ABSTRACT

THE EFFECT OF THE MATHEMATICS OF FINANCE ON THE DYNAMICS OF A CREDIT ECONOMY

by

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April, 2012

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The general equilibrium theory of J.M. Keynes was developed in the 1930s to help explain the great depression and prevent future economic downturns. Out of this came the IS-LM (investment saving/liquid money) model, introduced by J.R. Hicks in 1936. There is controversy about the success of Hicks's approach, not the least of which is the lack of dynamical aspects in the theory. The thesis considers three interest groups identified as bankers, capitalists and workers. A coupled system of differential equations describes the flow of money, capital and credit over time. A mathematical analysis of the high dimensional system reveals the existence of equilibrium points. Their stability properties are determined. By modifying the equations, cycles appear corresponding to periods of boom and bust. Thus, shocks to the system, which are theorized by neoclassical economists to be due to external events, are shown to be possible using dynamical endogenous equations.

THE EFFECT OF THE MATHEMATICS OF FINANCE ON THE DYNAMICS OF A CREDIT ECONOMY

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by

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DISCLAIMER

Due to the controversial nature of the topics discussed in this thesis, we urge the reader to keep an open mind. The work is intended to apply techniques of Mathematics to Finance and Economics from a basic modeling point of view. No claim is made that the work actually models any true world economy. This thesis presents a theoretical, idealized model for an economy. Any resemblance to true economies is purely coincidental.

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CHAPTER 1: Introduction

Economics, as an area of study, is a social science concerned with the understanding of production and consumption of goods and services. A simple approach to analyzing an economy starts with a system of bartering where goods and services are exchanged without the need for a currency. One of the problems with this perspective is that some entities are not immediately deliverable, like agricultural products that can only be harvested at certain times of the year. This requires a system of promissory notes or contracts, that include specifics like the quantity, quality and date of delivery for the products in question. Those who accept these notes are engaging in a system of accepting credit in exchange for their goods and services [6]. Thus markets will consist of three entities: goods, services and contracts. This leads to three related interest groups in a real economic system: companies (C), bankers (B) and all-the-rest-of-us (A).

Financial mathematics is a subject that deals with evaluating risk and developing policies that minimize the possibility of a default. The basic process that is considered is where a bank has funds that it is willing to lend. The banker estimates the risk that the borrower will not pay back the loan entirely, and assigns a premium, in the form of an interest rate, to the repayment plan. By loaning out to many individuals, a well-managed bank can remain solvent, even while some companies and workers go bankrupt. The problem is that this system is inherently unstable. If the bank over-estimates risk, and charges too high an interest rate, it will accumulate wealth without being productive. This leads to a situation of bank hoarding. On the other hand, if the bank underestimates risk it will not be able to recoup the funds that were lent out. This leads to a slowdown in lending activity until the bank eventually goes out of business.

One can see that in order for a financial sector to exist, it must be dynamic, adjusting its lending polcies to account for the variable successes of its borrowers. However, during economic upturns, risks, and thus interest rates, are low and borrowing is easy. As defaults begin to rise, due to excessive euphoria, banks are forced to adjust interest rates higher, making repayment more expensive and further increasing the incidents of bankruptcies.

This thesis will study some dynamical systems that model the interaction of the three groups: companies C, banks B and all-others A. In mathematical terminology, the systems are multi-dimensional, first-order, linear, constant-coefficient, autonomous linear differential equations. As such, they are explicitly solvable. It will be shown that, under reasonable conditions on the parameters, the only stable equilibria are those that correspond to bankruptcies of at least one of A, B or C, or cycles of booms and busts.

The thesis concentrates on endogenous models, which are of increasing interest to economists [9]. To provide some new insights on stability, the effect of delays is considered. The results are that delays lead to greater instability and more erratic oscillatory behavior.

The final section considers exogenous effects, like the different money supply policies of a federal-government reserve. Results are compared with data and possible alternative policies are suggested.

CHAPTER 2: Basic concepts

Each person is different. Thus a stratification of society using criteria of wealth, responsibility, or seniority will involve some questionable assumptions. We consider a trade-off that results in an interesting dynamic with a minimum of analysis.

2.1 All people

The simplest economic system started with tribes of people A. Let $A_i(t)$ represent the value of the resources that was obtained by people through income. In order to justify an outflow of activity, usually in the form of gainful employment, must be performed, which has value $A_o(t)$. Both quantities must be non-negative (although a promise of work may correspond to $A_o(t) < 0$ temporarily). However, for continued existence of the population, a minimum of $A_{min} > 0$ must be maintained by $A_o(t)$. This simplest economy satisfies

$$\textbf{hunter-gatherer society:} \qquad A_i(t) \simeq A_o(t) \geq A_{min} \ .$$

In this case no specific dynamics are needed. The excess resources through activities can be measured by $A_i(t) - A_o(t)$. Observe that if $A_o(t) < A_{min}$, then at time t the species will be heading toward extinction through starvation. An awareness of this fact could result in a reduction in A_i in order to raise A_o, and return A_o back safely away from A_{min} [1].

2.2 Management classes

About 12,000 years ago, agriculture began and with it a division of labor. A farm would consist of those who tended to the fields A, and those who would manage the resources C. In this system one small group of individuals are in a position to hoard the excess of production, and we assume that they do. Thus

agrarian society: $A_i(t) \gg A_o(t) \simeq A_{min}$, $C_i(t) \gg C_o(t) \simeq C_{min}$.

To survive, the A class must exchange its meager resources in order to get a diversity to meet their needs for survival. This uses a bartering system where the values of goods and services require a keen sense of relativeness by those living on the edge. As for the C class, the excess $C_i(t) - C_o(t) = A_i(t) - A_o(t) > 0$ can be used for self-glorification projects and campaigns.

2.3 Bankers

Finally, about 4000 years ago the minting of coins began. This provided a mechanism for delaying payment for products received, or soon to be purchased. This resulted in the notion of indebtedness and hence a credit economy. To manage debt, the B class is introduced to mediate the A and C classes, while satisfying

usuary society: $B_i(t) \gg B_o(t) \simeq B_{min} \simeq 0$.

As such, banking is very lucrative, while producing nothing. One justification for having $B_i(t) - B_o(t) \gg 0$ is that hoarding of wealth provides a reserve for future ventures of the C class. However, without a jubilee, there is no way to curb the ever-increasing wealth of B. One solution is to continually be pursuing growth, but this is unrealistic [3].

2.4 Socialist Utopia versus Capitalist Distopia

The future could be glorious, with all people participating in the vast wealth that human innovation can provide. This could be achieved by allowing young adults to enter the entrepreneurial class C. The objective would be to have them help develop and deliver new products by managing and advancing existing technology. Then, after a period of sufficient work experience, an adult should enter the investment class B. Eventual success would result in sufficient savings for the older adult to retire. Their expenses on such things as leisure, personal development and healthcare, would create a steady demand for new products from the society at large. In this brave new world, one would have an

$$\textbf{automated society:} \qquad A_0(t) \;=\; A_i(t) \;=\; A_{min} \;\gg\; 0 \;\;.$$

Unfortunately, at present, the future looks bleak. As work becomes more automated, there is less and less need for the A class. At the same time, there is a diminishing motivation for C to be bothered with maintaining the following requirement for A,

$$\textbf{functioning economic system:} \qquad A_i(t) \;\simeq\; A_o(t) \;\geq\; A_{min} \;>\; 0.$$

Hence, to avoid mass starvation, adjustments must be made either at the community level or at the various government levels. To obtain sufficient resources for A, taxation of B and C seems reasonable, since that is where the debt obligations and hoarded wealth reside. As an alternative, what often happens is that B lends money to A, thus delaying the effect of having a policy of insufficient income for work. Then C hires A to generate goods and services, while managing the profits from their sales. Of course, as part of this management, they take the lion's share:

Table 1:

Institution:	Capitalists	Banks	All others
Gift to society:	management	funds now + debt	goods & services
Gift to self:	hoarded wealth	IOUs	survival

Three classes: Contributions based on altruism and self-interest.

It will be shown that a credit economy has no intrinsic mechanism for sufficiency of A and thus cannot provide a functioning ecomony without it being forced on society. No matter how much money is put into the system, it will always eventually be hoarded by banks and successful managers of companies, who will themselves become bankers. One way that balance can be established is through a policy of high inflation, which erodes hoarded wealth. However, this also erodes retirement savings for those who have worked dilligently during their productive adult years. Thus, for fairness, only an economic policy of low inflation, with high taxation on B and C, can provide a stable environment for A. More specifically, to prevent the ravages of hoarding, an upper limit on wealth, and caps on salaries represents a policy that can provide a feedback mechanism that will stabilize the economic system and maintain a functioning environment.

CHAPTER 3: Mathematical modeling of financial-economic systems

The goal of modeling is to capture the basic descriptive features of a system in as simple a manner as possible. Here we want a system with several features:

(1) linear systems: express using vectors and matrices

(2) dynamical: determine rate-of-change equations

(3) asymptotically constant: at least one stable equilibrium point

(4) no negative accounts: evolution of the system maps

initial non-negative accounts $\to$ final non-negative accounts

We examine various choices of parameters and initial conditions that have solutions satisfying (1)-(4). We find evolutionary features that can be a result of external or internal effects. For example, consider a function $y(t) = \sin(t)$. It satisfies the initial conditions $y(0) = 0$, $y'(0) = 1$ and the two differential equations:

$$(X) \qquad y'' + 2y \;=\; \sin(t) \; , \qquad\qquad (N) \qquad y'' + y \;=\; 0 \; .$$

The (X) equation is exogenous (inhomogeneous) because it is forced by $\sin(t)$, whereas (N) is endogenous (homogeneous), no exernal forcing. Thus one cannot tell from a data set, corresponding to a phenomenon, whether it is being interally or due to external influences. If a complex behavior is generated by internal interactions, then a study of the governing system has the hope of leading to an understanding of the phenomenon. This is a point made in [9] but for nonlinear systems, which are more commonly seen in nature.

3.1 Linear finance

The mathematics of finance is quite linear. The components are the principle amount P_0 and the interest rate r. Policies for paying or charching interest can involve a discrete or continuous approach:

$$\begin{array}{rclrcl} \text{discrete payout: } P_k & = & P_0 \cdot (1+r)^k \ , & \text{continuous payout: } P(t) & = & e^{rt} \cdot P_0 \ , \\ \text{discrete charge: } P_k & = & P_0 \cdot (1-r)^k \ , & \text{continuous charge: } P(t) & = & e^{-rt} \cdot P_0 \ , \end{array}$$

This is discussed in detail in [2]. In this thesis continuous payout/charge will be discussed and analyzed. However, the numerical models used a discrete payout/charge method, also known as Euler's method. Since financial mathematics is linear, there is no need for a stability analysis. Instead we focus on the interaction between payout accounts and charging accounts.

3.2 Credit economy

The economy is a system whereby human activity takes place for the purpose of survival and the creation of goods and services. A credit economy is one where activity beyond basic survival is motivated through the generation of an IOU ("I owe you") Clearly someone borrows money, they should pay it back at some point in the future. However, a usurary policy says that there will be a charge associated with this agreement, and the amount of payback will be larger, the more time taken to return the funds. A debt bubble occurs when in fact there is not enough funds to every pay back the initial loan, plus the accumulated extra charges. However, a credit economy can still function if this fact is ignored.

