ABSTRACT

FOURIER ANALYSIS ON $SU(2)$

by

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The set SU(2) of 2x2 unitary matrices with determinant one forms a compact non-abelian Lie group diffeomorphic to the three dimensional sphere. This thesis surveys general theory concerning analysis on compact Lie groups and applies this in the setting of SU(2). Our principal reference is J. Faraut's book *Analysis on Lie Groups*. Fundamental results in representation theory with compact Lie groups include the Peter-Weyl Theorem, Plancherel Theorem and a criterion for uniform convergence of Fourier series.

On SU(2) we give explicit constructions for Haar measure and all irreducible unitary representations. For purposes of motivation and comparison we also consider analysis on U(1), the unit circle in the complex plane. In this context, the general theory specializes to yield classical results on Fourier series with periodic functions and the heat equation in one dimension. We discuss convergence behavior of Fourier series on SU(2) and show that Cauchy problem for the heat equation with continuous boundary data admits a unique solution.

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CHAPTER 1: Introduction to Matrix Groups

1.1 Matrix Groups

Let $M(n,\mathbb{K})$ be the set of $n \times n$ matrices with entries in $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Note that $M(n,\mathbb{K})$ is a vector space over $\mathbb{K}$ in the usual way, isomorphic to $\mathbb{K}^{n^2}$. We equip $M(n,\mathbb{K})$ with its usual vector topology. The subset of invertible matrices in $M(n,\mathbb{K})$ forms a group under matrix multiplication, known as the general linear group, denoted $GL(n,\mathbb{K})$. Another way to define the general linear group is

$$GL(n,\mathbb{K}) = \{X \in M(n,\mathbb{K}) \ : \ \det(X) \neq 0\}.$$

As $GL(n,\mathbb{K}) = \det^{-1}(\mathbb{K} - \{0\})$ and $\det : M(n,\mathbb{K}) \to \mathbb{K}$ is continuous, $GL(n,\mathbb{K})$ is an open subset of $M(n,\mathbb{K})$.

Definition 1.1. A closed subgroup of $GL(n,\mathbb{K})$ is called a *linear Lie group.*

There are a number of classic examples.

- $SL(n,\mathbb{K}) = \{g \in GL(n,\mathbb{K}) \ : \ \det(g) = 1\} = Ker(\det : GL(n,\mathbb{K}) \to \mathbb{K})$ is called the *special linear group.*
- $O(n) = \{g \in GL(n,\mathbb{R}) \ : \ \|gx\| = \|x\|, \text{ for all } x \in \mathbb{R}^n\}$ is the *orthogonal group* and $SO(n) = \{g \in O(n) \ : \ \det(g) = 1\} = O(n) \cap SL(n,\mathbb{R})$ the *special orthogonal group.* As is well known $g \in O(n)$ if and only if $g^{\intercal} g = I$.
- $U(n) = \{g \in GL(n,\mathbb{C}) \ : \ g^* g = I\}$ is the *unitary group* and $SU(n) = \{g \in U(n) \ : \ \det(g) = 1\}$ the *special unitary group.*

An important fact about linear Lie groups is that each such group $G \subset GL(n,\mathbb{K})$ is a smooth sub-manifold of $M(n,\mathbb{K})$ [2, Corollary 3.3.5]. This will allow us to apply calculus on manifolds to linear Lie groups.

We can extend the usual exponential map to a matrix valued function of a matrix variable.

Definition 1.2. The exponential of a matrix $X \in M(n, \mathbb{K})$ is the sum of the series

$$\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

Here we have that X^k is simply X matrix multiplied by itself k times with the usual convention that $X^0 = I$. Since matrices do not usually commute, we do not have, in general, that $\exp(X + Y) = \exp(X)\exp(Y)$. However

$$\exp(X)\exp(Y) = \exp(X + Y) \quad \text{provided} \quad XY = YX; \tag{1.1}$$

see [2, Proposition 2.2.3]. Indeed, if X, Y commute we can write

$$\begin{aligned}
\exp(X)\exp(Y) &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{1}{k!\ell!} X^k Y^\ell \\
&= \sum_{\ell=0}^{\infty} \frac{1}{m!} \left(\sum_{k+\ell=m}^{\infty} \frac{m!}{k!\ell!} X^k Y^\ell \right) \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} (X + Y)^m \\
&= \exp(X + Y).
\end{aligned}$$

We will also need the following important relation later.

Lemma 1.3. *For $X \in M(n, \mathbb{K})$ we have*

$$\det(\exp(X)) = e^{tr(X)}.$$

Proof. First observe that for a diagonal matrix $D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$ one has

$$\det(\exp(D)) = \det\left(\begin{bmatrix} e^{d_1} & & \\ & \ddots & \\ & & e^{d_n} \end{bmatrix}\right) = e^{d_1} \cdots e^{d_n} = e^{d_1+\cdots+d_n} = e^{tr(D)}$$

as claimed. Moreover if X is a diagonalizable matrix then $X = gDg^{-1}$ for some diagonal D and invertible matrix $g \in GL(n, \mathbb{K})$. As $X^k = gD^kg^{-1}$ we see that also $\exp(X) = g\,\exp(D)g^{-1}$ and now

$$\det(\exp(X)) = \det(g\,\exp(D)\,g^{-1}) = \det(\exp(D)) = e^{tr(D)} = e^{tr(g^{-1}Xg)} = e^{tr(X)}.$$

The result in general now follows from the fact that $\{X \in M(n, \mathbb{K}) \ : \ X \text{ is diagonalizable}\}$ is dense in $M(n, \mathbb{K})$ together with the continuity of the functions $f(X) = \det(\exp(X))$ and $g(X) = e^{tr(X)}$. Indeed given any $X \in M(n, \mathbb{K})$ we can find a sequence $\{X_m\}$ of diagonalizable matrices converging to X. Therefore $f(X) = \lim_{m\to\infty} f(X_m) = \lim_{m\to\infty} g(X_m) = g(X)$ by continuity. □

1.2 Lie Algebras

Using the afore mentioned exponential function we can now define the Lie algebra of a linear Lie group.

Definition 1.4. The associated Lie algebra for a linear Lie group G is

$$\mathfrak{g} = Lie(G) = \{X \in M(n, \mathbb{K}) \ : \ \text{for all } t \in \mathbb{K}, \quad \exp(tX) \in G\}$$

It is a non-trivial fact that $\mathfrak{g}$ is a vector subspace of $M(n, \mathbb{K})$ and that $\mathfrak{g}$ is closed under the commutator bracket, namely

$$[X, Y] = XY - YX.$$

That is, $[X, Y] \in \mathfrak{g}$ whenever $X, Y \in \mathfrak{g}$, [2, Theorem 3.2.1].

We can now apply this definition to the classical examples of linear Lie groups mentioned before.

- The Lie algebra of $GL(n, \mathbb{K})$ is

  $$gl(n, \mathbb{K}) = M(n, \mathbb{K}).$$

  Indeed, by Lemma 1.3 we have that for any $X \in M(n, \mathbb{K})$

  $$\det(\exp(tX)) = e^{tr(tX)} \neq 0.$$

- The Lie algebra for $SL(n, \mathbb{K})$ is

  $$sl(n, \mathbb{K}) = \{X \in M(n, \mathbb{K}) \ : \ tr(X) = 0\}.$$

  Indeed $X \in M(n, \mathbb{K})$ belongs to $Lie(SL(n, \mathbb{K}))$ if and only if $det(\exp(tX)) = 1$ for each t. But Lemma 1.3 shows

  $$\det(\exp(tX)) = e^{tr(tX)} = e^{t\,tr(X)}.$$

  Thus $X \in Lie(SL(n, \mathbb{K}))$ if and only if $tr(X) = 0$.

- The Lie algebra of both $O(n)$ and $SO(n)$ is

  $$so(n) = \{X \in M(n, \mathbb{R}) \ : \ X^{\intercal} = -X\},$$

  the skew symmetric matrices. First note that $X \in M(n, \mathbb{R})$ belongs to $Lie(O(n))$ if and only if $\exp(sX)^{\intercal} \exp(sX) = I$ for every $s \in \mathbb{R}$. Thus if $X \in Lie(O(n))$ we must have

  $$O = \frac{d}{ds}\bigg|_{s=0} (\exp(sX)^{\intercal} \exp(sX)) = X^{\intercal} I + IX$$

  and so $X^{\intercal} = -X$. Thus $Lie(O(n)) \subset \{X \in M(n, \mathbb{R}) \ : \ X^{\intercal} = -X\}$. On the other hand if $X^{\intercal} = -X$ then X commutes with $X^{\intercal}$ and we can apply Equation

1.1 to obtain

$$\exp(sX)^\intercal \exp(sX) = \exp(s(X^\intercal + X)) = \exp(0) = I.$$

Hence $Lie(O(n)) = \{X \in M(n, \mathbb{R}) \ : \ X^\intercal = -X\}$ as claimed.

- The Lie algebra of $U(n)$ is

$$u(n) = \{X : X^* = -X\},$$

the skew-Hermitian matrices. This can be shown just as for the Lie algebra of $SO(n)$, replacing $X^\intercal$ by X^*.

