln(|A|^2)\ln(q)+\ln^2(q)}{4\ln(q)[\ln(q)-\delta]}\right)\right]^2}{\left(\frac{1}{\sqrt{2\delta - 2\ln(q)}}\right)^2}\right] \tag{366}$$

$$\leq Be^{C_{10}}\left[1 + \sqrt{2\pi}\left(\frac{1}{\sqrt{2\delta - 2\ln(q)}}\right)\right] \tag{367}$$

$$= Be^{C_{10}}\left[1 + \sqrt{\frac{\pi}{\delta - \ln(q)}}\right] < \infty. \tag{368}$$

Now that we have bounded (351) by (368) which is finite. We will do the same for (352) in order to completely bound $|y(t)|$ and show absolute and uniform convergence of all the sums in $y(t)$ under the assumption of (355) and (356).

Recall the sums in (352) are of the form:

$$\sum_N \sum_M \frac{|c_{N,M}|}{q^{3N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)}\right]$$

$$= \sum_N \sum_M |c_{N,M}| \exp\left[-\frac{3N}{2}\cdot\ln(q)\right] \exp\left[\frac{\left[-2N\ln(q) + \ln(|A|^2 q^{-1})\right]^2}{4\ln(q)}\right]$$

$$= \sum_N \sum_M |c_{N,M}| \cdot \exp\left[\frac{4N^2\ln^2(q) - 4N\ln(|A|^2q^{-1})\ln(q) + \ln^2(|A|^2q^{-1}) - 6N\ln^2(q)}{4\ln(q)}\right]$$

$$= \sum_N \sum_M |c_{N,M}| \tag{369}$$

$$\cdot \exp\left[N^2\ln(q) - \frac{N\ln(|A|^2q^{-1})\ln(q)}{\ln(q)} + \frac{\ln^2(|A|^2q^{-1})}{4\ln(q)} - \frac{3N\ln(q)}{2}\right] \tag{370}$$

Again to gain convergence we see that $|c_{N,M}|$ must decay in N faster than an exponential with a leading coefficient of $N^2\ln(q)$ so we will make the same assumption on the $c_{N,M}$ as in (355)-(356), namely that:

$$|c_{N,M}| \leq \exp[-\delta(N-\mu)^2]\cdot g(M) \ \ \text{, where } \ \delta > \ln(q) \tag{371}$$

$$\text{and} \qquad \sum_M |g(M)| = B < \infty. \tag{372}$$

Then continuing from (369)-(370) and utilizing our condition for $|c_{N,M}|$ in (371), we get:

$$\sum_N \sum_M \frac{|c_{N,M}|}{q^{3N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)}\right] \tag{373}$$

$$\leq \sum_M |g(M)| \sum_N \exp\left[-\delta N^2 + 2\delta N\mu - \delta\mu^2\right] \tag{374}$$

$$\cdot \exp\left[N^2\ln(q) - \frac{N\ln(|A|^2q^{-1})\ln(q)}{\ln(q)} + \frac{\ln^2(|A|^2q^{-1})}{4\ln(q)} - \frac{3N\ln(q)}{2}\right] \tag{375}$$

Now using the bound (372) and combining like powers of N in (374)-(375) gives:

$$\sum_N \sum_M \frac{|c_{N,M}|}{q^{3N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)}\right] \tag{376}$$

$$\leq B\sum_N \exp\left[N^2[\ln(q)-\delta] - N\left(\frac{-4\delta\mu\ln(q) + 2\ln(|A|^2q^{-1})\ln(q) + 3\ln^2(q)}{2\ln(q)}\right)\right] \tag{377}$$

$$\cdot \exp\left[\frac{\ln^2(|A|^2q^{-1}) - 4\delta\mu^2\ln(q)}{4\ln(q)}\right] \tag{378}$$

$$= B\cdot e^{C_{11}} \sum_N \exp\left[[\ln(q)-\delta]\left(N^2 - N\left(\frac{-4\delta\mu\ln(q) + 2\ln(|A|^2q^{-1})\ln(q) + 3\ln^2(q)}{2\ln(q)[\ln(q)-\delta]}\right)\right)\right] \tag{379}$$

where in (379) $C_{11} = \dfrac{\ln^2(|A|^2q^{-1}) - 4\delta\mu^2\ln(q)}{4\ln(q)}$. Now completing the square on N in (379) gives:

$$
\begin{aligned}
&\sum_N \sum_M \frac{|c_{N,M}|}{q^{3N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)}\right] \\
&\leq B \cdot e^{C_{11}} \\
&\quad \cdot \sum_N \exp\left[[\ln(q)-\delta]\left(\left[N - \left(\frac{-4\delta\mu\ln(q)+2\ln(|A|^2q^{-1})\ln(q)+3\ln^2(q)}{4\ln(q)[\ln(q)-\delta]}\right)\right]^2\right)\right] \\
&\qquad \cdot \exp\left[-[\ln(q)-\delta]\left(\frac{-4\delta\mu\ln(q)+2\ln(|A|^2q^{-1})\ln(q)+3\ln^2(q)}{4\ln(q)[\ln(q)-\delta]}\right)^2\right] \\
&= B \cdot e^{C_{11}} e^{C_{12}} \qquad (380) \\
&\quad \cdot \sum_N \exp\left[[\ln(q)-\delta]\left[N - \left(\frac{-4\delta\mu\ln(q)+2\ln(|A|^2q^{-1})\ln(q)+3\ln^2(q)}{4\ln(q)[\ln(q)-\delta]}\right)\right]^2\right], \qquad (381)
\end{aligned}
$$

where in (380) $C_{12} = \dfrac{-\left[-4\delta\mu\ln(q)+2\ln(|A|^2q^{-1})\ln(q)+3\ln^2(q)\right]^2}{16\ln^2(q)[\ln(q)-\delta]}$. Then continuing from (380) and denoting $C_{13} = C_{11} + C_{12}$ we get that

$$
\begin{aligned}
&\sum_N \sum_M \frac{|c_{N,M}|}{q^{3N/2}} \exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)}\right] \\
&\leq B \cdot e^{C_{13}} \sum_N \exp\left[\frac{-\frac{1}{2}\left[N - \left(\frac{-4\delta\mu\ln(q)+2\ln(|A|^2q^{-1})\ln(q)+3\ln^2(q)}{4\ln(q)[\ln(q)-\delta]}\right)\right]^2}{\left(\frac{1}{\sqrt{2\delta-2\ln(q)}}\right)^2}\right] \qquad (382) \\
&\leq Be^{C_{13}}\left[1+\sqrt{2\pi}\left(\frac{1}{\sqrt{2\delta-2\ln(q)}}\right)\right] \qquad (383) \\
&= Be^{C_{13}}\left[1+\sqrt{\frac{\pi}{\delta-\ln(q)}}\right] < \infty, \qquad (384)
\end{aligned}
$$

where (383) was obtain by using Proposition 1 on the Gaussian in (382).

Now that we have bounded the sums in (352) by (384) we can look back to (351) and (352) and using the bounds (368) and (384) respectively we obtain the bound:

$$
\begin{aligned}
&|y(t)| \\
\leq\ & (q^{1/2})\left[1+\sqrt{\frac{\pi}{\ln(q)}}\right]\sum_N\sum_M \frac{|c_{N,M}|}{q^{N/2}}\exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-1}\right)}{4\ln(q)}\right] \\
&+|A|\left[1+\sqrt{\frac{\pi}{\ln(q)}}\right]\sum_N\sum_M \frac{|c_{N,M}|}{q^{3N/2}}\exp\left[\frac{\ln^2\left(|A|^2 q^{-2N-2}\right)}{4\ln(q)}\right] \\
\leq\ & Be^{C_{10}}(q^{1/2})\left(1+\sqrt{\frac{\pi}{\delta-\ln(q)}}\right)\left(1+\sqrt{\frac{\pi}{\ln(q)}}\right) && (385) \\
&+|A|Be^{C_{13}}\left(1+\sqrt{\frac{\pi}{\delta-\ln(q)}}\right)\left(1+\sqrt{\frac{\pi}{\ln(q)}}\right) < \infty && (386)
\end{aligned}
$$

Now that we have bounded $y(t)$ we need to bound the expression for $y'(t)$ in order to use Theorem 1. Recall from (246)-(247) that,

$$
\begin{aligned}
y'(t) &= Ay(qt) + f(t) && (387) \\
&= Ay(qt) + \sum_N\sum_M c_{N,M}\Phi_{N,M} && (388)
\end{aligned}
$$

Clearly since we have already shown in (329)-(330) that $|y(t)| < \infty$ with $y(t)$ converging uniformly and absolutely on all of $\mathbb{R}$, it follows that $|Ay(qt)| < \infty$, with the series $Ay(qt)$ converging absolutely and uniformly as well. So we will focus on bounding the $f(t)$ term in (387), namely $\sum_N\sum_M c_{N,M}\Phi_{N,M}$.