3.3 Alternatives

There are alternative financial-economic policies and theories to those in common use. A recent example is *MMT* (modern monetary theory) which advocates printing money as needed to continually avioding a bursting debt bubble. One supposed positive consequence is a zero level of unemployment [10].

Another example is called *RIBA*, which is sometimes used in Islamic economics. In this system interest rates are not to be charged on borrowed money [8]. There is a charge, but it is fixed over time. This raises questions, like how risk is assessed and evaluated [11]. A policy of capping debts or debt forgiveness may help to provide a negative feedback, that could stabilize the system. However this may come at the cost of lost economic activity.

CHAPTER 4: A basic 3-dimensional financial-economic system ABC

We begin a mathematical study by considering a society with three types of financial accounts: capitalists C, bankers B and all others A. In order to jump start the economy, a lump sum of base money $M_0 > 0$ is given to the banks. Initally, the capitalists have 0 funds, whereas all others have just enough to survive. Summing up, initially (at $t = 0$), we have

$$A(0) \;=\; A_{min} \;,\quad B(0) \;=\; M_0 \;,\quad C(0) \;=\; 0 \;.$$

Next, suppose the A class begins borrowing funds from the banks B at the rate $rb > 0$, relative to the amount of funds available for lending. Then the A and B accounts change at the rates:

$$A'(t) \;=\; rb \cdot B(t) \;,\quad B'(t) \;=\; -rb \cdot B(t) \;. \tag{4.1}$$

Once this process begins, A is in debt and so begins paying back at the rate $ra \simeq rb > 0$, but only on funds above what is needed to survive. This gives the rate equations

$$A'(t) \;=\; -ra \cdot (A(t) - A_{min}) \;\;,\quad B'(t) \;=\; ra \cdot (A(t) - A_{min}) \;\;. \tag{4.2}$$

The funds that populate the A accounts are used to make purchases from C, the ownership class, at a rate of $\alpha > 0$. Thus all's accounts are depleted through purchases

$$A'(t) \;=\; -\alpha \cdot (A(t) - A_{min}) \;\;,\quad C'(t) \;=\; \alpha \cdot (A(t) - A_{min}) \;\;.$$

This information can be displayed in a Godley table [9]. Here we only consider elements that lead to a conservation of the total amount of money in the system.

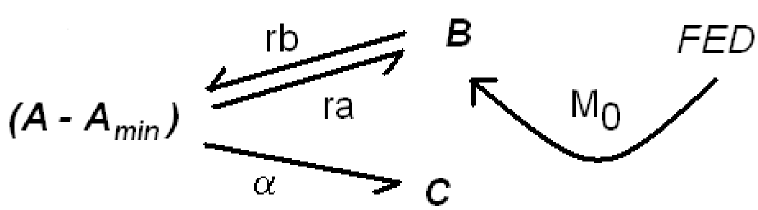
Table 2:

Account Name	All (excess)	Bankers	Capitalists
Symbol	A(t)-Amin	B(t)	C(t)
Initial Conditions	0	M0	0
Lend Money	rb	-- rb	
Consume	− α		α
Repay Loan	-- ra	ra	

Godley table: Typical ABC conservation of quantities table.

An alternative way of presenting the flow of money is with arrows that indicate the direction of money.

Diagram 1:



Flow diagram: Lump sum of M_0 supplied to the system. Then money flows from B to A with rate constant rb, which then flows to B with rate constant ra, and to C with rate constant α. Here the FED is the source of money, and C eventually absorbes all money.

We can now express this model as a 3-dimensional system of linear equations:

$$\frac{d}{dt}\begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} -(ra+\alpha)\cdot(A(t)-A_{min})+rb\cdot B(t) \\ ra\cdot(A(t)-A_{min})-rb\cdot B(t) \\ \alpha\cdot(A(t)-A_{min}) \end{pmatrix} , \quad \begin{pmatrix} A(0) \\ B(0) \\ C(0) \end{pmatrix} = \begin{pmatrix} A_{min} \\ M_0 \\ 0 \end{pmatrix} . \tag{4.3}$$

Using the assumption that A_{min} is a fixed positive constant, define the shifted variable

$$\overline{A}(t) = A(t) - A_{min} , \quad \overline{A}'(t) = A'(t) .$$

By rearrangement, the system can now be expressed as a matrix equation

$$\begin{pmatrix} \overline{A} \\ B \\ C \end{pmatrix}' = \begin{pmatrix} -(ra+\alpha) & rb & 0 \\ ra & -rb & 0 \\ \alpha & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \overline{A} \\ B \\ C \end{pmatrix} , \quad \begin{pmatrix} \overline{A} \\ B \\ C \end{pmatrix}_0 = \begin{pmatrix} 0 \\ M_0 \\ 0 \end{pmatrix} . \tag{4.4}$$

The following general result helps to understand the behavior of the system in (4.4).

4.1 Some basic results for ABC systems

In [9] Steve Keen argues for the need to consider the financial balance sheets of an economic system.

4.1.1 Conservation of total money

Here we make the following theoretical observation:

Theorem 4.1. *Consider the linear n-dimensional IVP (initial value problem):*

$$\frac{d\vec{Y}}{dt} = \mathcal{A} \cdot \vec{Y} , \quad \vec{Y}(0) = \vec{Y}_0 , \tag{4.5}$$

for some constant coefficient $n \times n$ matrix $\mathcal{A}$ and some constant initial vector $\vec{Y}_0 \in \mathbb{R}^n$. Then an invariant of the system is the sum of the components of the state vector $\vec{Y}(t)$.

Proof. The solution to the IVP in (4.5) is $\vec{Y}(t) = e^{\mathcal{A}t}\vec{Y}_0$. Since $\mathcal{A}$ is a bounded linear operator on $\mathbb{R}^n$, its exponentiation can be expressed using a Taylor series

$$\begin{aligned} \vec{Y}(t) &= \sum_{k=0} \frac{1}{k!}(\mathcal{A}t)^k \cdot \vec{Y}_0 \qquad (4.6) \\ &= \mathcal{I} \cdot \vec{Y}_0 + (t\mathcal{A}) \cdot \vec{Y}_0 + \frac{1}{2!}(t\mathcal{A})^2 \cdot \vec{Y}_0 + \frac{1}{3!}(t\mathcal{A})^3 \cdot \vec{Y}_0 + \ldots \\ &= \vec{Y}_0 + t\,(\mathcal{A} \cdot \vec{Y}_0) + \frac{t^2}{2!}\left(\mathcal{A}\left(\mathcal{A} \cdot \vec{Y}_0\right)\right) + \frac{t^3}{3!}\left(\mathcal{A}\left(\mathcal{A}\left(\mathcal{A} \cdot \vec{Y}_0\right)\right)\right) + \ldots . \end{aligned}$$

Now, express the matrix $\mathcal{A}$ in terms of its column vectors $\vec{a}_j \in \mathbb{R}^n$,

$$\mathcal{A} = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = (\vec{a}_1 \quad \vec{a}_2 \quad \dots \quad \vec{a}_n) \ , \ \vec{a}_j \equiv \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix} , \qquad (4.7)$$

Definition 4.2. For any n-dimensional vector, $\vec{v} = (v_1 \ v_2 \ \dots v_n)^T$, define the *sum* of $\vec{v}$ be the sum of its components, expressed notationally as

$$\sum \vec{v} \equiv \sum_{i=1}^{n} v_i = v_1 + v_2 + \cdots + v_n \ .$$

The sum of an $n \times n$ matrix $\mathcal{A}$, as given in (4.7), is defined to be the row vector:

$$\sum(\mathcal{A}) \equiv \sum(\vec{a}_1) + \sum(\vec{a}_2) + \cdots + \sum(\vec{a}_n) \ .$$

Now we show a useful property of the sum map $\sum$ on vector spaces.

Lemma 4.3. *The sum map $\sum : \mathbb{R}^n \to \mathbb{R}$ is a linear operator. Also, as a map from $n \times n$ matrices to n-dimensional row vectors, $\sum$ is linear.*

Proof. Verifying linearity requires showing two.

1) Consider two vectors $\vec{a}_1$ and $\vec{a}_2$ as defined in (4.7). Then compute the sum

$$\begin{aligned} \sum(\vec{a}_1 + \vec{a}_2) &= \sum\left(\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix}\right) = \sum \begin{pmatrix} a_{11} + a_{12} \\ a_{21} + a_{22} \\ \vdots \\ a_{n1} + a_{n2} \end{pmatrix} \\ &= (a_{11} + a_{12}) + (a_{21} + a_{22}) + \dots + (a_{n1} + a_{n2}) \\ &= (a_{11} + a_{21} + \dots + a_{n1}) + (a_{12} + a_{22}) + \dots + a_{n2}) \\ &= \sum(\vec{a}_1) + \sum(\vec{a}_2) \ , \end{aligned}$$

where rearrangement was used in the second last step.

2) Consider a vector $\vec{a}_1$ and a constant c_1. Compute the sum

$$\sum (c_1\vec{a}_1) = \sum \begin{pmatrix} c_1 a_{11} \\ c_1 a_{21} \\ \vdots \\ c_1 a_{n1} \end{pmatrix} = c_1 a_{11} + c_1 a_{21} + \cdots + c_1 a_{n1}$$
$$= c_1 \cdot (a_{11} \ + \ a_{21} \ + \ \ldots \ + \ a_{n1})$$
$$= c_1 \cdot \sum (\vec{a}_1) \ .$$

Combining these two results together gives the linearity statement

$$\sum (c_1\vec{a}_1 + c_2\vec{a}_2) \ = \ c_1 \cdot \sum (\vec{a}_1) \ + \ c_2 \cdot \sum (\vec{a}_2) \ ,$$

for arbitrary constants $c_1, \ c_2 \in \mathbb{R}$.

Finally, to show linearity as an operator on matrices, consider two constants c_1 and c_2, and two $n \times n$ matrices $\mathcal{A}$ and $\mathcal{B}$. Now compute the sum

$$\sum \ (c_1\mathcal{A} + c_2\mathcal{B}) \ =$$
$$\sum \big((c_1\vec{a}_1 + c_2\vec{b}_1) \ \ (c_1\vec{a}_2 + c_2\vec{b}_2) \ \ \ldots \ \ (c_1\vec{a}_n + c_2\vec{b}_n)\big)$$
$$= \big(\textstyle\sum(c_1\vec{a}_1 + c_2\vec{b}_1) \ \ \sum(c_1\vec{a}_2 + c_2\vec{b}_2) \ \ \ldots \ \ \sum(c_1\vec{a}_n + c_2\vec{b}_n)\big)$$
$$= \big((c_1 \textstyle\sum \vec{a}_1 + c_2 \sum \vec{b}_1) \ \ (c_1 \sum \vec{a}_2 + c_2 \sum \vec{b}_2) \ \ \ldots \ \ (c_1 \sum \vec{a}_n + c_2 \sum \vec{b}_n)\big)$$
$$= (c_1 \cdot \textstyle\sum(\vec{a}_1) \ \ c_1 \cdot \sum(\vec{a}_2) \ \ \ldots \ \ c_1 \cdot \sum(\vec{a}_n))$$
$$+ \ \big(c_2 \cdot \textstyle\sum(\vec{b}_1) \ \ c_2 \cdot \sum(\vec{b}_2) \ \ \ldots \ \ c_2 \cdot \sum(\vec{b}_n)\big)$$
$$= c_1 \cdot \sum(\mathcal{A}) \ + \ c_2 \cdot \sum(\mathcal{B}) \ .$$

The properties of constant multiplication and addition for column vectors were used in the steps of the proof. □

By the hypotheses of the theorem we have that $\sum \vec{a}_j = 0$ for the matrix $\mathcal{A}$ as defined in (4.7). To see how $\mathcal{A}$ acts, consider any constant vector $\vec{v} = (v_1 \ v_2 \ \ldots v_n)^T$.