- The Lie algebra of $SU(n)$ is

$$su(n) = \{X \in u(n) \ : \ tr(X) = 0\}.$$

Indeed $X \in M(n, \mathbb{C})$ belongs to $Lie(SU(n))$ if and only if $\exp(tX) \in U(n) \cap SL(n, \mathbb{C})$ for each t. From above this is equivalent to $X \in u(n)$ (i.e. $X^* = -X$) and $tr(X) = 0$.

1.3 Representations

A *topological group* is a group G together with a topology for which the maps

$$G \times G \to G \ \ (x, y) \mapsto xy, \qquad G \to G \ \ x \mapsto x^{-1}$$

are continuous. Note that, in particular, every linear Lie group is a topological group.

Let G be a topological group and V a complex vector space.

Definition 1.5. A *representation* of G on a finite dimensional complex vector space V is a group homomorphism $\pi : G \to GL(V)$ for which $g \mapsto \pi(g)v$ is continuous for each $v \in V$. ($GL(V)$ is the group of invertible linear operators on V.)

Suppose (π_1, V) and (π_2, W) are two representations of G in vector spaces V and W. Then we say that π_1 and π_2 are *equivalent*, and write $\pi_1 \simeq \pi_2$, if there exists a continuous isomorphism $A : V \to W$ where

$$A\pi_1(g) = \pi_2(g)A.$$

In this case we call $A : V \to W$ an *intertwining operator* and say that A *intertwines* π_1 and π_2.

Suppose (π, V) is a representation of G and $W \subset V$ a subspace of V. We say that W is *invariant* if for every $g \in G$ and $w \in W$, we have $\pi(g)w \in W$. The restriction $\pi(g)|_W$ is now a representation of G on W.

Definition 1.6. A representation is said to be *irreducible* if there are no closed invariant subspaces except $\{0\}$ and the whole space.

Let (π, V) be a representation of a linear Lie group G in a finite-dimensional vector space V. The *derived representation* of the Lie algebra $\mathfrak{g}$ for G is defined as

$$d\pi(X) = \frac{d}{dt}\pi(\exp(tX))|_{t=0}$$

for $X \in \mathfrak{g}$. It is a non-trivial fact that $d\pi(X) : V \to V$ is a well-defined linear operator on V and that the resulting map $d\pi : \mathfrak{g} \to gl(V)$ is a *Lie algebra representation* of $\mathfrak{g}$. That is, $d\pi$ is a linear mapping and

$$d\pi([X, Y]) = d\pi(X)d\pi(Y) - d\pi(Y)d\pi(X).$$

(See [2, Page 53].)

Next we will see the relationship between equivalent representations of a linear Lie group and its Lie algebra.

Lemma 1.7. *If $\pi_1 \simeq \pi_2$ then $d\pi_1 \simeq d\pi_2$. Conversely if $d\pi_1 \simeq d\pi_2$ then $\pi_1|_{G_\circ} \simeq \pi_2|_{G_\circ}$ where $G_\circ$ denotes the connected component of the identity element in G.*

Proof. Let $\pi_1 : G \to GL(V_1)$ and $\pi_2 : G \to GL(V_2)$ and suppose $\pi_1 \simeq \pi_2$. Let $T : V_1 \to V_2$ be an intertwining operator. Then for $X \in \mathfrak{g}$ we have

$$\begin{aligned} d\pi_2(X) = \left.\frac{d}{dt}\right|_0 \pi_2(\exp(tX)) = \left.\frac{d}{dt}\right|_0 T\pi_1(\exp(tX))T^{-1} &= T \left.\frac{d}{dt}\right|_0 \pi_1(\exp(tX))T^{-1} \\ &= Td\pi_1(X)T^{-1}. \end{aligned}$$

Thus $d\pi_1 \simeq d\pi_2$.

Conversely suppose $d\pi_1 \simeq d\pi_2$ and $T : V_1 \to V_2$ is an intertwining operator. $X \in \mathfrak{g}$ then

$$\begin{aligned} \pi_2(\exp(X)) = Exp(d\pi_2(X)) &= Exp(Td\pi_1(X)T^{-1}) \\ &= TExp(d\pi_1(X))T^{-1} \\ &= T\pi_1(\exp(X))T^{-1} \end{aligned}$$

As $\exp(\mathfrak{g})$ generates $G_\circ$ (see [2, Page 43]) we see that $\pi_2(g) = T\pi_1(g)T^{-1}$ for all $g \in G_\circ$. □

Notice that if G is connected then we get that $\pi_1 \simeq \pi_2$ if and only if $d\pi_1 \simeq d\pi_2$. There is also a relationship between irreducibility of a representation and its derived representation.

Lemma 1.8. *If $d\pi : \mathfrak{g} \to gl(V)$ is irreducible then so is $\pi : G \to GL(V)$. Conversely if $\pi|_{G_\circ}$ is irreducible then so is $d\pi$.*

Proof. Let $d\pi : \mathfrak{g} \to gl(V)$ be irreducible and $W \subset V$ be some $\pi(G)$-invariant closed subspace. Then for $X \in \mathfrak{g}$ and $w \in W$ we have

$$d\pi(X)(w) = \frac{d}{dt}\bigg|_0 \pi(\exp(tX))(w)$$

also belongs to W. So W is $d\pi(\mathfrak{g})$-invariant. Hence $W = V$ or $W = \{0\}$, as $d\pi$ is irreducible.

Conversely, let $\pi|_{G_\circ}$ be irreducible and let $W \subset V$ be a closed $d\pi(\mathfrak{g})$-invariant subspace. Let $X \in \mathfrak{g}$ and $w \in W$ then

$$\pi(\exp(X))(w) = Exp(d\pi(X))(w)$$

which is in W. As $\exp(\mathfrak{g})$ generates $G_\circ$, W is a $\pi(G_\circ)$-invariant subspace. Thus $W = V$ or $W = \{0\}$. Hence $d\pi$ is irreducible. □

Again we get that if G is connected then the previous lemma shows π is irreducible if and only if $d\pi$ is irreducible.

Next suppose that V is a *hermitian* vector space. That is a complex vector space equipped with a hermitian inner product $\langle \cdot, \cdot \rangle$. An operator A on V is unitary if $\langle Av, Aw \rangle = \langle v, w \rangle$ for all $v, w \in V$.

Definition 1.9. A representation π of G on V is said to be unitary if for every $g \in G$ we have that $\pi(g)$ is a unitary operator.

Lemma 1.10. *Suppose that π is a representation of a compact linear Lie group G on a finite dimensional vector space V. Then V admits a $\pi(G)$-invariant hermitian inner product. Thus we may always suppose that π is unitary.*

A proof will be given below at the end of Section 1.5. An important fact is that every *irreducible* representation of a compact Lie group is finite dimensional [2, Theorem 6.3.2].

1.4 Adjoint Representation

Let G be a linear Lie group with Lie algebra $\mathfrak{g}$. For $X \in \mathfrak{g}$ and $g \in G$ we define $Ad_g(X)$ via

$$Ad_g(X) = gXg^{-1}.$$

As $(gXg^{-1})^k = gX^kg^{-1}$ for each $k = 0, 1, 2, \ldots$ it follows that for each $t \in \mathbb{R}$

$$\exp(tAd_g(X)) = \exp(Ad_g(tX)) = g\exp(tX)g^{-1}$$

by definition of the exponential map. Now as $\exp(tX) \in G$ we see that $Ad_g(X) \in \mathfrak{g}$ by definition of the Lie algebra. It is clear that $Ad_g : \mathfrak{g} \to \mathfrak{g}$ is linear and invertible with $(Ad_g)^{-1} = Ad_{g^{-1}}$. Moreover we see

$$Ad_{gh}(X) = (gh)X(gh)^{-1} = ghXh^{-1}g^{-1} = Ad_g(Ad_h(X)).$$

So

$$Ad : G \to GL(\mathfrak{g}), \quad Ad(g) = Ad_g$$

is a representation of G in $\mathfrak{g}$. This is the *adjoint representation* for the linear Lie group G. From above we see that this representation carries the important property

$$\exp(Ad_g(X)) = g\exp(X)g^{-1}.$$

Proposition 1.11. *The derived representation for* $Ad : G \to GL(\mathfrak{g})$ *is given by*

$$dAd(X)(Y) = [X, Y].$$

Proof. Indeed for $X, Y \in \mathfrak{g}$ we compute

$$\begin{aligned}
dAd(X)(Y) &= \frac{d}{dt}\bigg|_0 Ad(\exp(tX))(Y) \\
&= \frac{d}{dt}\bigg|_0 \big(\exp(tX)Y\exp(tX)^{-1}\big) \\
&= \frac{d}{dt}\bigg|_0 \big(\exp(tX)Y\exp(-tX)\big) \\
&= \frac{d}{dt}\bigg|_0 \big((I + tX + O(t^2))Y(I - tX + O(t^2)\big) \\
&= \frac{d}{dt}\bigg|_0 \big(Y + t(XY - YX) + O(t^2)\big) \\
&= XY - YX \\
&= [X, Y].
\end{aligned}$$

□

1.5 Haar Measure

It is well documented that there exists on any linear Lie group G a measure μ which is left invariant, that is which satisfies

$$\int_G f(kx)\, d\mu(x) = \int_G f(x)\, d\mu(x)$$

for each $k \in G$ and measurable function $f : G \to \mathbb{C}$. Such a *Haar measure* is unique up to a scalar multiple. If G is compact, then we can normalize so that

$$\int_G d\mu(x) = 1.$$

Moreover, for G compact, it is known that μ is also right-invariant, that is [2, Proposition 5.1.3]

$$\int_G f(xk)\, d\mu(x) = \int_G f(x)\, d\mu(x).$$

Later we will see this explicitly for the Haar measure of $SU(2)$.