Recall in this case (243) is our choice of $\Phi_{N,M}$ and then using (371) for our condition on

$c_{N,M}$ we can rely on (42) to obtain:

$$\left|\sum_N \sum_M c_{N,M} \Phi_{N,M}\right| \tag{389}$$

$$\leq \sum_N \sum_M \exp\left[-\delta(N-\mu)^2\right] \cdot |g(M)| q^{\frac{N}{2}} \left|{}_q Cos(q^N t - Mb)\right| \tag{390}$$

$$\leq \sum_M |g(M)| \sum_N \exp\left[-\delta N^2 + 2\delta N \mu - \delta\mu^2\right] \exp\left[\frac{N}{2}\ln(q)\right] \tag{391}$$

$$= B \sum_N \exp\left[-\delta\left(N^2 - N\left(\frac{4\delta\mu + \ln(q)}{2\delta}\right) + \mu^2\right)\right] \tag{392}$$

$$= B \sum_N \exp\left[-\delta\left[\left(N - \frac{4\delta\mu + \ln(q)}{4\delta}\right)^2 - \left(\frac{4\delta\mu + \ln(q)}{4\delta}\right)^2 + \mu^2\right]\right] \tag{393}$$

$$= B \cdot e^{C_{14}} \sum_N \exp\left[\frac{-\frac{1}{2}\left(N - \frac{4\delta\mu + \ln(q)}{4\delta}\right)^2}{\left(\frac{1}{\sqrt{2\delta}}\right)^2}\right] \tag{394}$$

$$\leq B \cdot e^{C_{14}} \left[1 + \sqrt{2\pi}\left(\frac{1}{\sqrt{2\delta}}\right)\right] \tag{395}$$

$$= B \cdot e^{C_{14}} \left[1 + \sqrt{\frac{\pi}{\delta}}\right] < \infty, \tag{396}$$

where $C_{14} = \dfrac{[4\delta\mu + \ln(q)]^2}{16\delta} - \delta\mu^2$ and Proposition 1 was used to obtain (395). Thus we can now say from using (389)-(396) in (387)-(388) that $|y'(t)| < \infty$. Specifically that, since $y'(t)$ converges absolutely and uniformly on the whole real line, and we have shown that $y(t)$ also converges at each point on the real line. Then by Theorem 1 we are allowed to pass the derivatives through the infinite sums as we did in (271)-(274). This means that our solution $y(t)$, (253)-(254), converges absolutely and uniformly on the entire real line as needed. This rigorously allows the exchange of derivatives with infinite sums in (271)-(274) and allows us to conclude that our solution (253)-(254) does satisfy the MADE (53).

We have reached the main Theorem of this thesis.

Theorem 5. *Let* $f(t) \in L^2(\mathbb{R})$ *, with*

$$f(t) = \sum_N \sum_M c_{N,M} \Phi_{N,M}$$

where for $N, M \in \mathbb{Z}$

$$\Phi_{N,M} = q^{N/2} \, {}_qCos(q^N t - Mb) \tag{397}$$

$$or \qquad \Phi_{N,M} = q^{N/2} \, {}_qSin(q^N t - Mb). \tag{398}$$

Let $|c_{N,M}| \leq \exp\left[-\delta(N-\mu)^2\right] \cdot g(M)$ *, with* $\delta > \ln(q)$ *and* $\sum_M |g(M)| = B < \infty$*.*
A solution of

$$y'(t) - Ay(qt) = f(t)$$

when $f(t)$ *is expanded in the* (397) *expansion is given by:*

$$y(t) = \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^n \left(\frac{A}{q^N}\right)^{2n} \left(\frac{1}{q^{n^2}}\right) {}_qSin\left(\frac{q^{2n} q^N t - Mb}{q^{n+1}}\right) \tag{399}$$

$$+ \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{n+1} \left(\frac{A}{q^N}\right)^{2n+1} \left(\frac{1}{q^{n(n+1)}}\right) {}_qCos\left(\frac{q^{2n} q^{N+1} t - Mb}{q^{n+1}}\right) \tag{400}$$

and a solution of

$$y'(t) - Ay(qt) = f(t)$$

when $f(t)$ *is expanded with the* (398) *expansion is given by:*

$$y(t) = \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{n+1} \left(\frac{A}{q^N}\right)^{2n} \left(\frac{1}{q^{n(n+1)}}\right) {}_qCos\left(\frac{q^{2n}q^N t - Mb}{q^n}\right) \tag{401}$$

$$+ \sum_N \sum_M \sum_{n=0}^{\infty} c_{N,M} \frac{q^{\frac{N}{2}}}{q^N} (-1)^{n+1} \left(\frac{A}{q^N}\right)^{2n+1} \left(\frac{1}{q^{(n+1)^2}}\right) {}_qSin\left(\frac{q^{2n}q^{N+1} t - Mb}{q^{n+1}}\right). \tag{402}$$