Then $\mathcal{A}\vec{v}$ is in the range space of $\mathcal{A}$, which is also the image of $\mathcal{A}$:

$$Range(\mathcal{A}) \;=\; Image(\mathcal{A}) \;=\; span(\mathcal{A}) \;\equiv\; \left\{ \sum_{j=1}^{n} c_j \vec{a}_j \mid \forall c_j \in \mathbb{R} \right\} .$$

This means that $\mathcal{A}\vec{v}$ can be written as a linear combination of the vectors $\{\vec{a}_1,\ \vec{a}_2,\ \ldots,\ \vec{a}_n\}$. Observe that taking the sum of the components of $\mathcal{A}\vec{v}$ gives the following

$$\mathcal{A}\vec{v} \;=\; (\vec{a}_1 \quad \vec{a}_2 \quad \ldots \quad \vec{a}_n) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \;=\; v_1\vec{a}_1 + v_2\vec{a}_2 + \cdots + v_n\vec{a}_n \ . \tag{4.8}$$

Lemma 4.4. *Suppose* $\sum(\mathcal{A}) = \vec{0}$*. Then* $\sum(\mathcal{A}^k\vec{v}) = 0$ *for each* $k \in \mathbb{N}$ *and* $\vec{v} \in \mathbb{R}^n$*,*

Proof. For the $k = 1$ case, use (4.8) and the previous Lemma to compute,

$$\begin{aligned} \sum(\mathcal{A}\vec{v}) &= v_1 \sum(\vec{a}_1) + v_2 \sum(\vec{a}_2) + \cdots + v_n \sum(\vec{a}_n) \\ &= v_1 \cdot 0 + v_2 \cdot 0 + \cdots + v_n \cdot 0 \\ &= \vec{v} \cdot \vec{0} \;=\; 0 \ . \end{aligned} \tag{4.9}$$

Now, for $k > 1$, write

$$\sum\left(\mathcal{A}^k\vec{v}\right) \;=\; \sum\left(\mathcal{A}\mathcal{A}^{k-1}\vec{v}\right) \;=\; \sum\left(\mathcal{A}\left(\mathcal{A}^{k-1}\vec{v}\right)\right) \;=\; \left(\mathcal{A}^{k-1}\vec{v}\right) \cdot \vec{0} \;=\; 0 \ ,$$

since $\vec{v} \in \mathbb{R}^n$ implies $\mathcal{A}^{k-1}\vec{v} \in \mathbb{R}^n$. □

To complete the theorem, apply the sum to the system in (4.10), and use the previous two Lemmas along with the condition that $\sum \mathcal{A} = \vec{0}$,

$$\begin{aligned} \sum\left(\vec{Y}(t)\right) &= \sum\left(\mathcal{I} \cdot \vec{Y}_0\right) \;+\; \sum_{k=1} \frac{t^k}{k!} \sum\left(\mathcal{A}^k \cdot \vec{Y}_0\right) \\ &= \sum\left(\vec{Y}_0\right) \ . \end{aligned} \tag{4.10}$$

Thus $\sum \vec{Y}(t) \;=\; \sum(\vec{Y}_0) = 0$ for all $t \geq 0$. □

The evolution of the system in (4.5) preserves the total content of the state, which in the cases considered here corresponds to the total amount of liquid money.

4.1.2 Preservation of liquid accounts

Next we address the positivity. In particular, we consider when non-negative accounts stay non-negative. Typically, a negative account is considered to be in default, which leads to bankruptcy. In practice negative accounts have been allowed through a policy of overdrafts. These are temporary mechanisms used to maintain legitimacy, but we will not attempt to incorporate such activity in our models.

Theorem 4.5. *Consider the IVP (initial value problem)*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -1 & b & 0 \\ a & -b & 0 \\ 1-a & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} , \quad \begin{pmatrix} x \\ y \\ c \end{pmatrix}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} , \tag{4.11}$$

for $a,\ b \in (0,1)$*. This system has no solution where* $x(t) \geq 0$*,* $y(t) \geq 0$*,* $z(t) \geq 0$ *for all* $t \geq 0$*.*

Proof. Define the 3×3 matrix $\mathcal{A} \equiv \begin{pmatrix} -1 & b & 0 \\ a & -b & 0 \\ 1-a & 0 & 0 \end{pmatrix}$. It has the characteristic polynomial

$$\begin{aligned} p_{\mathcal{A}}(\lambda) &= \det(\mathcal{A} - \lambda \mathcal{I}) = \det \begin{pmatrix} -1-\lambda & b & 0 \\ a & -b-\lambda & 0 \\ 1-a & 0 & -\lambda \end{pmatrix} \\ &= (1+\lambda) \cdot (b+\lambda) \cdot \lambda \ + \ b \cdot \lambda \cdot a \ + \ 0 \\ &= \lambda^3 \ + \ (1+b+ab) \cdot \lambda^2 \ + \ b \cdot \lambda \\ &= \lambda \cdot (\lambda^2 \ + \ (1+b+ab) \cdot \lambda \ + \ b) \ . \end{aligned}$$

The spectrum of $\mathcal{A}$, is the set of eigenvalues

$$\sigma(\mathcal{A}) \equiv \left\{ 0, \ \left[-(1+b+ab) \pm \sqrt{(1+b+ab)^2 - 4b} \right] /2 \right\} \ .$$

Thus there is one vanishing eigenvalue and 2 eigenvalues with negative real parts. In fact this system can never have an imaginary part because

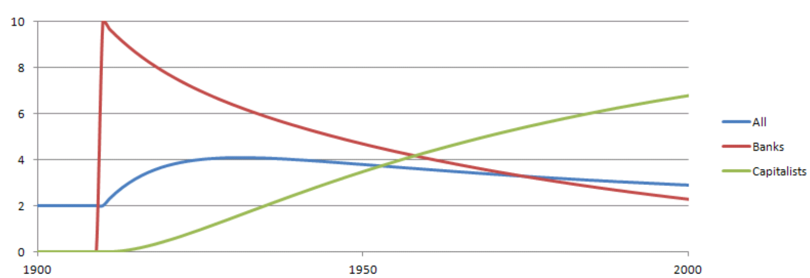
$$(1+b+ab)^2 - 4b \;=\; (1-b)^2 + 2ab\cdot(1+b) + a^2b^2 \;>\; 0\;,$$

for $a > 0,\ b > 0$. Thus, the component of the initial conditions corresponding to the negative eigenvalues will vanish over time, and the asymptotic solution will tend towards the zero-eigenvector direction. Thus we only solve the zero-eigenvector equation:

$$\mathcal{A}\vec{v} \;=\; \vec{0} \quad\Longrightarrow\quad \begin{pmatrix} -1 & b & 0 \\ a & -b & 0 \\ 1-a & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad\Longrightarrow\quad \begin{cases} x = y \\ ax = by \\ (1-a)x = 0 \end{cases} .$$

The third equation implies $x = 0$ and the first equation gives $y = 0$. Thus, the equilibrium vector is $(0,\ 0,\ M_0)^T$ for some $M_0 \in \mathbb{R}$. □

Figure 1:



A-B-C model parameters: A_{min}, $M_0 = 10$, $t_* = 1910$, $ra = 0.04$, and $\alpha = 0.05$.

The model describes as dynamical system where money M_0 is given to the banks, which they eventually lend to all workers in the society. However, as the workers pay off their debts while making purchases from capitalists, their buying power eventually fades. The banks expell all of their reserves, while the proceeds of purchases go to the capitalists. This model preserves the total amount of money M_0 in the system. Approach to equilibrium proceeds in the form of exponential decay.

4.2 Two unrealistic ABC models

As a comparison to other models, we consider a couple of unreasonable cases. These will demonstrate that one cannot just choose any values for parameters and call the resulting dynamical system a model for a sustainable economy.

4.2.1 Borrowing increases with A

First , suppose that

- A borrows from B with rate constant a proportional to A (not B)
- A purchases from C with rate constant c proportional to B (not A)
- C pays, on borrowed startup costs, to B at rate b proportional to C

Here we assume no minimum, i.e. $A_{min} = 0$. The model, as a 3-dimensional system, of linear equations:

$$\frac{d}{dt}\begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} a \cdot A(t) - c \cdot B(t) \\ -a \cdot A(t) + b \cdot C(t) \\ c \cdot B(t) - b \cdot C(t) \end{pmatrix} , \quad \begin{pmatrix} A(0) \\ B(0) \\ C(0) \end{pmatrix} = \begin{pmatrix} 0 \\ M_0 \\ 0 \end{pmatrix} , \tag{4.12}$$

or as a system of matrix equations

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix}' = \begin{pmatrix} a & -c & 0 \\ -a & 0 & b \\ 0 & c & -b \end{pmatrix} \cdot \begin{pmatrix} A \\ B \\ C \end{pmatrix} , \quad \begin{pmatrix} A \\ B \\ C \end{pmatrix}_0 = \begin{pmatrix} 0 \\ M_0 \\ 0 \end{pmatrix} . \tag{4.13}$$

The characteristic polynomial is

$$p(\lambda) = -\lambda \left[\lambda^2 + (b-a)\lambda - (ab + bc + ac)\right] .$$

There is an obvious zero root of this polynomial. The other two roots are

$$\lambda \;=\; \frac{(a-b)}{2} \;\pm\; \frac{1}{2}\,\sqrt{(a-b)^2+4(ab+bc+ac)}\;, \tag{4.14}$$

assuming $a,\ b,\ c$ are positive, nonzero. Then the system has at least one strictly positive eigenvalue. This implies unbounded growth, unless initial conditions are chosen inside the linear span of the eigenspace for the 0 and negative eigenvalues.