Proof of Lemma 1.10. Let π be a representation of a compact Lie group G on a finite dimensional complex vector space V and let $\langle \cdot, \cdot \rangle$ be any hermitian inner product on V. Then

$$\langle u, v \rangle' = \int_G \langle \pi(g)u, \pi(g)v \rangle \, d\mu(g)$$

is a new inner product for which π is unitary. Indeed, for any fixed $g_\circ \in G$ we have

$$\begin{aligned} \langle \pi(g_\circ)u, \pi(g_\circ)v \rangle' &= \int_G \langle \pi(g)\pi(g_\circ)u, \pi(g)\pi(g_\circ)v \rangle \, d\mu(g) \\ &= \int_G \langle \pi(gg_\circ)u, \pi(gg_\circ)v \rangle \, d\mu(g) \\ &= \int_G \langle \pi(g)u, \pi(g)v \rangle \, d\mu(g) = \langle u, v \rangle' \end{aligned}$$

by right-invariance of the Haar measure on G. □

CHAPTER 2: Compact Lie Groups

Let G be a compact linear Lie group and (π, V_π) be an irreducible, unitary representation in a finite dimensional complex vector space. Now let $\langle\cdot|\cdot\rangle$ be a $\pi(G)$-invariant hermitian inner product on V_π. Also, let μ be the normalized Haar measure on G.

2.1 Schur Orthogonality

Lemma 2.1 (Schur's Lemma)**.** *If an operator $T : V_\pi \to V_\pi$ commutes with π then T is a scalar operator.*

Proof. As $V = V_\pi$ is complex then T must have at least one eigenvalue, say λ. Now let V_λ be the eigenspace $V_\lambda = \{v \in V : T_v = \lambda v\}$. Since V_λ is an eigenspace then $V_\lambda \neq 0$. We will show that V_λ is $\pi(G)$-invariant. Indeed for $v \in V_\lambda$ we get

$$(T\pi(g))(v) = (\pi(g)T)(v)$$

since T commutes with π and thus

$$\pi(g)T(v) = \pi(g)\lambda v = \lambda\pi(g)v.$$

Now since π is irreducible then $V_\lambda = V$. Thus $T_\lambda = \lambda v$, $\forall v \in V$. So we get that T is a scalar operator. □

We will now define a special operator that commutes with the representation π that will help us later. Let $v \in V_\pi$ and define the operator $K_v : V_\pi \to V_\pi$ by

$$K_v(w) = \int_G \langle w, \pi(g)v\rangle \pi(g)v \, d\mu(g)$$

So now if we look at the inner product $\langle K_v(w), w'\rangle$ we see

$$\langle K_v(w), w'\rangle = \int\limits_G \langle w, \pi(g)v\rangle\langle \pi(g)v, w'\rangle \, d\mu(g). \tag{2.1}$$

Lemma 2.2. *K_v commutes with π.*

Proof. We have

$$K_v\pi(g_\circ)(w) = \int\limits_G \langle \pi(g_\circ)w, \pi(g)v\rangle \pi(g)v \, d\mu(g)$$

and now since $\langle\cdot|\cdot\rangle$ is $\pi(G)$-invariant we can apply $\pi(g_\circ^{-1})$ on both sides to write

$$K_v\pi(g_\circ)(w) = \int\limits_G \langle w, \pi(g_\circ^{-1})\pi(g)v\rangle \pi(g)v \, d\mu(g) = \int\limits_G \langle w, \pi(g_\circ^{-1}g)v\rangle \pi(g)v \, d\mu(g).$$

We can now apply change of variables and by taking $g \to g_\circ g$ we get

$$K_v\pi(g_\circ)(w) = \int\limits_G \langle w, \pi(g)v\rangle \pi(g_\circ g)v \, d\mu(g) = \int\limits_G \langle w, \pi(g)v\rangle \pi(g_\circ)\pi(g)v \, d\mu(g)$$

and so by taking the constant operator $\pi(g_\circ)$ out, we get that the integral is equal to $\pi(g_\circ)K_v(w)$. □

Thus by using lemma 2.1 we see that the operator K_v must be a scalar operator, say $K_v = \lambda(v)I$ where $\lambda(v) \in \mathbb{C}$. Using this fact will help us show our next theorem.

Theorem 2.3 (Schur Orthogonality)**.** *Let π be an irreducible unitary $\mathbb{C}$-linear representation of G on a complex vector space V with dimension d_π. Then for $v, w \in V$*

$$\int\limits_G |\langle \pi(g)v, w\rangle|^2 \, d\mu(g) = \frac{1}{d_\pi}\|v\|^2\|w\|^2$$

where $\langle\cdot|\cdot\rangle$ is the inner product in the space V.

Proof. Let $v, w \in V$. If $v = 0$ or $w = 0$ then the proof is clear. Assume $v, w \neq 0$. We know that $K_v = \lambda(v)I$ for some scalar $\lambda(v) \in \mathbb{C}$ and applying Equation 2.1

$$\begin{aligned}\int_G |\langle \pi(g)v, w\rangle|^2 \, d\mu(g) = \int_G \langle \pi(g)v, w\rangle \overline{\langle \pi(g)v, w\rangle} \, d\mu(g) &= \int_G \langle w, \pi(g)v\rangle \langle \pi(g)v, w\rangle \, d\mu(g) \\ &= \langle K_v w | w\rangle = \langle \lambda(v) w | w\rangle \\ &= \lambda(v) \|w\|^2. \end{aligned} \tag{2.2}$$

But we also have

$$\int_G |\langle \pi(g)v, w\rangle|^2 \, d\mu(g) = \int_G |\langle v, \pi(g^{-1})w\rangle|^2 \, d\mu(g)$$

and by replacing $g \to g^{-1}$ and using the fact that μ is right invariant as G is compact we get

$$\int_G |\langle \pi(g)v, w\rangle|^2 \, d\mu(g) = \int_G |\langle \pi(g)w, v\rangle|^2 \, d\mu(g) = \lambda(w)\|v\|^2.$$

Now by equation 2.2 we have that $\lambda(w)\|v\|^2 = \lambda(v)\|w\|^2$. That is $\lambda(v) = \frac{\lambda(w)}{\|w\|^2}\|v\|^2$ independent of $w \neq 0$. This shows that $\lambda(v) = \lambda_\circ \|v\|^2$ for some $\lambda_\circ$ constant.

Now choose an orthonormal basis for V_π, say $\{e_1, ..., e_{d_\pi}\}$. Since π is unitary for any $g \in G$ we have

$$\|v\|^2 = \|\pi(g)v\|^2 = \sum_{j=1}^{d_\pi} |\langle \pi(g)v, e_j\rangle|^2$$

and applying (2.2) we obtain

$$\|v\|^2 = \sum_{j=1}^{d_\pi} \int_G |\langle \pi(g)v, e_j\rangle|^2 \, d\mu(g) = \sum_{j=1}^{d_\pi} \lambda_\circ \|v\|^2 \|e_j\|^2 = \sum_{j=1}^{d_\pi} \lambda_\circ \|v\|^2 = \lambda_\circ d_\pi \|v\|^2.$$

This gives $\lambda_\circ = \frac{1}{d_\pi}$ and $\lambda(v) = \frac{1}{d_\pi}\|v\|^2$. Substituting $\lambda(v)$ into equation 2.2 yields

$$\int_G |\langle \pi(g)v, w\rangle|^2 \, d\mu(g) = \lambda_\circ \|v\|^2\|w\|^2 = \frac{1}{d_\pi}\|v\|^2\|w\|^2. \quad \square$$

As explained in [2, page 105] Schur Orthogonality together with polarization yields

$$\int_G \langle \pi(g)u|v\rangle \overline{\langle \pi(g)u'|v'\rangle} \, d\mu(g) = \frac{1}{d_\pi}\langle u|u'\rangle \overline{\langle v|v'\rangle}$$

for $u, v, u', v' \in V$. Then for the chosen orthonormal basis in the preceding proof, $\{e_1, ..., e_{d_\pi}\}$, we consider the *matrix coefficients* $\pi_{i,j}(g) = \langle \pi(g)e_j, e_i\rangle$ and look at the inner product of two such matrix coefficients

$$\begin{aligned} \langle \pi_{i,j}, \pi_{k,l}\rangle &= \int_G \pi_{i,j}(g)\overline{\pi_{k,l}(g)} \, d\mu(g) \\ &= \int_G \langle \pi(g)e_j, e_i\rangle \overline{\langle \pi(g)e_l, e_k\rangle} \, d\mu(g) \\ &= \frac{1}{d_\pi}\langle e_j, e_l\rangle \langle e_i, e_k\rangle \\ &= \frac{1}{d_\pi}\delta_{j,l}\delta_{i,k} \end{aligned}$$

where δ is the Kronecker delta.