Proof. We have shown that (399)-(400) converge absolutely and uniformly on $\mathbb{R}$ via the work (337)-(338) up through (385)-(386). Then we showed that $y'(t)$ also converges uniformly and absolutely on $\mathbb{R}$ via (389)-(396). We have shown (401)-(402) to converge absolutely and uniformly via the work (280)-(281) up through (329)-(330). Then we showed that $y'(t)$ also converges uniformly and absolutely via (333)-(336). After applying Theorem 1 in both cases we got that under the decay conditions for $|c_{N,M}|$, given in (371)-(372), one can pass the derivatives through the infinite sums in both (262)-(265) and (271)-(274). This provides a rigorous proof that $y(t)$ is a valid solution to $y'(t) - Ay(qt) = f(t)$. □

Now we provide, pictures for a a picture of the main MADE, (53), using the frame expansion (397), with $b = 1$, and the condition on the $c_{N,M}$'s from (371)-(372), letting $\delta = \frac{1}{2}$, $\mu = 0$, and $G(M) = \frac{1}{M^6+1}$. So $|c_{N,M}| < \exp\left[-\frac{1}{2}N^2\right]\frac{1}{M^6+1}$, and note that since $\delta = .5$ the condition of $\delta > \ln(q)$ holds for the three figures below, since $\frac{1}{2} > \ln(1.3) > \ln(1.2) > \ln(1.15)$. Also note that

$$\sum_M |g(M)| = \sum_M \left|\frac{1}{M^6+1}\right| < \infty$$

so (372) holds.

In Figure 5 - Figure 7 below the dashed line is the forcing term

$$f(t) = \sum_N \sum_M c_{N,M} q^{N/2} {}_q Cos(q^N t - M),$$

where q's value is given in the figure's caption and $c_{N,M} = \exp\left[-\frac{1}{2}N^2\right]\frac{1}{M^6+1}$. The solid line is the solution (399)-(400) of the MADE

$$y'(t) - 2y(qt) = f(t),$$

for the $f(t)$ given above and q's value given in the figure's caption.