4.2.2 Borrowing decreases with A

The requirement that $b > a$ in Eq.(4.14) ensures the existence of at least one negative eigenvalue for the system in Eq.(4.13). To obtain two negative eigenvalues consider making thereplacement $a \to -a$. Then the system becomes

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix}' \;=\; \begin{pmatrix} -a & -c & 0 \\ a & 0 & b \\ 0 & c & -b \end{pmatrix} \cdot \begin{pmatrix} A \\ B \\ C \end{pmatrix}\;,\quad \begin{pmatrix} A \\ B \\ C \end{pmatrix}_0 \;=\; \begin{pmatrix} 0 \\ M_0 \\ 0 \end{pmatrix}\;. \tag{4.15}$$

Besides the zero eigenvalue, the roots of $p(\lambda)$ are

$$\frac{-(a+b)}{2} \;\pm\; \frac{1}{2}\,\sqrt{(a+b)^2-4(ab-bc+ac)} \;=\; \frac{-(a+b)\;\pm\;\sqrt{(b-a)^2+(b-a)c}}{2}\;,$$

which are both negative for $b > a \geq 0$, and $c \geq 0$ sufficiently small. However, the equilibrium state is a solution to the 0-eigenvector equation

$$\begin{pmatrix} -a & -c & 0 \\ a & 0 & b \\ 0 & c & -b \end{pmatrix} \cdot \begin{pmatrix} A \\ B \\ C \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \;\Longrightarrow\; \begin{cases} -aA-cB=0 \\ aA+bC=0 \\ cB-bC \end{cases} \;\Longrightarrow\; \begin{cases} B=-(a/c)A \\ C=(-a/b)A \\ \end{cases}\;. \tag{4.16}$$

The system in general will tend toward a zero eigenvector, which is in the eigenspace

$$\mathcal{E}_0 \equiv \left\{ s \cdot \begin{pmatrix} bc \\ -ab \\ -ac \end{pmatrix} \;\middle|\; s \in \mathbb{R} \right\}$$

The problem here is that for non-zero a, b and c, we cannot have all accounts being non-zero. For the dynamical system this means that one can still have

$$A(t) + B(t) + C(t) = M_0 > 0 \ ,$$

while asymptotically at least one account goes bankrupt. Further discussion of this situation is presented in this thesis.

4.3 A basic ABC with delayed payments

To help consumers cope with loan payments a delay in payment $\tau > 0$ is sometimes allowed. This results in an adjustment to the system in (4.17) so that

$$\frac{d}{dt}\begin{pmatrix} \overline{A}(t) \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} -ra \cdot \overline{A}(t-\tau) - \alpha \cdot \overline{A}(t) + rb \cdot B(t) \\ ra \cdot \overline{A}(t-\tau) - rb \cdot B(t) \\ \alpha \cdot \overline{A}(t) \end{pmatrix} \ , \quad \begin{pmatrix} \overline{A}(t_-) \\ B(t_-) \\ C(t_-) \end{pmatrix} = \begin{pmatrix} 0 \\ M_0 \\ 0 \end{pmatrix} \ , \tag{4.17}$$

where $t_- \leq 0$. To solve this equation, knowledge is needed about past values of the accounts. Here we assume that all accounts were frozen until $t = 0$. When $\tau > 0$ the accounts no longer sum to M_0 and the totals exhibit an oscillatory behavior.

We see that there more of a spike in the All accounts, with the presence of a delay. Furthermore, an oscillation, or small boom and bust cycle does appear, endogenously.

Figure 2

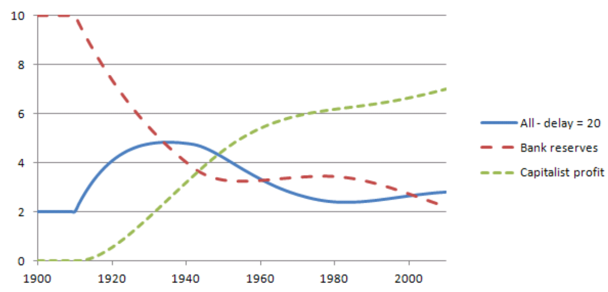
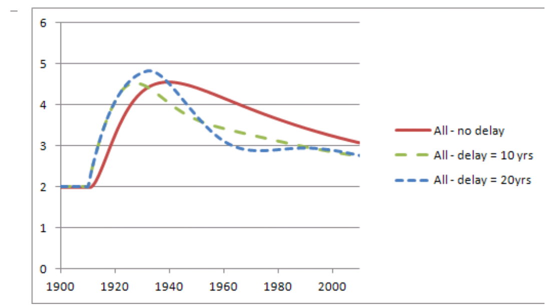


Figure 3



ABC model parameters: A_{min}, $M_0 = 10$, $ra = 0.2$, $rb = 0.2$, $\alpha = 0.10$, $\tau \leq 20$.
Left: ABC with delay $\tau = 20yrs$ payback, **Right:** All, with $\tau = 0, 10, 20yrs$.

4.4 Behind the scenes indebtedness to banks

In this thesis we assume that rather than declarations of bankruptcy, debt is simply restructured. However, this does not deter from the fact that "loans" mean "obligations" also called *IOUs*. Over time the amount owed to the banks is far beyond what can ever be paid, but these IOUs may still be recorded as an asset by the banks contributing to net worth. If there is little of value in the form of collateral to back up these assets, then the loans are unsecured and the increase in debt represents a debt bubble. The bubble bursts when a banker stops restructuring and asks for a repayment of the IOU, in full.

To model this dynamic, recall B lends to A at rate $rb > 0$, from equation (4.1). Thus at time t the amount loaned out to the workers, over a small period Δt, is

$$\Delta L_i \;=\; rb \cdot B(t) \cdot \Delta t \;.$$

This loan is paid back, during each time period Δt, by the amount of

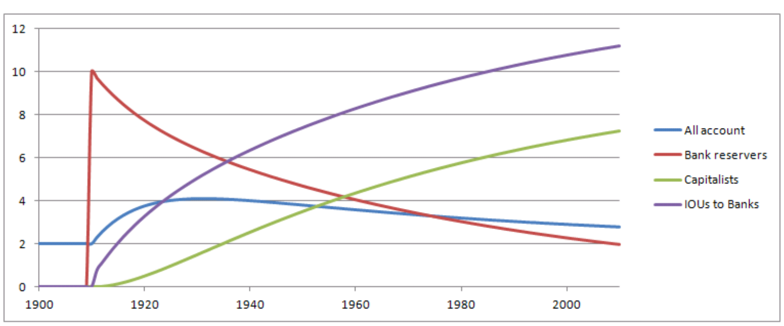
$$\Delta L_o \;=\; ra \cdot (A(t) - A_{min}) \cdot \Delta t \;,$$

due to (4.2). Assuming a debt-free environment initially, the debt bubble is measured using the integral

$$L(t) \;=\; \int_0^t (dL_i \;-\; dL_o) \;=\; \int_{u=0}^t (rb \cdot B(u) \;-\; ra \cdot (A(u) - A_{min}))\; du \;.$$

As the bank reserves get depleted, they are replenished with IOUs to an amout that exceeds the original amount put into the system by the FED. Thus, at least on paper, the banks could become independent of the FED by paying them back (assuming that the FED charges 0% interest.) Indeed, returning to the previous example in section 4.1 we see that after several decades the IOUs to the banks actually exceed the amount accumulated by the capitalists (see Figure 4). This shows that in a credit economy both the banks and capitalists grow in worth, whereas the workers loose buying power (compare with Figure 1).

Figure 4:



A-B-C model parameters: A_{min}, $M_0 = 10$, $ra = 0.2$, $rb = 0.2$, $\alpha = 0.10$, $\tau \leq 20$.

Missing from this discussion is the actual value of the goods (durables and perishables) and services (in the form of functioning instritutions) provided to the society by the activities of workers. These products provide the collateral that backs up the IOUs owed to the banks. However, this leads to another concern, not considered here, which are the liquid funds required to maintain the status and supply of goods and services. With an inclusion of depreciation, a rationale for introducing more funds into the system becomes apparent. Thus the FED has an important *exogenous* role to play in averting the bursting of a debt bubble.

CHAPTER 5: A simple 4-dimensional financial-economic system for BC

We begin a mathematical study by considering a futuristic world consisting of only bankers B and capitalists C. Nothing can happen without a set of rules for engagement, thus any interaction between these groups must occur with some government in the background. One role of government is to enforce contractual agreements often expressed in terms of coinage and paper money. Another role of government is to provide *base money* $M_0 > 0$, periodically, to the system. This can be done in bursts, which will be modeled with the use of delta functions:

$$\delta_{t_*}(t) \;=\; \delta_0(t - t_*) \;\equiv\; \begin{cases} \infty \;, & t = t_* \\ 0 \;, & t \neq t_* \end{cases} \;, \quad \int_{-\infty}^{\infty} \delta_{t_*}(t)\; dt \;\equiv\; 1 \;. \tag{5.1}$$

Suppose that, in the begining ($t < t_*$), there is no money:

initial conditions (IC): $B_{i,0} \;=\; B_{o,0} \;=\; C_{i,0} \;=\; C_{o,0} \;=\; 0.$

Dynamical equations now need to be constructed. Assume that a policy is in place whereby if a bank has funds on hand $B_i > 0$, it will quickly transfer these to its lending account B_o at the rate $b^i_o \simeq 0$. The notation b^i_o is used to represent the transfer rate from B_i to B_o. Once B_o is solvent, assume that the bank steadily lends out these funds to C, at a rate $b_i \gg 0$. The companies will deposit these funds in their C_i accounts, earning interest $c_i > 0$ while having to pay back the bank with interest on the loans $c_o > c_i$. This leads to the coupled linear system of ordinary differential equations (4-dimensional system of ODEs):

$$\frac{d}{dt}\begin{pmatrix} B_i \\ B_o \\ C_i \\ C_o \end{pmatrix} = \begin{pmatrix} -b^i_o \cdot B_i + rd \cdot (C_i + C_o) - rc \cdot C_o \\ b^i_o \cdot B_i - bc \cdot B_o \\ bc \cdot B_o + (rc - c^i_o) \cdot C_i \\ (c^o_i - rd) \cdot C_i - rd \cdot C_o \end{pmatrix} \;, \; \textbf{IC} : \begin{pmatrix} B_i \\ B_o \\ C_i \\ C_o \end{pmatrix}_0 = \begin{pmatrix} B_{i,0} \\ B_{o,0} \\ C_{i,0} \\ C_{o,0} \end{pmatrix} \;. \tag{5.2}$$

This represents an endogenous system since the forcing is $\vec{F} = \vec{0}$.

5.0.1 $B - C$ example with forcing

First we express the system (5.2) in matrix form, and inject $M_0 > 0$ into this economy through B_i. This represents an exogenous forcing vector on the system. expressed as

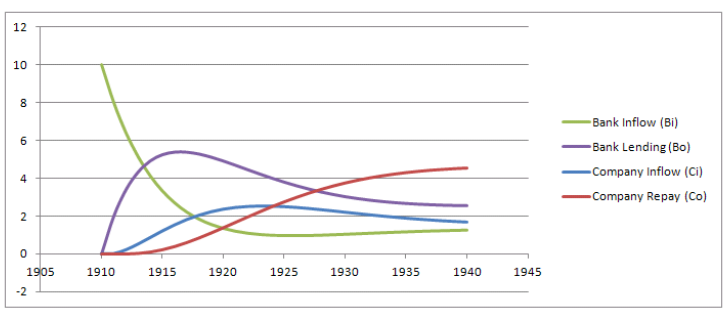
$$\begin{pmatrix} B_i \\ B_o \\ C_i \\ C_o \end{pmatrix}' = \begin{pmatrix} -b_o^i & 0 & rd - rc & rd \\ b_o^i & -bc & 0 & 0 \\ 0 & bc & rc - c_o^i & 0 \\ 0 & 0 & c_i^o - rd & -rd \end{pmatrix} \begin{pmatrix} B_i \\ B_o \\ C_i \\ C_o \end{pmatrix} + M_0 \begin{pmatrix} \delta_{t*} \\ 0 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} B_{i,0} \\ B_{o,0} \\ C_{i,0} \\ C_{o,0} \end{pmatrix} = \vec{0} \, . \quad (5.3)$$

The relative size of these parameters is

$$\infty \; > \; (\text{fast}) \quad b_o^i \; \simeq \; c_o^i \; \gg \; bc \; \gg \; rd \; > \; rc \quad (\text{slow}) \; > \; 0 \, . \qquad (5.4)$$

Note that the sum $B_i(t) + B_o(t) + C_i(t) + C_o(t) = M_0$ for all $t > t_*$.