Definition 2.4. We let $\mathcal{M}_\pi$ denote the subspace of $L^2(G)$ spanned by the matrix coefficients $\{\pi_{i,j} : 1 \leq i, j \leq d_\pi\}$.

It is not hard to see that $\mathcal{M}_\pi$ does not depend on the choice of orthonormal basis in V. We have shown that the $\pi_{i,j}$'s are pair-wise orthogonal functions in $L^2(G)$ with $\|\pi_{i,j}\|^2 = \frac{1}{d_\pi}$. Hence $\{\sqrt{d_\pi}\pi_{i,j} : 1 \leq i, j \leq d_\pi\}$ is an orthonormal basis for $\mathcal{M}_\pi$.

Theorem 2.5. *Suppose $\pi \not\simeq \pi'$ are inequivalent irreducible representations of G.*

Then $\mathcal{M}_\pi \perp \mathcal{M}_{\pi'}$ *in* $L^2(G)$*, i.e.,*

$$\int_G \langle \pi(g)u, v\rangle_V \overline{\langle \pi'(g)u', v'\rangle_{V'}}\, d\mu(g) = 0$$

where $\langle\cdot|\cdot\rangle_V$ *and* $\langle\cdot|\cdot\rangle_{V'}$ *are the respective inner products for* V *and* V'*.*

Proof. Let $u, v \in V$ and $u', v' \in V'$ be fixed. Define the operator $T : V \to V'$ via

$$T(x) = \langle x, v\rangle_V v'.$$

We can clearly see that $T \in \mathcal{L}(V, V')$ and we will also define $\widetilde{T} : V \to V'$ via

$$\widetilde{T} = \int_G \pi'(g^{-1})T\pi(g)\, d\mu(g).$$

We show that $\widetilde{T}$ intertwines the representations π and π'. Then by Schur's Lemma we have that $\widetilde{T} = 0$ since we assumed $\pi \not\simeq \pi'$. Indeed, for $g_\circ \in G$ we have

$$\widetilde{T}\pi(g_\circ) = \int_G \pi'(g^{-1})T\pi(g)\pi(g_\circ)\, d\mu(g) = \int_G \pi'(g^{-1})T\pi(gg_\circ)\, d\mu(g).$$

Take $g \to gg_\circ^{-1}$, i.e., $g^{-1} \to g_\circ g^{-1}$, to obtain

$$\begin{aligned}\int_G \pi'(g^{-1})T\pi(gg_\circ)\, d\mu(g) = \int_G \pi'(g_\circ g^{-1})T\pi(g)\, d\mu(g) = \pi(g_\circ)\int_G \pi'(g^{-1})T\pi(g)\, d\mu(g) \\ = \pi'(g_\circ)\widetilde{T}.\end{aligned}$$

Therefore we have that $\widetilde{T} = 0$ so

$$\begin{aligned}\widetilde{T}(u) &= \int_G \pi'(g^{-1}) T \pi(g)(u)\, d\mu(g) \\ &= \int_G \pi'(g^{-1}) \langle \pi(g)u, v\rangle_V v'\, d\mu(g) = \int_G \langle \pi(g)u, v\rangle_V \pi'(g^{-1}) v'\, d\mu(g) = 0\end{aligned}$$

and hence also

$$\begin{aligned}\int_G \langle \pi(g)u, v\rangle_V \overline{\langle \pi'(g)u', v'\rangle_{V'}}\, d\mu(g) &= \int_G \langle \pi(g)u, v\rangle_V \langle v', \pi'(g)u'\rangle_{V'}\, d\mu(g) \\ &= \int_G \langle \pi(g)u, v\rangle_V \langle \pi'(g^{-1})v', u'\rangle_{V'}\, d\mu(g) \\ &= \langle \widetilde{T}(u), u'\rangle_{V'} \\ &= 0.\end{aligned}$$

Here we have the $\mathcal{L}^2$ inner product of two matrix coefficients always being zero for arbitrary $u, v \in V$ and $u', v' \in V'$. $\square$

2.2 The Peter-Weyl Theorem

Before we can understand the Peter-Weyl Theorem we will first need to understand the right regular representation. For $f \in L^2(G)$ and $g \in G$ define $R(g)f \in L^2(G)$ via

$$(R(g)f)(x) = f(xg).$$

Then we have $R : G \to U(L^2(G))$ where $U(L^2(G))$ is the group of unitary operators on the space $L^2(G)$. Indeed,

$$\|R(g)f\|^2 = \int_G |(R(g)f)(x)|^2 \, d\mu(x) = \int_G |f(xg)|^2 \, d\mu(x).$$

Then by the right invariance of the Haar measure we get

$$\int_G |f(xg)|^2 \, d\mu(x) = \int_G |f(x)|^2 \, d\mu(x) = \|f\|^2.$$

Definition 2.6. $\widehat{G}$ will denote the set of equivalence classes of irreducible unitary representations of the compact group G.

This means that if π_1 and π_2 are two equivalent irreducible unitary representations of G then they are considered in the same class in $\widehat{G}$.

Lemma 2.7. *For $\pi \in \widehat{G}$ the subspace $\mathcal{M}_\pi \subset L^2(G)$ is $R(G)$-invariant.*

Proof. Given $u, v \in V_\pi$ we have the matrix coefficient

$$\phi_{u,v}(x) = \langle \pi(x)u, v\rangle_\pi$$

as before. Then

$$(R(g)\phi_{u,v})(x) = \phi_{u,v}(xg) = \langle \pi(xg)u, v\rangle_\pi = \langle \pi(x)\pi(g)u, v\rangle_\pi = \phi_{\pi(g)u,v}(x)$$

which is another matrix coefficient. As R is a linear operator the result follows. □

Let $\mathcal{M} = \sum_{\pi \in \widehat{G}} \mathcal{M}_\pi$, that is, the subspace of all functions $f \in L^2(G)$ that can be written as finite sums, i.e.

$$f = f_{\pi_1} + f_{\pi_2} + \cdots + f_{\pi_k},$$

where $f_{\pi_j} \in \mathcal{M}_{\pi_j}$ and $\pi_1, ..., \pi_k \in \widehat{G}$. Note that since $\mathcal{M}_\pi \perp \mathcal{M}_{\pi'}$ for $\pi \not\simeq \pi'$ we have a direct sum

$$\mathcal{M} = \bigoplus\nolimits_{\pi \in \widehat{G}} \mathcal{M}_\pi.$$

We let $\overline{\mathcal{M}} = \overline{\bigoplus}_{\pi \in \widehat{G}} \mathcal{M}_\pi$, the closure of $\mathcal{M}$ in $L^2(G)$. In view of Lemma 2.7 the subspace $\mathcal{M}$, and hence also, $\overline{\mathcal{M}}$ is $R(G)$-invariant

Lemma 2.8. *The orthogonal complement* $(\overline{\mathcal{M}})^\perp$ *is* $R(G)$*-invariant.*

Proof. Let $\psi \in (\overline{\mathcal{M}})^\perp$, $\phi \in \overline{\mathcal{M}}$ and $g \in G$. We have

$$\langle R(g)\psi, \phi \rangle_2 = \langle R(g^{-1})R(g)\psi, R(g^{-1})\phi \rangle_2 = \langle \psi, R(g^{-1})\phi \rangle_2 = 0$$

since $R(g)$ preserves $\overline{\mathcal{M}}$. Thus $R(g)$ preserves $(\overline{\mathcal{M}})^\perp$ as claimed. □

We now have all the tools we need to prove the Peter-Weyl Theorem.

Theorem 2.9 (Peter-Weyl)**.**

$$L^2(G) = \widehat{\bigoplus}_{\pi \in \widehat{G}} \mathcal{M}_\pi.$$

Proof. We must show that $\mathcal{M} = \bigoplus_{\pi \in \widehat{G}} \mathcal{M}_\pi$ is dense in $L^2(G)$, i.e.,

$$\overline{\mathcal{M}} = \widehat{\bigoplus}_{\pi \in \widehat{G}} \mathcal{M}_\pi = L^2(G).$$

.

Indeed, suppose $\overline{\mathcal{M}} \neq L^2(G)$, then $\overline{\mathcal{M}}^{\perp} \neq \{0\}$. One can show [2, Theorem 6.3.2] that $\overline{\mathcal{M}}^{\perp}$ must contain a non-zero $R(G)$-irreducible subspace $Y \subset \overline{\mathcal{M}}^{\perp}$. Note that, in particular, $R|_Y$ belongs to one of the classes in $\widehat{G}$, i.e. $R|_Y \simeq \pi$ for some $\pi \in \widehat{G}$.