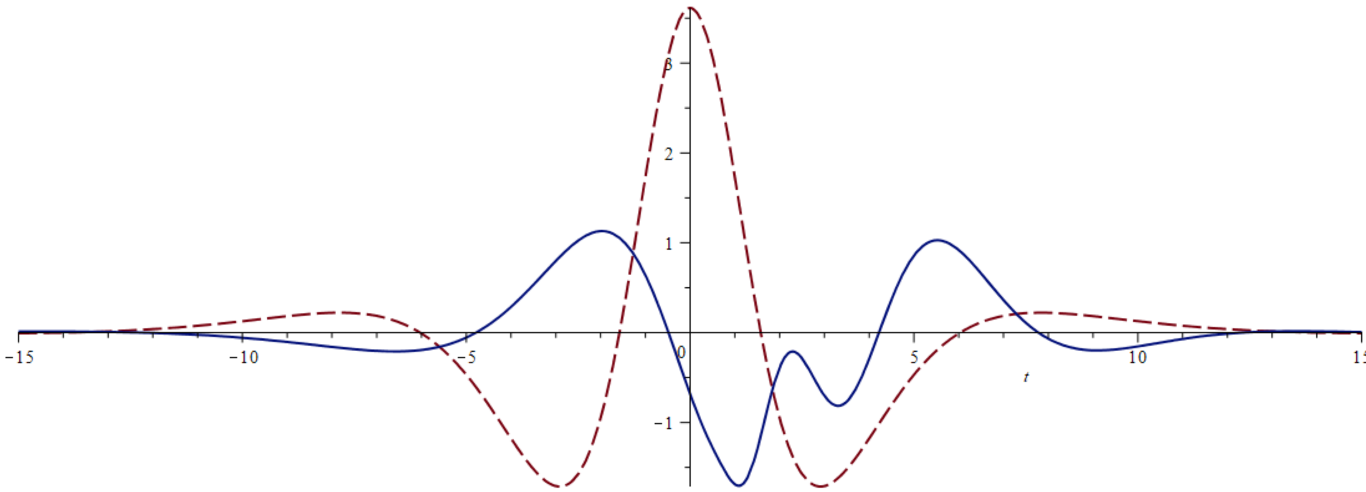

Figure 5: A general forcing solution with q = 1.3

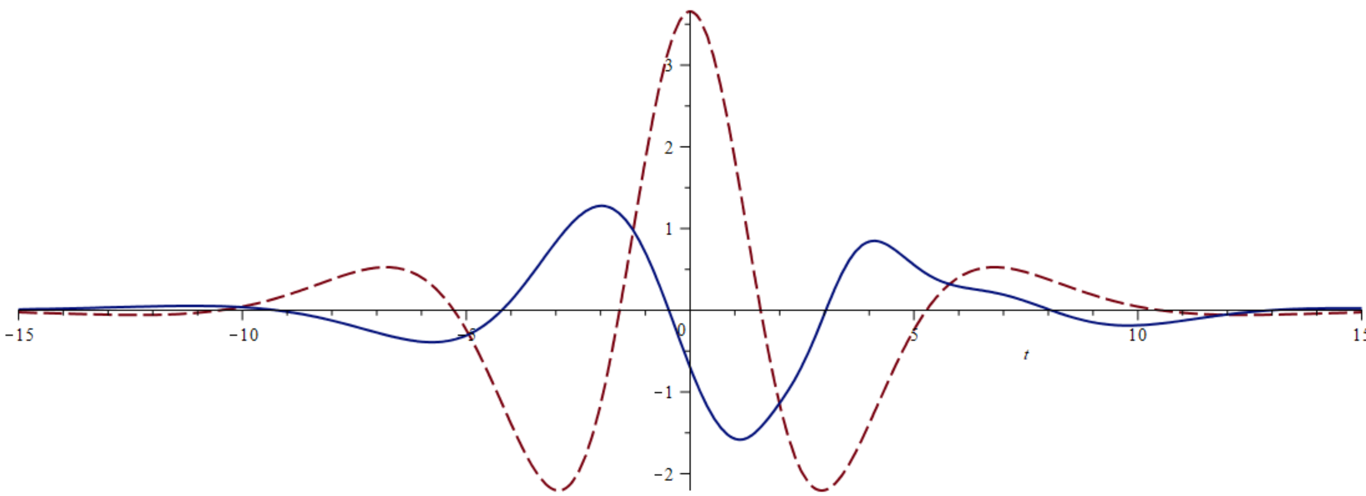

Figure 6: A general forcing solution with q = 1.2

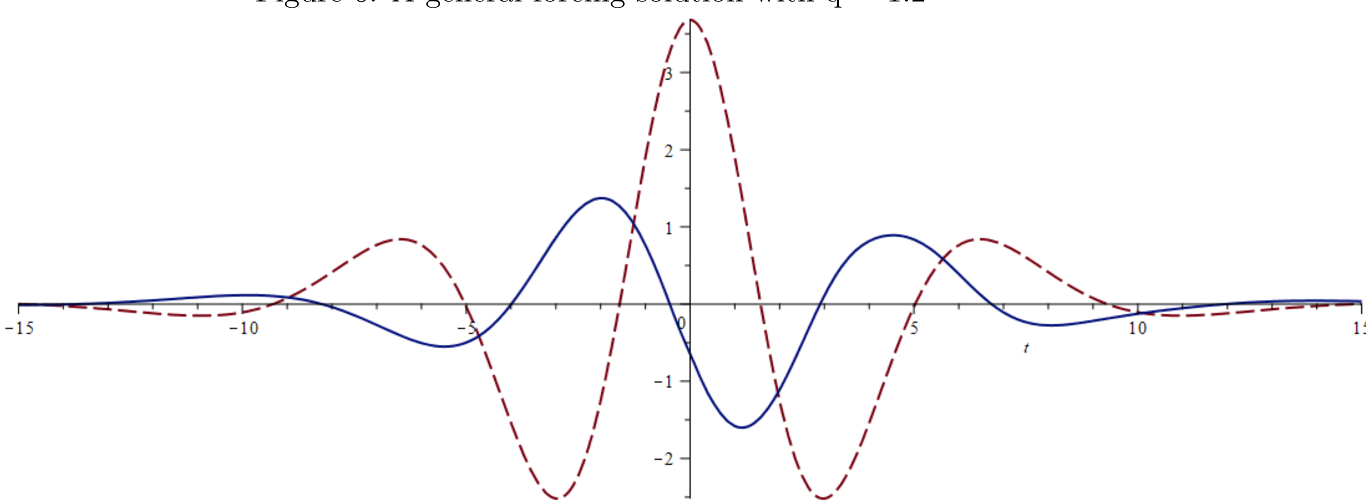

Figure 7: A general forcing solution with q = 1.15

5 References

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