Figure 5:



B-C model parameters: $M_0 = 10$, $t_* = 1910$, $b_o^i = 0.2$, $c_o^i = 0.2$, $bc = 0.10$, $rd = 0.05$ and $rc = 0.03$.

Observe that Figure 1 suggests the existence of an asymptotic level for all accounts, wherein are all accounts are non-zero. Also, note that the company's credit account C_o retains the largest amount of reserves in spite of the fact that the bank has claim to all funds in the system, with interest. However, contrary to our intuition, the cycle of borrowing and repayment leads to a stable system in perpetuity. It is kept that way by only requiring the company to pay what it can afford, so bankruptcy is made impossible.

It should not be forgotten that the government entity, which at $t_* = 1$ infused funds $M_0 = 10$ into the system through $\vec{F}$, has ultimate claim to all funds in this system. This is set up so that the company will use the funds in C_o to support the costs of creating goods and services for profit. The profit represents private wealth and it would be generated by the credit economy if a business is successful.

5.0.2 Equilibria of the $B - C$ model

An equilibrium is a constant vector of the variables, like the initial condition vector, where the rate of change vanishes. Thus we solve the system of equations, from (5.2),

$$\vec{0} = \begin{pmatrix} -b_o^i B_i + rd(C_i + C_o) - rcC_o \\ b_o^i \cdot B_i - bc \cdot B_o \\ bc \cdot B_o + (rc - c_o^i) \cdot C_i \\ (c_i^o - rd) \cdot C_i - rd \cdot C_o \end{pmatrix} \implies \begin{cases} b_o^i B_i = rdC_i + (rd - rc)C_o \\ b_o^i \cdot B_i = bc \cdot B_o \\ bc \cdot B_o = -(rc - c_o^i) \cdot C_i \\ rd \cdot C_o = (c_i^o - rd) \cdot C_i \end{cases} \tag{5.5}$$

A trivial solution to (5.5) is $(B_i,\ B_o,\ C_i,\ C_o)^T = \vec{0}$, which only occurs if $M_0 = 0$.

Theorem 1. *Let* $M_0 = B_i + B_o + C_i + C_o > 0$*. The system in* (5.2)*, under the conditions in* (5.4)*, has a unique solution, which represents a stable equilibrium for the system in* (5.2)*.*

Proof. The matrix in (5.3) can be expressed as $\mathcal{A} \equiv \begin{pmatrix} -a & 0 & d & e \\ a & -b & 0 & 0 \\ 0 & b & -(c+d) & 0 \\ 0 & 0 & c & -e \end{pmatrix}$, where the components satisfy: $a \simeq c \gg b \gg e > d > 0$ due to (5.4). Now observe that the sum of the columns of $\mathcal{A}$ is 0, thus $\mathcal{A}$ has a 0 eigenvalue. The characteristic polynomial simplifies to

$$p_{\mathcal{A}}(\lambda) \;=\; (\lambda+a)(\lambda+b)(\lambda+d)(\lambda+e) \;-\; abde\ . \tag{5.6}$$

Lemma 5.1. *The polynomial $p_{\mathcal{A}}$ in (5.6) has a zero root and at least one strictly negative root. If the remaining two zeros are real, then they are negative. However, if these roots of $p_{\mathcal{A}}(\lambda)$ are complex, then they will each have a negative real part.*

Proof. Expanding the expression in (5.6) it is clear that λ is a factor of $p_{\mathcal{A}}(\lambda)$. This ensures a zero root. The derivative at $\lambda = 0$ satisfies

$$p'_{\mathcal{A}}(0) \;=\; bde + ade + abe + abd \;>\; 0\ , \tag{5.7}$$

so $\lambda = 0$ is non-degenerate. Also, since $p_{\mathcal{A}}$ is a forth degree polynomial, it must tend to $+\infty$ as $|\lambda|$ increases. Hence (5.7) ensures that $p_{\mathcal{A}}(\lambda) < 0$ for some $\lambda \simeq 0$. Thus there is one stricly negative eigenvalue.

Now consider the rational polynomial

$$q_{\mathcal{A}}(\lambda) \;=\; \frac{p_{\mathcal{A}}(\lambda)}{\lambda(\lambda+a)} \;\simeq\; \frac{(\lambda+b)(\lambda+d)(\lambda+e) \;-\; bde}{\lambda}\ .$$

All coefficients of the quadratic polynomial on the right are positive, so the vertex is negative, and this is the real part, if the roots are complex. □

To complete the theorem, it is sufficient to find a 0 eigenvector $\vec{v}_0 = (v_1,\ v_2,\ v_3,\ v_4)^T$

for A in order to determine the asymptotic value of the solution to (5.3).

$$0\cdot\vec{v}_0 \;=\; A\cdot\vec{v}_0 \;=\; \begin{pmatrix} -a & 0 & d & e \\ a & -b & 0 & 0 \\ 0 & b & -(c+d) & 0 \\ 0 & 0 & c & -e \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \;=\; \begin{pmatrix} -av_1+dv_3+ev_4 \\ av_1-bv_2 \\ bv_2-(c+d)v_3 \\ cv_3-ev_4 \end{pmatrix} \quad (5.8)$$

The first component gives the equation $v_1 = (d/a)v_3 + (e/a)v_4$. The eigenvector can be expressed in terms of v_2 as

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \;=\; \begin{pmatrix} (b/a)\cdot v_2 \\ v_2 \\ b/(c+d)\cdot v_2 \\ (c/e)\cdot v_3 \end{pmatrix} \;=\; \begin{pmatrix} b/a \\ 1 \\ b/(c+d) \\ bc/[e(c+d)] \end{pmatrix}\cdot v_2 \;=\; \begin{pmatrix} be(c+d) \\ ae(c+d) \\ abe \\ abc \end{pmatrix}\cdot\frac{v_2}{ae(c+d)}\,. \quad (5.9)$$

As a check, note that the 0-eigenvector $\vec{v}_0 = (be(c+d),\ ae(c+d),\ abe,\ abc)^T$ satisfies the first component equation $av_1 = dv_3 + ev_4$.

All the components of v_0 are strictly positive. The equilibrium vector is N_0v_0 must have a sum of its components equal to $M_0 > 0$. In this case the normalization constant is

$$N_0 \;\equiv\; \frac{M_0}{abc+abe+ace+ade+bce+bde} \;=\; \frac{M_0}{b_o^i bc\cdot c_i^o \;+\; (b_o^i+bc)(c_i^o-rc)rd}\,.$$

The positivity of all components, and the condition that $c_i^o \gg rc$ ensures a unique equilibrium for the system (5.2). □

First find a formula for Let B_i, B_o, C_i and C_o. in terms of M_0 and the parameters. Then setup the eigenvalue problem. We need to show that all eigenvalues are negative, which implies stability.

CHAPTER 6: A simple 4-dimensional financial-agrarian system for AB

Next we consider an ancient society consisting of only bankers B and agrarians A. Once again the main role of government is to provide a trusted currency of amount M_0. Suppose that, in the begining ($t < t_*$), there is no money:

initial conditions (IC): $B_{i,0} = B_{o,0} = 0$ and $A_{i,0} \geq A_{o,0} = A_{min}$.

One reason for a subsistence populace to enter into a credit economy agreement is to improve productivity and make $A_{i,0} \gg A_{min}$. Funding the work required to finish a project requires a commitment to reimburse laborers into the future for services delivered now. Hence banking allows a relationship of the form

microfinance: intense activity now $\longleftrightarrow$ resource commitment long-term

This model begins with a bank that has funds on hand $B_i > 0$, that it transfers to its lending account B_o at the rate $b_o^i \simeq 0$. Once B_o is solvent, assume that the bank lends out these funds to A, at a rate $b_i \gg 0$, so they can complete their project. Assuming a success, the new productivity will be A_{prod} and the productivity gain is $A_{prod} - A_{min} > 0$. However, to achieve a benefit, the productivity rate π must exceed the payback rate rd. This requirement suggests that an increase in the excess account, must be used to pay the debt obligations to the bank.

The agrarians will spend all borrowed funds that enter their A_i accounts, and will earn no interest. The obligation to the bank will have an interest rate $rd > 0$. This leads to the linear autonomous system of ODEs:

$$\frac{d}{dt}\begin{pmatrix} A_i \\ A_o \\ B_i \\ B_o \end{pmatrix} = \begin{pmatrix} -rd(A_i - A_{min}) + \pi(A_o - A_{min}) - a_o^i(A_i - A_o) + baB_o \\ -\pi \cdot (A_o - A_{min}) + a_o^i \cdot (A_i - A_o) \\ -b_o^i \cdot B_i + rd \cdot (A_i - A_{min}) \\ b_o^i \cdot B_i - ba \cdot B_o \end{pmatrix} . \tag{6.1}$$

The total money in the system $M_0 > 0$ will remain constant.

6.0.3 $A - B$ example with initial lending agreement

The system (6.1) can be expressed in matrix form by first redefining variables

$$\overline{A}_i \;\equiv\; A_i - A_{min}\;,\quad \overline{A}_o \;\equiv\; A_o - A_{min} \qquad \Longrightarrow \qquad \overline{A}_i{}' \;=\; A_i'\;,\quad \overline{A}_o{}' \;=\; A_o'\;. \qquad (6.2)$$

Then we obtain the initial value problem:

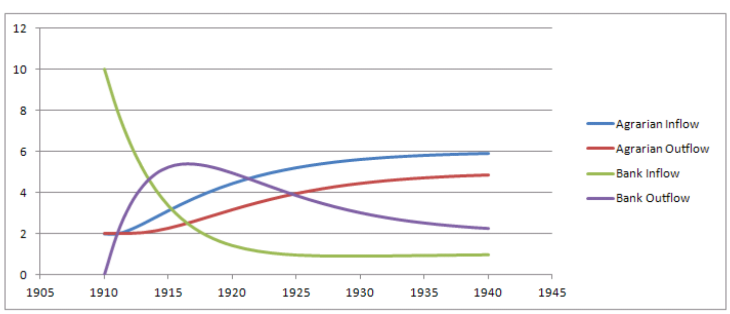
$$\begin{pmatrix} \overline{A}_i \\ \overline{A}_o \\ B_i \\ B_o \end{pmatrix}' = \begin{pmatrix} -rd - a_o^i & a_o^i + \pi & 0 & ba \\ a_o^i & -a_o^i - \pi & 0 & 0 \\ rd & 0 & -b_o^i & 0 \\ 0 & 0 & b_o^i & -ba \end{pmatrix} \begin{pmatrix} \overline{A}_i \\ \overline{A}_o \\ B_i \\ B_o \end{pmatrix}\;, \quad \begin{pmatrix} A_{i,0} \\ A_{o,0} \\ B_{i,0} \\ B_{o,0} \end{pmatrix} = \begin{pmatrix} A_{min} \\ A_{min} \\ M_0 \\ 0 \end{pmatrix}\;, \qquad (6.3)$$

where the relative size of these parameters is considered to be

$$\infty \;>\; (\text{fast}) \quad a_o^i \;\simeq\; b_o^i \;\gg\; ba \;\gg\; \pi \;>\; rd \quad (\text{slow}) \;>\; 0\;. \qquad (6.4)$$

Note that the sum $A_i(t) + A_o(t) + B_i(t) + B_o(t) = M_0 + 2A_{min}$ for all $t > t_*$.