Now, let $f \in Y$ and $f \neq 0$. Also, let $R_Y = R|_Y$ and define $F : G \to \mathbb{C}$ by

$$F(g) = \langle R(g)f, f\rangle_2 = \int_G f(xg)\overline{f(x)}\, d\mu(x).$$

Note that $F \in \mathcal{M}_\pi$ thus $F \in \overline{\mathcal{M}}$, however, $f \in Y \subset \overline{\mathcal{M}}^{\perp}$.

Now for $u, v \in V_\pi$ we have

$$\langle F, \phi_{u,v}\rangle = \int_G F(g)\overline{\langle \pi(g)u, v\rangle_\pi}\, d\mu(g) = \int_G \int_G f(xg)\overline{f(x)\langle \pi(g)u, v\rangle_\pi}\, d\mu(x)\, d\mu(g)$$

By change of variables let $g \to x^{-1}g'$ then

$$\begin{aligned}\int_G \int_G f(xg)\overline{f(x)\langle \pi(g)u, v\rangle_\pi}\, d\mu(x)\, d\mu(g) &= \int_G \overline{f(x)} \int_G f(g')\overline{\langle \pi(x^{-1}g')u, v\rangle_\pi}\, d\mu(g')\, d\mu(x) \\ &= \int_G \overline{f(x)} \int_G f(g')\overline{\langle \pi(g')u, \pi(x)v\rangle_\pi}\, d\mu(g')\, d\mu(x).\end{aligned}$$

And now we have that

$$\int_G f(g')\overline{\langle \pi(g')u, \pi(x)v\rangle_\pi}\, d\mu(g') = \langle f, \phi_{u,\pi(x)v}\rangle_2 = 0$$

since $f \in Y \subset \overline{\mathcal{M}}^{\perp}$ and $\phi_{u,\pi(x)v} \in \overline{\mathcal{M}}$. Now we have that $\langle F, \phi_{u,v}\rangle = 0$ for every

$\phi_{u,v} \in \mathcal{M}_\pi$ and as $F \in \mathcal{M}_\pi$ this gives $F = 0$. In particular

$$F(e) = \langle R(e)f, f\rangle_2 = \langle f, f\rangle_2 = 0$$

so $f \equiv 0 \in L^2(G)$ which is a contradiction. Thus $\overline{\mathcal{M}} = L^2(G)$. □

2.3 The Plancherel Theorem

For each $\pi \in \widehat{G}$ we let V_π denote the representation space for π and $d_\pi = \dim(V_\pi)$. It is known that each d_π is finite [2, Theorem 6.3.2]. If f is an integrable function on G then its Fourier coefficient $\widehat{f}(\pi)$ is the operator acting on V_π defined by

$$\widehat{f}(\pi) = \int_G f(g)\pi(g^{-1})\, d\mu(g). \tag{2.3}$$

For each $\pi \in \widehat{G}$ we choose an orthonormal basis say

$$\mathcal{B}_\pi = \{e_{\pi,1}, ...e_{\pi,d_\pi}\}$$

for V_π. Now define $\phi_{\pi,i,j} = \phi_{\pi,e_{\pi,i},e_{\pi,j}}$ where $\phi_{\pi,e_{\pi,i},e_{\pi,j}}(x) = \langle \pi(x)e_{\pi,j}, e_{\pi,i}\rangle_\pi$. Then by Schur orthogonality and the Peter-Weyl theorem, we get that

$$\mathcal{B} = \{\sqrt{d_\pi}\phi_{\pi,i,j} : \pi \in \widehat{G},\ 1 \le i, j \le d_\pi\}$$

is a Hilbert basis for $L^2(G)$. By using this Hilbert basis for $L^2(G)$ we can produce the Plancherel theorem, but first we need to see the relationship between the inner products in $L^2(G)$ and V_π.

Lemma 2.10. *For $u, v \in V_\pi$ and $f \in L^2(G)$ we have that $\langle f, \phi_{\pi,u,v}\rangle_2 = \langle \widehat{f}(\pi)v, u\rangle_\pi$*

Proof. Indeed,

$$\langle \widehat{f}(\pi)v, u\rangle_\pi = \int\limits_G \langle f(x)\pi(x^{-1})v, u\rangle_\pi \, d\mu(x)$$

and since π is unitary

$$\begin{aligned}
\int\limits_G \langle f(x)\pi(x^{-1})v, u\rangle_\pi \, d\mu(x) &= \int\limits_G f(x)\langle v, \pi(x)u\rangle_\pi \, d\mu(x) \\
&= \int\limits_G f(x)\overline{\langle \pi(x)u, v\rangle_\pi} \, d\mu(x) \\
&= \int\limits_G f(x)\overline{\phi_{\pi,u,v}(x)} \, d\mu(x) \\
&= \langle f, \phi_{\pi,u,v}\rangle_2.
\end{aligned}$$

□

For $A, B \in L(V_\pi)$ the Hilbert-Schmidt inner product is defined by

$$(A|B) = tr(AB^*)$$

and

$$|||A||| = (A|A)^{1/2}$$

is the associated Hilbert-Schmidt norm.

Theorem 2.11 (Plancherel Theorem)**.** *Let $f \in L^2(G)$.*

1. $f(g) = \sum_{\pi\in\widehat{G}} d_\pi tr(\widehat{f}(\pi)\pi(g))$ *in the L^2 sense.*

2. $\|f\|_2^2 = \sum_{\pi\in\widehat{G}} d_\pi |||\widehat{f}(\pi)|||^2$.

Proof. (1) Using the Hilbert basis $\mathcal{B} = \{\sqrt{d_\pi}\phi_{\pi,i,j} : \pi \in \widehat{G},\ 1 \le i, j \le d_\pi\}$ we get

$$f(g) = \sum_{\pi\in\widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \langle f, \phi_{\pi,i,j}\rangle_2 \phi_{\pi,i,j}(g)$$

(in the L^2-sense) and by using the previous lemma we get

$$\sum_{\pi\in\widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \langle f, \phi_{\pi,i,j}\rangle_2 \phi_{\pi,i,j}(g) = \sum_{\pi\in\widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \langle \widehat{f}(\pi)e_{\pi,j}, e_{\pi,i}\rangle_\pi \langle \pi(g)e_{\pi,i}, e_{\pi,j}\rangle_\pi.$$

We now consider $\widehat{f}$ and $\pi(g)$ as matrices and see

$$\begin{aligned}\sum_{\pi\in\widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \langle \widehat{f}(\pi)e_{\pi,j}, e_{\pi,i}\rangle_\pi \langle \pi(g)e_{\pi,i}, e_{\pi,j}\rangle_\pi &= \sum_{\pi\in\widehat{G}} \sum_{i=1} d_\pi([\widehat{f}(\pi)]_{\mathcal{B}_\pi}[\pi(g)]_{\mathcal{B}_\pi})_{i,i} \\ &= \sum_{\pi\in\widehat{G}} d_\pi tr(\widehat{f}(\pi)\pi(g)). \qquad (2.4)\end{aligned}$$

(2) Let $\mathcal{B}$ be the Hilbert basis for $L^2(G)$ as before. We have

$$\|f\|^2 = \sum_{\pi\in\widehat{G}} \sum_{i,j=1}^{d_\pi} |\langle f, \sqrt{d_\pi}\phi_{\pi,i,j}\rangle_2|^2 = \sum_{\pi\in\widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} |\langle f, \phi_{\pi,i,j}\rangle_2|^2.$$

Now look at

$$\begin{aligned}\sum_{i,j=1}^{d_\pi} |\langle f, \phi_{\pi,i,j}\rangle_2|^2 &= \sum_{i,j=1}^{d_\pi} \langle f, \phi_{\pi,i,j}\rangle_2 \overline{\langle f, \phi_{\pi,i,j}\rangle_2} \\ &= \sum_{i,j=1}^{d_\pi} \langle \widehat{f}(\pi)e_{\pi,j}, e_{\pi,i}\rangle_\pi \overline{\langle \widehat{f}(\pi)e_{\pi,j}, e_{\pi,i}\rangle_\pi} \\ &= \sum_{i,j=1}^{d_\pi} \langle \widehat{f}(\pi)e_{\pi,j}, e_{\pi,i}\rangle_\pi \langle e_{\pi,i}, \widehat{f}(\pi)e_{\pi,j}\rangle_\pi \\ &= \sum_{i,j=1}^{d_\pi} \langle \widehat{f}(\pi)e_{\pi,j}, e_{\pi,i}\rangle_\pi \langle \widehat{f}(\pi)^*e_{\pi,i}, e_{\pi,j}\rangle_\pi.\end{aligned}$$

Now considering $\widehat{f}(\pi)$ and $\widehat{f}(\pi)^*$ as matrices as before

$$\sum_{i,j=1}^{d_\pi} \langle \widehat{f}(\pi) e_{\pi,j}, e_{\pi,i}\rangle_\pi \langle \widehat{f}(\pi)^* e_{\pi,i}, e_{\pi,j}\rangle_\pi = \sum_{i,j=1}^{d_\pi} ([\widehat{f}(\pi)]_{\mathcal{B}_\pi})_{i,j} ([\widehat{f}(\pi)^*]_{\mathcal{B}_\pi})_{j,i}$$
$$= tr(\widehat{f}(\pi)\widehat{f}(\pi)^*) = |||\widehat{f}(\pi)|||^2.$$

Thus $\|f\|^2 = \sum_{\pi\in\widehat{G}} d_\pi |||\widehat{f}(\pi)|||^2$. □

2.4 Absolute and Uniform Convergence of Fourier Series

Let f be a continuous function on G. We have just seen that the Fourier series converges to f in the L^2-norm. Below is given a sufficient condition for uniform convergence of the Fourier series. But first we need the following relation. As above $\pi \in \widehat{G}$ acts on V_π with dimension d_π.