Figure 6:



A-B model parameters: $M_0 = 10$, $A_{min} = 2$, $t_* = 1910$, $a_o^i = 0.2$, $b_o^i = 0.2$, $ba = 0.10$, $\pi = 0.07$ and $rd = 0.05$.

The numerical simulation in Figure 2 clearly indicates that all the accounts are non-vanishing asymptotically. The agrarian's input account A_i obtains the largest amount of reserves, while the bank never regains full possesion of its original investment M_0. The system results in the desired objective for the agrarians where in $A_i > A_o > A_{min}$. Thus the community will have an excess, sustained by a continual process of borrowing. However, the wealth of the community erodes over time due to the new credit economy.

The role of government in this debt trap, comes from providing currency that backs future exchange for goods. Although it appears that there is development and excess, ownership remains with the banks. Some of the difference $A_i - A_o$ can be siphoned out of the system into private hands, leaving the obligations created by the credit economy for others to deal with.

6.0.4 Equilibrium of the $A - B$ model

The 4×4 matrix in (6.3) has a 0 eigenvalue since its columns sum to zero, so its determinant vanishes, which is the product of the eigenvalues. The 0 eigenvector that is the asymptotic solution to (6.1) will solve

$$\overline{A}_i + \overline{A}_o + B_i + B_o = M_0 \qquad \text{and} \qquad \begin{cases} (rd + a_o^i)\overline{A}_i = (\pi + a_o^i)\overline{A}_o + baB_o \\ (\pi + a_o^i)\overline{A}_o = a_o^i a_o^i \overline{A}_i \\ b_o^i B_i = rd \cdot \overline{A}_i \\ ba \cdot B_o = b_o^i \cdot B_i \end{cases} \tag{6.5}$$

A trivial solution to (6.5) is $(B_i,\ B_o,\ C_i,\ C_o)^T = \vec{0}$, which only occurs if $M_0 = 0$.

Theorem 2. *Let* $M_0, A_{min} > 0$ *and suppose* $A_i + A_o + B_i + B_o = M_0 + 2A_{min}$. *The system in* (6.1), *under the conditions in* (6.4), *has a unique solution, which represents a stable equilibrium for the system in* (6.1).

Proof. The matrix in (6.3) can be expressed as

$$\mathcal{A} \equiv \begin{pmatrix} -a & b & 0 & e \\ a-d & -b & 0 & 0 \\ d & 0 & -c & 0 \\ 0 & 0 & c & -e \end{pmatrix} , \tag{6.6}$$

where the components satisfy: $a \simeq b \simeq c \gg e \gg d > 0$ due to (6.4). The matrix $\mathcal{A}$ has a 0 eigenvalue. To analyze this case, consider the special case that $a = b = c > e$ and $d = 0$. Then the characteristic polynomial for $\mathcal{A}$ becomes

$$p_{\mathcal{A}}(\lambda) \;=\; \lambda \cdot (\lambda + 2a) \cdot (\lambda + a) \cdot (\lambda + e) , \tag{6.7}$$

giving that $\sigma(\mathcal{A}) = \{0, \; -e, \; -a, \; -2a\}$. The 0 eigenvector $\vec{v}_0 = (v_1, \; v_2, \; v_3, \; v_4)^T$ for $\mathcal{A}$ is found by solving the equation

$$0 \cdot \vec{v}_0 \;=\; A \cdot \vec{v}_0 \;=\; \begin{pmatrix} -a & a & 0 & e \\ a & -a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & a & -e \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \;=\; \begin{pmatrix} -av_1 + av_2 + ev_4 \\ av_1 - av_2 \\ -cv_3 \\ cv_3 - ev_4 \end{pmatrix} \tag{6.8}$$

The third component gives $v_3 = 0$, and using this in the fourth equation gives $v_4 = 0$. The second equation gives $v_1 = v_2$ and this is now compatible with the first equation. The zero eigenvector, or asymptotic equilibrium, can be expressed as

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \;=\; \begin{pmatrix} v_2 \\ v_2 \\ 0 \\ 0 \end{pmatrix} \;=\; v_2 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \;=\; \begin{pmatrix} M_0/2 \\ M_0/2 \\ 0 \\ 0 \end{pmatrix} , \tag{6.9}$$

using the condition that the components add to $M_0 > 0$. The fact that the equilibrium has the banker accounts vanish, and the two worker accounts are nearly the same is seen in Figure 6.

To obtain the general result, recall the condition in (6.4). Consider the components in (6.6) to be perturbations of the specal case where $a = b = c > e$, $d = 0$. The specturm will change as the components change, with the restrictions that the 0 eigenvalue remains fixed, and the negative eigenvalues remain real and negative. □

This model has no capitalists, so it may seem to represent a worker's paradise. However, as discussed in Chapter 3, and displayed in Figure 4, there is an growing debt bubble due to continued borrowing.

CHAPTER 7: A simple 4-dimensional capitalist-worker systems AC

Consider an economic system that is comprised of workers A and capitalists C. Typically there is payment for labor, which is a service, in order to create a product or commodity, which are goods. The capitalist brings the finished good to a market where the price received, called the revenue, is expected to exceed the costs. The profit is the revenue less the cost, and it is kept with C to be used in two ways:

Say-Walrus action: profits are used to buy goods, services or property

Minsky-Marx action: profits add to personal wealth and bank creation

A Say-Walrus action will put money back into the economic system, requiring workers in different areas of the economy to finish goods needed by the capitalists. This gives a tension to the system whereby wages can be negotiated based on demand for labor. This will increase the cost of production, so if C wants to maintain profits, prices will rise, resulting in inflation. This affects demand for products, so sales will decrease resulting in a slowdown of the economy. Then, the demand for labor decreases and workers are forced to accept lower wages. Capitalists are then expected to lower their prices resulting in deflation to the point where sales increase. This completes the cycle that is partially expressed in a diagram:

Phillips Curve: inflation versus unemployment (often inversely related)

When an economy experiences extensive Minsky-Marx actions, then money is taken out of circulation. So more money has to be printed, diluting the value of the currency and thus reducing the value of profits, and motivating capitalists to increase prices. Thus hoarding can lead to an inflationary spiral. However, when wage earners do not participate in the inflationary decrease in value of money, then their buying power errodes over time and A is reduced to A_{min}.

The desire to make a purchase does not imply an actual rate of purchasing, known as the velocity of money, measured as $V = \#exchanges/(1\$ \times 1month)$.

Equation of Exchange: $M_0 \cdot V =$ Price $\times$ Output

One has to have purchasing power, measured using the difference $A_i - A_{min}$.

purchase action: "intent to purchase" with sufficient "cash on hand"

Purchases made using money in the form of MZM (money with zero maturity), implies that in effect a barter has taken place. However, if credit is used, there is an assumption that future resources will present themselves sufficient to pay the cost of the product, along with extra interest charges. The interest charged is compensation to the lender for having taken a risk in providing monetary resources now. The more that credit is used for purchases, the less one has a bartering economy, and the more one has a credit economy. This leads to a risk that there will eventually be a default in payments.

To frame an economic system mathematically, imagine pure bartering, where even when things are made, accounts are used for purchases within the system. In particular, assume no hoarding by workers and a total money supply of $M_0 > 0$.

Workers continuously produce items of value A_i, which must meet or exceed A_{min} in order for them to survive. Capitalists can only survive while workers exist and are active. The excess of labor is shared between A and C and this suggests an accumulation parameter $\alpha \in [0, 1]$ where the asymptotic values are

$$A_o(\infty) = A_{min} + \alpha \cdot (A_o(\infty) - A_{min}) \ , \tag{7.1}$$

$$C_i(\infty) = (1 - \alpha) \cdot (A_o(\infty) - A_{min}) \ . \tag{7.2}$$

If there is no excess, then there is no capitalism. With excess in production there is an opportunity for a subclass that does no work, but receives a portion of the output. Once taken and funnelled into the C_i account, funds are transferred to the output account C_o, which at this point represents status. This leads to the linear autonomous system of ODEs:

$$\begin{pmatrix} A_i \\ A_o \\ C_i \\ C_o \end{pmatrix}' = \begin{pmatrix} \pi \cdot (A_o - A_{min}) - a_o^i \cdot (A_i - A_o) \\ -(\alpha + \pi) \cdot (A_o - A_{min}) + a_o^i \cdot (A_i - A_o) \\ \alpha \cdot (A_o - A_{min}) - c_o^i \cdot C_i \\ c_o^i \cdot C_i \end{pmatrix} , \quad \begin{pmatrix} A_{i,0} \\ A_{o,0} \\ C_{i,0} \\ C_{o,0} \end{pmatrix} = \begin{pmatrix} M_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} . \quad (7.3)$$

The total money in the system $M_0 > 0$ will remain constant.

7.0.5 $A - C$ example of workers invested with capitalists

The system (7.3) will again be described using the new variables

$$\overline{A}_i \equiv A_i - A_{min} , \quad \overline{A}_o \equiv A_o - A_{min} \quad \Longrightarrow \quad \overline{A}_i' = A_i' , \quad \overline{A}_o' = A_o' . \quad (7.4)$$

Then we obtain the initial value problem:

$$\begin{pmatrix} \overline{A}_i \\ \overline{A}_o \\ C_i \\ C_o \end{pmatrix}' = \begin{pmatrix} -a_o^i & a_o^i + \pi & 0 & 0 \\ a_o^i & -(a_o^i + \alpha + \pi) & 0 & 0 \\ 0 & \alpha & -c_o^i & 0 \\ 0 & 0 & c_o^i & 0 \end{pmatrix} \begin{pmatrix} \overline{A}_i \\ \overline{A}_o \\ C_i \\ C_o \end{pmatrix} , \quad \begin{pmatrix} A_{i,0} \\ A_{o,0} \\ C_{i,0} \\ C_{o,0} \end{pmatrix} = \begin{pmatrix} M_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \quad (7.5)$$

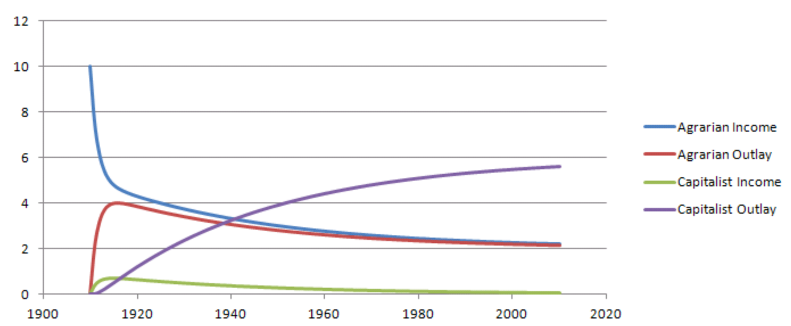
where the relative size of these parameters is considered to be

$$\infty > \text{(fast)} \quad a_o^i \simeq c_o^i \gg \pi > \alpha \quad \text{(slow)} > 0 . \quad (7.6)$$

Note that the sum $A_i(t) + A_o(t) + C_i(t) + C_o(t) = M_0 + 2A_{min}$ for all $t > t_*$. Also the dimension of the range space is only 2, so the matrix in (7.5) has 2 zero eigenvalues, both which are negative due to (7.5).