Lemma 2.12. *For each $g \in G$ one has $|||\pi(g)||| = \sqrt{d_\pi}$.*

Proof. As $\pi(g)$ is a unitary operator on V_π, hence normal, the Spectral Theorem [5, Theorem 6.25] ensures that there is an orthonormal basis for V_π, say $\mathcal{B}$, depending on g, for which $[\pi(g)]_{\mathcal{B}}$ is a diagonal matrix,

$$[\pi(g)]_{\mathcal{B}} = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_{d_\pi} \end{bmatrix}$$

say. Moreover we must have $|d_j| = 1$ since $\pi(g)$ is unitary. Thus we get

$$|||\pi(g)|||^2 = tr(\pi(g)\pi(g)^*) = tr \begin{bmatrix} |d_1|^2 & & & \\ & |d_2|^2 & & \\ & & \ddots & \\ & & & |d_{d_\pi}|^2 \end{bmatrix} = d_\pi$$

and finally $|||\pi(g)||| = \sqrt{d_\pi}$. □

Proposition 2.13. *Let f be a continuous function on G such that*

$$\sum_{\pi \in \widehat{G}} d_\pi^{3/2} |||\widehat{f}(\pi)||| < \infty.$$

Then the Fourier series

$$\sum_{\pi \in \widehat{G}} d_\pi tr(\widehat{f}(\pi)\pi(g))$$

converges absolutely and uniformly to $f(g)$ on G.

Proof. Recall for $A, B \in L(V_\pi)$ that $(A|B) = tr(AB^*)$. Now note

$$|tr(\widehat{f}(\pi)\pi(g))| = |(\widehat{f}(\pi)|\pi(g)^*)| \leq |||\widehat{f}(\pi)|||\ |||\pi(g)^*|||$$

by the Cauchy-Schwartz inequality. As $\pi(g)$ is unitary we have $|||\pi(g)^*||| = |||\pi(g^{-1})||| = \sqrt{d_\pi}$ by the previous lemma and thus

$$|d_\pi tr(\widehat{f}(\pi)\pi(g))| \leq d_\pi^{3/2} |||\widehat{f}(\pi)|||.$$

As by hypothesis $\sum_{\pi \in \widehat{G}} d_\pi^{3/2} |||\widehat{f}(\pi)||| < \infty$, the M-test implies that $\sum_{\pi \in \widehat{G}} d_\pi tr(\widehat{f}(\pi)\pi(g))$ converges uniformly and absolutely on G. It remains to show that

$$\sum_{\pi \in \widehat{G}} d_\pi tr(\widehat{f}(\pi)\pi(g)) = f(g).$$

Let

$$h(g) = \sum_{\pi \in \widehat{G}} d_\pi tr(\widehat{f}(\pi)\pi(g)).$$

We will show that $h = f$. Since the series converges uniformly and each term is continuous, then h must be continuous. In particular we get that $h \in L(G)$ and will show that $\widehat{h}(\pi) = \widehat{f}(\pi)$ for each $\pi \in \widehat{G}$.

Fix $\pi \in \widehat{G}$ and let $\mathcal{B} = \{e_1, ..., e_{d_\pi}\}$ be an orthonormal basis for V_π. Then

$$\begin{aligned}\langle \widehat{h}(\pi)e_k, e_l\rangle_\pi &= \int_G h(x)\langle \pi(x^{-1})e_k, e_l\rangle_\pi \, d\mu(x) \\ &= \int_G \sum_{\pi\in\widehat{G}} d_{\pi'} tr(\widehat{f}(\pi')\pi'(x))\langle \pi(x^{-1})e_k, e_l\rangle_\pi \, d\mu(x).\end{aligned}$$

Note that $\langle \pi(x^{-1})e_k, e_l\rangle_\pi = \overline{\langle \pi(x)e_l, e_k\rangle_\pi} = \overline{\phi_{\pi,k,l}(x)}$ and that $tr(\widehat{f}(\pi')\pi'(x)) \in \mathcal{M}_{\pi'}$. Indeed, let $T(x) = tr(\widehat{f}(\pi')\pi'(x))$ and $A = [\widehat{f}(\pi')]_\mathcal{B}$ and $B(x) = [\pi'(x)]_\mathcal{B}$. Then

$$T(x) = tr(AB(x)) = \sum_{i,j=1}^{d_\pi} a_{i,j}b_{j,i}(x)$$

is a linear combination of matrix coefficients for π' and thus $T(x) \in \mathcal{M}_{\pi'}$.

Now, since $\phi_{\pi,l,k} \in \mathcal{M}_\pi$, Schur orthogonality yields

$$\langle \widehat{h}(\pi)e_k, e_l\rangle_\pi = \int_G \sum_{\pi'\in\widehat{G}} d_{\pi'} tr(\widehat{f}(\pi')\pi'(x))\overline{\phi_{\pi,l,k}} \, d\mu(x) = \int_G d_\pi tr(\widehat{f}(\pi)\pi(x))\overline{\phi_{\pi,l,k}} \, d\mu(x).$$

But we have shown that $tr(\widehat{f}(\pi)\pi(x)) = \sum_{i,j=1}^{d_\pi} \langle f, \phi_{\pi,i,j}\rangle_2 \phi_{\pi,i,j}(x)$ by Equation 2.4, so we get

$$\langle \widehat{h}(\pi)e_k, e_l\rangle_\pi = d_\pi \sum_{i,j=1}^{d_\pi} \langle f, \phi_{\pi,i,j}\rangle_2 \langle \phi_{\pi,i,j}, \phi_{\pi,l,k}\rangle_2.$$

By Theorem 2.3 we have $\langle \phi_{\pi,i,j}, \phi_{\pi,l,k}\rangle_2 = \frac{1}{d_\pi}\delta_{i,l}\delta_{j,k}$ so,

$$d_\pi \sum_{i,j=1}^{d_\pi} \langle f, \phi_{\pi,i,j}\rangle_2 \langle \phi_{\pi,i,j}, \phi_{\pi,l,k}\rangle_2 = \langle f, \phi_{\pi,l,k}\rangle_2 = \langle \widehat{f}(\pi)e_k, e_l\rangle_2.$$

Thus we have $\langle \widehat{h}(\pi)e_k, e_l\rangle_\pi = \langle \widehat{f}(\pi)e_k, e_l\rangle_\pi$ for all $\pi \in \widehat{G}$ so $\widehat{h} = \widehat{f}$. Finally, applying Theorem 2.11 part (2) to the norm of the difference between f and h, we see,

$$\|f-h\|^2 = \sum_{\pi\in\widehat{G}} d_\pi |||\widehat{f}(\pi) - \widehat{h}(\pi)|||^2 = 0,$$

i.e. $\int_G |f(x)-h(x)|^2\, d\mu(x) = 0$. Now as f and h are continuous then $f = h$. □

2.5 Fourier Series for Central Functions

Let G be a linear Lie group and π a representation of G.

Definition 2.14. A function $f : G \to \mathbb{C}$ is central if and only if

$$f(gxg^{-1}) = f(x), \quad \text{for all} \quad g, x \in G.$$

We denote the space of integrable central functions as $L^1(G)^{Ad}$.

Lemma 2.15. *If $f \in L^1(G)^{Ad}$ then for each irreducible unitary representation $\pi \in \widehat{G}$, $\widehat{f}(\pi)$ is a scalar operator.*

Proof. By Schur's Lemma (2.1) we just need to show $\widehat{f}(\pi)$ commutes with π, i.e. $\widehat{f}(\pi)\pi(g) = \pi(g)\widehat{f}(\pi)$ for all $g \in G$. By definition we have

$$\widehat{f}(\pi)\pi(g) = \int_G f(x)\pi(x^{-1})\pi(g)\, d\mu(x).$$

If we now substitute $x \to gxg^{-1}$ then $x^{-1} \to gx^{-1}g^{-1}$ so the integral becomes

$$\int_G f(gxg^{-1})\pi(gx^{-1}g^{-1})\pi(g)\, d\mu(x) = \int_G f(x)\pi(gx^{-1}g^{-1})\pi(g)\, d\mu(x)$$

as f is central. Also as π is a homomorphism

$$\begin{aligned}\int_G f(x)\pi(gx^{-1}g^{-1})\pi(g)\,d\mu(x) &= \int_G f(x)\pi(g)\pi(x^{-1})\pi(g^{-1})\pi(g)\,d\mu(x)\\ &= \int_G f(x)\pi(g)\pi(x^{-1})\,d\mu(x)\\ &= \pi(g)\widehat{f}(\pi).\end{aligned}$$

□

Definition 2.16. The *character* of $\pi \in \widehat{G}$ is $\chi_\pi : G \to \mathbb{C}$ where

$$\chi_\pi(x) = tr(\pi(x)).$$

Lemma 2.17. $\overline{\chi_\pi(x)} = \chi_\pi(x^{-1})$.

Proof. For $x \in G$ we calculate

$$\overline{\chi_\pi(x)} = tr(\overline{\pi(x)}) = tr((\pi(x)^*)^\intercal) = tr(\pi(x^{-1})^\intercal) = tr(\pi(x^{-1})) = \chi_\pi(x^{-1}).$$

□

Theorem 2.18. *For $f \in L^2(G)^{Ad}$,*

$$f(g) = \sum_{\pi\in\widehat{G}} \langle f, \chi_\pi\rangle \chi_\pi(g)$$

converges in the L^2-sense. Moreover if f is continuous, central, and

$$\sum_{\pi\in\widehat{G}} d_\pi |\langle f, \chi_\pi\rangle_2| < \infty$$

then the series converges absolutely and uniformly to f on G.

Proof. Let $f \in L^2(G)^{Ad}$. We know the Fourier series

$$\sum_{\pi\in\widehat{G}} d_\pi tr(\widehat{f}(\pi)\pi(g))$$

converges to f in the L^2-sense. But since f is central the previous lemma shows $\widehat{f}(\pi)$ is a scalar operator, $\widehat{f}(\pi) = C_\pi(f)I_{V_\pi}$ say. Thus

$$\sum_{\pi\in\widehat{G}} d_\pi tr(\widehat{f}(\pi)\pi(g)) = \sum_{\pi\in\widehat{G}} d_\pi C_\pi(f) tr(\pi(g)) = \sum_{\pi\in\widehat{G}} d_\pi C_\pi(f)\chi_\pi(g).$$

We now calculate

$$\begin{aligned}\langle f, \chi_\pi\rangle_2 &= \int_G f(g)\overline{\chi_\pi(g)}\, d\mu(g) = \int_G f(g)\chi_\pi(g^{-1})\, d\mu(g) = \int_G f(g) tr(\pi(g^{-1}))\, d\mu(g)\\ &= tr\left(\int_G f(g)\pi(g^{-1})\, d\mu(g)\right)\\ &= tr(\widehat{f}(\pi))\\ &= d_\pi C_\pi(f).\end{aligned}$$

So $C_\pi(f) = \langle f, \chi_\pi\rangle_2/d_\pi$ and $\sum_{\pi\in\widehat{G}} \langle f, \chi_\pi\rangle\chi_\pi(g)$ converges to f in $L^2(G)$ as stated.

Next assume that f is a continuous central function satisfying $\sum_{m\in\widehat{G}} d_\pi|\langle f, \chi_\pi\rangle_2| < \infty$. As

$$\begin{aligned} d_\pi^{\frac{3}{2}}|||\widehat{f}(\pi)||| &= d_\pi^{\frac{3}{2}}\left|\left|\left|\frac{\langle f, \chi_\pi\rangle_2}{d_\pi} I_{V_\pi}\right|\right|\right| = d_\pi^{\frac{3}{2}}\frac{|\langle f, \chi_\pi\rangle_2|}{d_\pi}|||I_{V_\pi}||| = d_\pi^{\frac{3}{2}}\frac{|\langle f, \chi_\pi\rangle_2|}{d_\pi} d_\pi^{1/2}\\ &= d_\pi|\langle f, \chi_\pi\rangle|,\end{aligned}$$

we have that $\sum_{\pi\in\widehat{G}} d_\pi^{\frac{3}{2}}|||\widehat{f}(\pi)||| < \infty$. Proposition 2.13 now ensures that the Fourier series converges absolutely and uniformly to f on G. $\square$

2.6 Laplace Operator

As before let G denote a compact linear Lie group. Let $\langle \cdot, \cdot \rangle$ be the Hilbert-Schimdt inner product on $\mathfrak{g}$, that is $\langle X, Y \rangle = tr(XY^*)$, and $\{X_1, ..., X_n\}$ an orthonormal basis for $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$. Define $\Delta : C^\infty(G) \to C^\infty(G)$ via

$$\Delta = \sum_{j=1}^{n} \rho(X_j)^2$$

where $(\rho(X)f))(g) = \frac{d}{dt}\big|_0 f(g \exp(tX))$ for $f \in C^\infty(G)$ and $g \in G$. Δ is known as the *Laplace Operator.*

Notice that

$$\begin{aligned}(\rho(X)^2 f)(g) = \rho(X)\rho(X)f(g) &= \frac{d}{dt}\bigg|_0 \rho(X)f(g \exp(tX)) \\ &= \frac{d}{dt}\bigg|_0 \frac{d}{ds}\bigg|_0 f(g \exp(tX)\exp(sX)) \\ &= \frac{d}{dt}\bigg|_0 \frac{d}{ds}\bigg|_0 f(g \exp((t+s)X)) \\ &= \frac{d^2}{dt^2}\bigg|_0 f(g \exp(tX)).\end{aligned}$$

Therefore we can write

$$(\Delta f)(g) = \sum_{j=1}^{n} \frac{d^2}{dt^2}\bigg|_0 f(g \exp(tX_j)).$$

Proposition 2.19. *The Laplace operator Δ does not depend on the choice of orthonormal basis.*

Proof. Let $\{Y_1, ..., Y_n\}$ be another orthonormal basis for $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ then

$$Y_j = \sum_{i=1}^{n} a_{ij} X_i$$

for some matrix $A = [a_{ij}] \in O(n)$. Now we observe

$$\begin{aligned}
\sum_{j=1}^{n} \rho(Y_j)^2 = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} \rho(X_i) \right)^2 &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} \rho(X_i) \right) \left(\sum_{k=1}^{n} a_{kj} \rho(X_k) \right) \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} \left(\sum_{j=1}^{n} a_{ij} a_{kj} \right) \rho(X_i) \rho(X_k) \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} (\sum_{j=1}^{n} A_{ij} A^{\intercal}_{jk}) \rho(X_i) \rho(X_k) \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} (AA^{\intercal})_{ik} \rho(X_i) \rho(X_k) \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} \delta_{ik} \rho(X_i) \rho(X_k) \\
&= \sum_{i=1}^{n} \rho(X_i)^2.
\end{aligned}$$

□

Proposition 2.20. *Δ is left G-invariant, i.e.*

$$\Delta(f \circ L(g)) = (\Delta f) \circ L(g)$$

where $L(g) : G \to G$ is left multiplication by g, i.e. $L(g)(x) = gx$.

Proof. First observe that

$$\begin{aligned}
((\rho(X))(f \circ L(g))(x) = \left.\frac{d}{dt}\right|_0 (f \circ L(g))(x \exp(tX)) &= \left.\frac{d}{dt}\right|_0 f(gx \exp(tX)) \\
&= (\rho(X)f)(gx)
\end{aligned}$$

$$= ((\rho(X)f) \circ L(g))(x).$$

Thus we have

$$\begin{aligned}\rho(X)^2(f \circ L(g)) = \rho(X)(\rho(X)(f \circ L(g))) &= \rho(X)((\rho(X) \circ f) \circ L(g)) \\ &= (\rho(X)\rho(X)f) \circ L(g) \\ &= (\rho(X)^2 f) \circ L(g).\end{aligned}$$

Finally, observe that

$$\Delta(f \circ L(g)) = \sum_{j=1}^{n} \rho(X_j)^2(f \circ L(g)) = \sum_{j=1}^{n}(\rho(X_j)^2 f) \circ L(g) = (\Delta f) \circ L(g). \quad \square$$

We will show that Δ is right G-invariant under certain conditions but first we will need the following lemma.