The results of a numerical simulation are presented in Figure 7 where all initial money M_0 is given to A. Since the capitalists only allow the agrarians to retain the mininum required to sustain themselves, the A_i and A_o accounts approch A_{min} asymptotically. The capitalist's input account C_i obtains the largest amount of re-

Figure 7:



A-C model parameters: $M_0 = 10$, $A_{min} = 2$, $t_* = 1910$, $a_o^i = 0.2$, $c_o^i = 0.2$, $\pi = 0.07$ and $\alpha = 0.05$.

serves, while the agrarians never regain full possesion of their original wealth M_0. All excess in the community is eventually taken by the capitalists. The wealth of the community erodes over time due to the presence of a meritorious class, known as the capitalists.

The role of government in the exploitation of workers comes in the form of minimum wage legislation and support programs like food stamps. Putting more money into cirulation can provide only a temporary relief to A.

7.0.6 Equilibria of the $A - C$ model

The 4×4 matrix in (7.5) can have one or two linearly independent 0 eigenvectors. The 0 eigenvectors are asymptotic solutions to (7.3) and will solve

$$\overline{A}_i + \overline{A}_o + C_i + C_o = M_0 \qquad \text{and} \qquad \begin{cases} (\alpha + a_o^i)\overline{A}_i = (\pi + a_o^i)\overline{A}_o \\ (\pi + a_o^i)\overline{A}_o = a_o^i \cdot \overline{A}_i \\ c_o^i C_i = \alpha \cdot \overline{A}_i \\ 0 = c_o^i \cdot C_i \end{cases} \tag{7.7}$$

In the long run the capitalists can take no more from the economy if workers are going to survive. Thus C_i approaches 0, whereas the outlay account C_o approaches a large constant, which represents the money that the capitalists are sitting on. A trivial solution to (7.7) is $(B_i,\ B_o,\ C_i,\ C_o)^T = \vec{0}$, which only occurs if $M_0 = 0$.

Theorem 3. *Let $M_0, A_{min} > 0$ and suppose $A_i = M_0 + A_{min}$, $A_o = A_{min}$, and $C_i = C_o = 0$ initially. If $\alpha > 0$ then the system in (7.3), under the conditions in (7.6), has a unique solution, which represents a stable equilibrium for the system in (7.3). In particular, the matrix in (7.7) has a 0 eigenvalue with algebraic multiplicity 2, but geometric multiplicity 1.*

Proof. The matrix in (7.5) can be expressed as $\mathcal{A} \equiv \begin{pmatrix} -a & b & 0 & 0 \\ c & -b & 0 & 0 \\ a-c & 0 & -d & 0 \\ 0 & 0 & d & 0 \end{pmatrix}$, where the components satisfy: $a \simeq c \gg b \gg d > 0$ due to (7.6). Now observe that the sum of the columns of $\mathcal{A}$ is 0, thus $\mathcal{A}$ has a 0 eigenvalue. The characteristic polynomial simplifies to

$$p_{\mathcal{A}}(\lambda) \;=\; \lambda \cdot (\lambda + d) \cdot \left(\lambda^2 + (a+b) \cdot \lambda + (a-c) \cdot b\right) \;. \tag{7.8}$$

This implies $\lambda = 0$ or $\lambda = -d$. There are also two other strictly negative eigenvalues,

$$\lambda \;=\; \frac{-(a+b)}{2} \;\pm\; \frac{1}{2} \cdot \sqrt{(a-b)^2 + 4bc}\;,$$

assuming that $c < a$.

For the equilibrium, it is sufficient to find a 0 eigenvector $\vec{v}_0 = (v_1,\ v_2,\ v_3,\ v_4)^T$

for $\mathcal{A}$ in order to determine the asymptotic value of the solution to (7.5).

$$0 \cdot \vec{v}_0 \;=\; \mathcal{A} \cdot \vec{v}_0 \;=\; \begin{pmatrix} -a & b & 0 & 0 \\ c & -b & 0 & 0 \\ a-c & 0 & -d & 0 \\ 0 & 0 & d & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \;=\; \begin{pmatrix} -av_1 + bv_2 \\ cv_1 - bv_2 \\ (a-c)v_1 - dv_3 \\ dv_3 \end{pmatrix} \tag{7.9}$$

The first component gives the equation $v_1 = (b/a)v_2$ and the second $v_1 = (c/a)v_2$. Thus if $a \neq c$ then $v_1 = v_2 = 0$. However, if $a = c$ then any $v_1 = v_2 \neq 0$ is allowed. If $d > 0$ then $v_3 = 0$ is required, and any $v_4 \neq 0$ is allowed. Thus, when $a \neq c$ then the only eigenvector is a multiple of $(0\ 0\ 1\ 0)^T$. But if $a = c$, then the eigenvector can be expressed as

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \;=\; \begin{pmatrix} (b/a) \cdot v_2 \\ v_2 \\ 0 \\ v_4 \end{pmatrix} \;=\; (v_2/a) \cdot \begin{pmatrix} b \\ a \\ 0 \\ 0 \end{pmatrix} \;+\; v_4 \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \;. \tag{7.10}$$

All the components of v_0 are strictly positive. The equilibrium vector is $N_0 v_0$ must have a sum of its components equal to $M_0 > 0$. In this case the normalization constant is

$$N_0 \;\equiv\; \frac{M_0}{abc + abe + ace + ade + bce + bde} \;=\; \frac{M_0}{b_o^i bc \cdot c_i^o \;+\; (b_o^i + bc)(c_i^o - rc)rd} \;.$$

The positivity of all components, and the condition that $c_i^o \gg rc$ ensures a unique equilibrium for the system (5.2). □

Thus even without credit in the economy, there are problems for the A class in this model.

CHAPTER 8: A 6-dimensional worker-banker-capitalist system ABC

Finally we consider an economic system that is comprised of workers A, bankers B and capitalists C. Since payment for labor does not inherently allow for excess income in the working class, credit is one way that new products and services can be obtained. However this incurs debt obligation to bankers that can only be resolved through sufficient wages. This sets up a relationship between the three parties, A, B and C.

8.1 Lending only to Consumers

To achieve a credit economy, suppose a government issues money $M_0 > 0$ and delivers it to the bank's input account B_i. The initial condition for the system is assumed to be

$$A_{i,0} = A_{o,0} = A_{min} , \quad B_{i,0} = M_0 , \; B_{o,0} = C_{i,0} = C_{o,0} = 0 .$$

The models in the previous chapters can be combined into the 6-dimensional system of coupled linear system of ordinary differential equations, where lending is only provided to A:

$$\frac{d}{dt}\begin{pmatrix} A_i \\ A_o \\ B_i \\ B_o \\ C_i \\ C_o \end{pmatrix} = \begin{pmatrix} -\alpha(A_i - A_{min}) + \pi(A_o - A_{min}) - a_o^i(A_i - A_o) + rb \cdot B_o \\ -(ra + \pi)\cdot(A_o - A_{min}) + a_o^i \cdot (A_i - A_o) \\ ra \cdot (A_o - A_{min}) - b_o^i \cdot B_i \\ b_o^i \cdot B_i - rb \cdot B_o \\ \alpha \cdot (A_i - A_{min}) - c_o^i \cdot C_i \\ c_o^i \cdot C_i \end{pmatrix} . \qquad (8.1)$$

The flow of money modeled by this system is

government → banks → agrarian-consumer-worker → capitalist

The apparent goal of such a system is to transfer wealth from the government to the capitalist, while temporarily stimulating economic activity. One negative outcome of this process is to create debt for the consumer, which are financial obligations held by the banks. On top of this government is charged with protecting the wealth of capitalists, and enforcing any and all obligations to bankers.

8.1.1 $A - B - C$ example with only consumer debt

First we express the system (8.1) in matrix form, and inject $M_0 > 0$ into this economy through B_i. This represents an exogenous forcing vector on the system expressed as

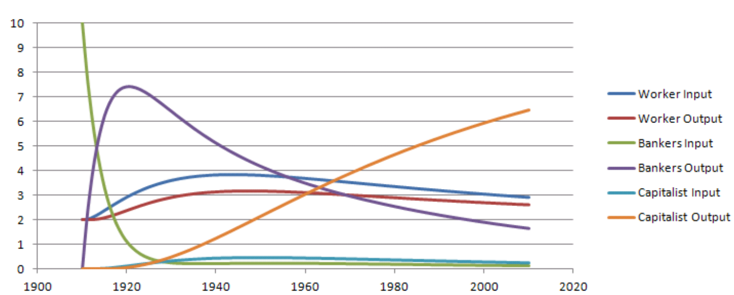
$$\begin{pmatrix} \overline{A}_i \\ \overline{A}_o \\ B_i \\ B_o \\ C_i \\ C_o \end{pmatrix}' = \begin{pmatrix} -a^i_o & a^i_o + \pi & 0 & rb & 0 & 0 \\ a^i_o & -(a^i_o + \alpha + \pi + ra) & 0 & 0 & 0 & 0 \\ 0 & ra & -b^i_o & 0 & 0 & 0 \\ 0 & 0 & b^i_o & -rb & 0 & 0 \\ 0 & \alpha & 0 & 0 & -c^i_o & 0 \\ 0 & 0 & 0 & 0 & c^i_o & 0 \end{pmatrix} \begin{pmatrix} \overline{A}_i \\ \overline{A}_o \\ B_i \\ B_o \\ C_i \\ C_o \end{pmatrix} , \begin{pmatrix} A_{i,0} \\ A_{o,0} \\ B_{i,0} \\ B_{o,0} \\ C_{i,0} \\ C_{o,0} \end{pmatrix} = \begin{pmatrix} A_{min} \\ A_{min} \\ M_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{8.2}$$

The relative sizes of these parameters are

$$\infty > \text{(fast)} \quad a^i_o \simeq b^i_o \simeq c^i_o \gg \alpha \simeq \pi \gg ra \simeq rb \quad \text{(slow)} > 0 . \tag{8.3}$$

Note that the sum $A_i(t) + A_o(t) + B_i(t) + B_o(t) + C_i(t) + C_o(t) = M_0 + 2A_{min}, \forall t > 0$.

Figure 8:



A-B-C model parameters: $M_0 = 10$, $t_* = 1910$, $b^i_o = 0.2$, $c^i_o = 0.2$, $a^i_o = 0.2$, $\alpha = 0.05$, $ba = 0.10$, and $\pi = 0.03$.

The numerical results in Figure 8 show the asymptotic level for all accounts, wherein are all accounts vanish except for C_o.

8.1.2 Equilibria of the $A - B - C$ model

Setting the left hand side of (8.2) to $\vec{0}$

$$C_i \ = \ \overline{A}_i \ = \ \overline{A}_o \ = \ B_o \ = \ B_i \ = \ 0 \ , \ \ C_o \ = \ M_0 \ , \tag{8.4}$$

which follows by inspection.

Theorem 4. *Let* $M_0 = B_i + B_o + C_i + C_o > 0$. *The system in* (8.1), *under the conditions in* (8.3), *has a unique solution, which represents a stable equilibrium for the system in* (8.1).

Proof. It is sufficient to consider the matrix in (8.2) to be expressed as

$$\mathcal{A} \ \equiv \ \begin{pmatrix} -1 & 1+b & 0 & a & 0 & 0 \\ 1 & -1-2b-a & 0 & 0 & 0 & 0 \\ 0 & a & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -a & 0 & 0 \\ 0 & b & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{8.5}$$

where the components satisfy: $1 \geq b \geq a \geq 0$ due to (8.3).

Lemma 8.1. *The matrix* $\mathcal{A}$ *is non-degenerate. Also,* $\mathcal{A}$ *has the two eigenvalues 0, -1 which are independent of the parameters* a *and* b*. Furthermore,* $\mathcal{A}$ *has an eigenvalue in the open set* $(-a, 0)$ *and another less than* -1*. Finally, for* $b \in (0,1)$*, and* $a \in (0, 0.17)$*,* $\mathcal{A}$ *has 2 more real eignvalues that are strictly negative.*

Proof. Now observe that the sum of the rows of $\mathcal{A}$ is 0, thus $\mathcal{A}$ has a 0 eigenvalue. The characteristic polynomial has only 2 free parameters which allows some analysis. In particular, we can write

$$p_{\mathcal{A}}(\lambda) \;=\; \lambda \cdot (\lambda+1) \cdot q_{\mathcal{A}}(\lambda) \tag{8.6}$$

$$\begin{aligned} q_{\mathcal{A}}(\lambda) \;=\; \Big[&\lambda^4 + 2(a+b+1)\lambda^3 \\ &+(4a+a^2+3b+2ab+1)\lambda^2 + (2a^2+3ab+2a+b)\lambda + ab\Big] \;. \end{aligned} \tag{8.7}$$

This implies that $0, -1 \in \sigma(\mathcal{A})$. We now consider the quartic polynomial $q_{\mathcal{A}}(\lambda)$ which has strictly positive coefficients. Note that for $\lambda < 0$, the polynomial has negative values

$$q_{\mathcal{A}}(-a) \;=\; q_{\mathcal{A}}(-1) \;=\; -a^2 \;.$$

Since $q_{\mathcal{A}}(0) = ab > 0$, there must be a zero for $q_{\mathcal{A}}$ in $(-a, 0)$ and $(-5, -1)$, using the conditions that $a, b \in (0, 1)$.

In the special case that $a = 0$,

$$q_{\mathcal{A}}(\lambda) \;=\; \lambda^4 + 2(b+1)\lambda^3 + (3b+1)\lambda^2 + b\lambda \;=\; \lambda(\lambda+1)(\lambda^2 + (2b+1)\lambda + b) \;,$$

and there are clearly zeros at $\lambda = -1,\ 0$. The other two eigenvalues are

$$-b - \frac{1}{2} \;\pm\; \frac{1}{2}\sqrt{4b^2+1} \;,$$

and both of these are negative. Now, as a is increased, the roots remain negative for $a \leq 0.267$, as long as $b \in (a, 1)$. To see this, it is sufficieint to find $\lambda \in (-1, -a)$ where $q_{\mathcal{A}}(\lambda) > 0$. The choice $\lambda = -1/2$ reduces the polynomial to

$$q_{\mathcal{A}}(-1/2) \;=\; 48 \;-\; 128a \;-\; 192a^2 \;.$$

Finally, note that there is no need to look for another real root, because a complex root, should it exist, must come with a complex conjugate pair. □

To verify that positive accounts will remain as such, the 0 eigenvector needs to be found. Thus solve $\vec{v}_{eq} = (v_1,\ v_2,\ v_3,\ v_4,\ v_5,\ v_6)^T$ for $\mathcal{A}$ in (8.2)

$$0 \cdot \vec{v}_{eq} \ = \ \mathcal{A} \cdot \vec{v}_{eq} \ \ \Longrightarrow \ \begin{cases} v_1 = (1+b)\ v_2 \\ v_1 = (1+a+2b)\ v_2 \\ v_3 = a\ v_2 \\ v_3 = a\ v_4 \\ v_5 = b\ v_2 \\ v_5 = 0 \end{cases} . \tag{8.8}$$

Thus the logical conclusions are that $v_5 = 0 \implies v_2 = 0 \implies v_3 = 0 \implies v_4 = 0 \implies v_1 = 0$. This gives the only eigenvector, with sum equal to $M_0 > 0$, is $\vec{v}_{eq} = (0,\ 0,\ 0,\ 0,\ 0,\ M_0)^T$. □

8.2 A complex eigenvalue example for the 6-d ABC

For the matrix $\mathcal{A}$ in (8.5) the spectrum contains only real non-positive eigenvalues if $1 \geq b > a \geq 0$ for a sufficiently small. However, in the extreme case that $a = b = 1$ the matrix takes the form

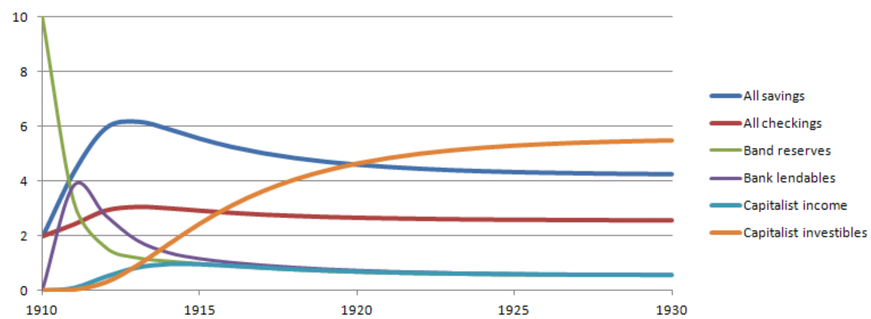
$$\mathcal{A}(a=1,\ b=1) \ = \ \mathcal{A}_{1,1} \ \equiv \ \begin{pmatrix} -1 & 2 & 0 & a & 0 & 0 \\ 1 & -4 & 0 & 0 & 0 & 0 \\ 0 & a & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & b & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} . \tag{8.9}$$

The spectrum of this matrix is found numerically to be

$$\sigma(\mathcal{A}_{1,1}) \simeq \{0,\ -1,\ -0.13586,\ -4.5804,\ -1.14188+i0.55053,\ -1.14188-i0.55053\}\ .$$

There are two complex eigenvalues, and all eigenvalues have non-positive real parts. The corresponding systems will converge to an equilibrium exponentially, however there is a possibility for oscillations. In Figure 9 an example is given.

Figure 9:



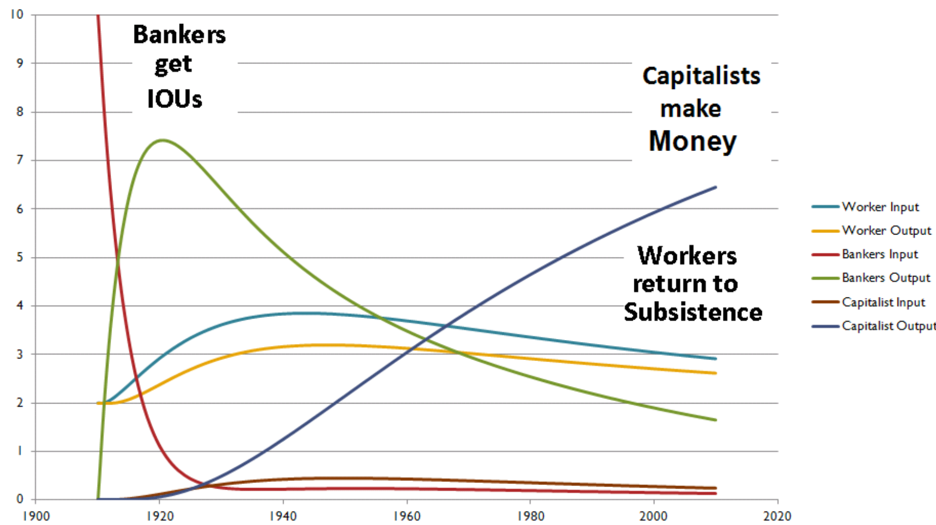
Complex eigenvalue case: System converges to equilibrium without the appearance of oscillations. Convergence is fast.

Here we have an example where even with complex eigenvalues, oscillations are so damped that they do not appear in the system. Instead convergence to equilibrium is fast.

CHAPTER 9: Concluding Remarks

This thesis considered several dynamical models involving the interaction between different groups in a financial-economic system. Even for fairly simple systems, a mathematical analysis requires a delicate analysis of matrices and polynomials. The conclusions result in a situation where money gets absorbed by one class, and the lives of workers get indentured into an unfixabe set of obligations.

Figure 10:



A-B-C conclusions: Banks accumulate loan contracts, Capitalists hoard cash from profitable sales, and All others return to basic survival as loans are paid back.

Additions to the models examined here will not drastically alter the dynamical behavior of the systems considered here.

CHAPTER 10: Glossary

The following are some terms relevant to the thesis:

bank: It also defines what a bank is: it is a third party whose record-keeping is trusted by all parties as recording the transfers of credit money that effect sales of commodities. The bank makes a legitimate living by lending money to other agents. Thus it simultaneously creates loans and deposits. The bank charges a higher rate of interest on loans than on deposits in order to make a profit.

deflation: $(P_S/P_*) < 1$, i.e. negative natural rate of interest, negative inflation in prices and/or wages

deleveraging: paying off debts through endowments or the selling off of assests

endowment: salary or other steady source of income

FED: The U.S. Federal Reserve acts as a gatekeeper of the economy by setting economic policy. In particular, it lends out money to banks at a *prime* interest rate, which it chooses in order to regulate inflation

hedge fund: an investment that is solvent where profits meet or exceed debt repayment obligations

horizontalism: an approach to money creation theory pioneered by Basil Moore which states that private bank reserves are not managed by central banks. Instead reserves will be provided on demand at the bank rate set by the central bank. This inverts the mainstream textbook money multiplier relationship between deposits and loans since loans are said to cause deposits which in turn cause reserves.

inflation: increase in prices and/or wages over a unit period of time (bad in the 1970s, good for the 2010s)

IS-LM: Investment-saving, liquidity-money model, developed by Hicks in 1936 as a way of explaining some ideas of Keynes.

leveraging: borrowing money based on assests of endowment projections

MZM: money with zero maturity, or simply money available on demand

Ponzi scheme: after Charles Ponzi who used invested money to pay investors, while producing nothing of value and creating no profit

quantitative easing: also expressed as $QE1$, $QE2$, etc; a dramatic increase in the level of base money M_0 = currency and commercial bank accounts with the Federal Reserve, in an attempt to cause a substantial expansion in the money supply, with the goal of stabilizing prices by creating inflation

speculative fund: an investment that is temporarily insolvent where profits do not meet debt repayment obligations, but with bridge loans, may become solvent i nthe near future

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