Lemma 2.21. $\rho(X)(f \circ R(g)) = (\rho(Ad(g^{-1})X)f) \circ R(g)$

Proof. For $x \in G$ we calculate

$$\begin{aligned}(\rho(X)(f \circ R(g))(x) = \frac{d}{dt}\Big|_0 (f \circ R(g))(x\exp(tX)) &= \frac{d}{dt}\Big|_0 f(x\exp(tX)g) \\ &= \frac{d}{dt}\Big|_0 f(xg(g^{-1}\exp(tX)g)) \\ &= \frac{d}{dt}\Big|_0 f(xg\exp(t\,Ad(g^{-1})X)) \\ &= (\rho(Ad(g^{-1})X)f)(xg) \\ &= ((\rho(Ad(g^{-1})X)f) \circ R(g))(x). \square\end{aligned}$$

Proposition 2.22. *As $\langle\cdot,\cdot\rangle$ is $Ad(G)$-invariant then Δ is right G-invariant.*

Proof. Since the Hilbert-Schmidt inner product $\langle \cdot, \cdot \rangle$ is $Ad(G)$-invariant then

$$\Delta(f \circ R(g)) = \sum_{j=1}^{n} \rho(X_j)^2 (f \circ R(g)) = \sum_{j=1}^{n} \rho(X_j)((\rho(Ad(g^{-1})X_j)f) \circ R(g))$$
$$= \sum_{j=1}^{n} (\rho(Ad(g^{-1})X_j)^2 f) \circ R(g).$$

Now if we let $Y_j = Ad(g^{-1})X_j$ then as $\langle \cdot, \cdot \rangle$ is $Ad(G)$-invariant we get that $\{Y_j\}$ is another orthonormal basis for $\mathfrak{g}$. Now by the previous proposition we have

$$\sum_{j=1}^{n} \rho(Y_j)^2 = \sum_{j=1}^{n} \rho(Ad(g^{-1})X_j)^2 = \Delta.$$

Thus we have

$$\Delta(f \circ R(g)) = \Delta f \circ R(g)$$

as desired. □

Corollary 2.23. *Let $f, g \in C^\infty(G)$ then*

$$\langle \Delta f, g \rangle_2 = \langle f, \Delta g \rangle_2$$

where $\langle \cdot, \cdot \rangle_2 = \int_G f(x)\overline{g(x)}\, d\mu(x)$, the inner product for the space $L^2(G)$. In other words we have that Δ is self-adjoint.

Proof. Let $X \in \mathfrak{g}$ then

$$\langle \rho(X)f, g \rangle_2 = \int_G \rho(X)f(x)\overline{g(x)}\, d\mu(x) = \frac{d}{dt}\bigg|_0 \int_G f(x \exp(tX))\overline{g(x)}\, d\mu(x).$$

We now perform a change of variable with $x \mapsto x\exp(-tX)$ then the integral becomes

$$\begin{aligned}\frac{d}{dt}\Big|_0 \int_G f(x)\overline{g(x\ \exp(-tX))}\,d\mu(x) &= \int_G f(x)\overline{(\rho(-X)g)(x)}\,d\mu(x)\\ &= \langle f, \rho(-X)g\rangle_2\\ &= -\langle f, \rho(X)g\rangle_2.\end{aligned}$$

Thus we have shown that $\rho(X)^* = -\rho(X)$. Now we get that

$$\Delta^* = \sum_{j=1}^{n}(\rho(X_j)^2)^* = \sum_{j=1}^{n}(\rho(X_j)^*)^2 = \sum_{j=1}^{n}(-\rho(X_j))^2 = \Delta. \quad \square$$

Lemma 2.24. $\langle \Delta f, f\rangle_2 \leq 0$ *for all* $f \in L^2(G)$.

Proof. We have

$$\begin{aligned}\langle \Delta f, f\rangle_2 = \sum_{j=1}^{n}\langle \rho(X_j)\rho(X_j)f, f\rangle_2 &= -\sum_{j=1}^{n}\langle \rho(X_j)f, \rho(X_j)f\rangle_2\\ &= -\sum_{j=1}^{n}\|\rho(X_j)f\|_2^2 \leq 0.\end{aligned}$$ $\square$

Thus Δ is a negative self-adjoint operator and any eigenvalue for Δ is a non-positive real number.

Let $\pi : G \to GL(V)$ be a representation of G on a finite dimensional complex vector space V. Then we also have $d\pi : \mathfrak{g} \to gl(V)$ where

$$d\pi(X)(v) = \frac{d}{dt}\Big|_0 \pi(v\ \exp(tX)).$$

We now define the *Casimir Operator* as $\Omega_\pi : V \to V$ as

$$\Omega_\pi = \sum_{j=1}^{n} d\pi(X_j)^2$$

where $\{X_1, ..., X_n\}$ is, as before, an orthonormal basis for $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ and $\langle \cdot, \cdot \rangle$ is $Ad(G)$-invariant.

Proposition 2.25. *The Casimir Operator Ω_π does not depend on the choice of orthonormal basis.*

Proof. As the Casimir operator is a sum of derived representations similar to the Laplace operator. The proof follows directly from Proposition 2.22. □

Proposition 2.26. *Ω_π commutes with π, i.e.,*

$$\Omega_\pi \circ \pi(g) = \pi(g) \circ \Omega_\pi$$

for all $g \in G$.

Proof. First notice that for $g \in G$ we have $\pi(g)\pi(g^{-1}) = id$. Thus we can do the following,

$$d\pi(X)\pi(g) = \pi(g)\pi(g^{-1})d\pi(X)\pi(g) = \pi(g)d\pi(Ad(g^{-1})X).$$

From this we get

$$d\pi(X)^2\pi(g) = \pi(g)d\pi(Ad(g^{-1})X)^2.$$

Now observe that

$$\Omega_\pi \circ \pi(g) = (\sum_{j=1}^{n} d\pi(X_j)^2) \circ \pi(g) = \pi(g) \circ \sum_{j=1}^{n} d\pi(Ad(g^{-1})X_j)^2.$$

Since $\langle \cdot, \cdot \rangle$ is $Ad(G)$-invariant then $\{Ad(g^{-1})X_j\}$ is an orthonormal basis for $\mathfrak{g}$. Thus we get

$$\Omega_\pi \circ \pi(g) = \pi(g) \circ \Omega_\pi$$

as desired. □

Now we will suppose that π is an *irreducible* representation of G. Recall that V_π is necessarily finite dimensional as G is compact. By Lemma 2.1 and the previous proposition it follows that $\Omega_\pi = \lambda \, id$ for some $\lambda \in \mathbb{C}$.

Lemma 2.27. *$\Omega_\pi = -k_\pi \, id$ where $k_\pi \in \mathbb{R}^+$.*

Proof. We already have that $\Omega_\pi = \lambda \, id$ where $\lambda \in \mathbb{C}$. First we calculate

$$\begin{aligned}
\langle d\pi(X)v, w\rangle = \left.\frac{d}{dt}\right|_0 \langle \pi(\exp(tX))v, w\rangle &= \left.\frac{d}{dt}\right|_0 \langle v, \pi(\exp(tX))^{-1}w\rangle \\
&= \left.\frac{d}{dt}\right|_0 \langle v, \pi(\exp(-tX))w\rangle \\
&= \langle v, d\pi(-X)w\rangle \\
&= -\langle v, d\pi(X)w\rangle.
\end{aligned}$$

Thus

$$\begin{aligned}
\langle \Omega_\pi v, w\rangle = \left\langle \sum_{j=1}^n d\pi(X_j)^2 v, w \right\rangle = \sum_{j=1}^n \langle d\pi(X_j)^2 v, w\rangle &= \sum_{j=1}^n -\langle d\pi(X_j)v, d\pi(X_j)w\rangle \\
&= \sum_{j=1}^n \langle v, d\pi(X_j)^2 w\rangle \\
&= \langle v, \Omega_\pi w\rangle.
\end{aligned}$$

Hence Ω_π is a self-adjoint operator and its eigenvalues lie in $\mathbb{R}$. Now observe that

$$\langle \Omega_\pi v, v\rangle = \sum_{j=1}^{n} \langle d\pi(X_j)^2 v, v\rangle = -\sum_{j=1}^{n} \langle d\pi(X_j)v, d\pi(X_j)v\rangle = -\sum_{j=1}^{n} \|d\pi(X_j)v\|^2 \leq 0.$$

Finally

$$\langle \Omega_\pi v, v\rangle = \lambda\langle v, v\rangle \leq 0, \quad \text{for all} \quad v \in V.$$

So $\lambda = -k_\pi$ for some $k_\pi \geq 0$. □

Proposition 2.28. *Let π be an irreducible representation of G on V_π and $k_\pi \in \mathbb{R}^+$ be as in Lemma 2.27. Then*

$$\Delta(f) = -k_\pi f$$

for every function $f \in \mathcal{M}_\pi$. In particular each matrix coefficient π_{ij} is an eigenfunction for Δ with eigenvalue $-k_\pi$.

Proof. By definition of $\mathcal{M}_\pi$ and linearity of Δ it suffices to verify that $\Delta\pi_{ij} = -k_\pi\pi_{ij}$ for each $i, j \in \{1, \ldots, n\}$.

Observe

$$\begin{aligned}
\Delta\pi_{ij}(g) = \sum_{k=0}^{m} \left.\frac{d^2}{dt^2}\right|_0 \pi_{ij}(g\exp(tX_k)) &= \sum_{k=0}^{m} \left.\frac{d^2}{dt^2}\right|_0 \langle \pi(g\exp(tX_k))e_j, e_i\rangle \\
&= \sum_{k=0}^{m} \left.\frac{d^2}{dt^2}\right|_0 \langle \pi(g)\pi(\exp(tX_k))e_j, e_i\rangle \\
&= \sum_{k=0}^{m} \langle \pi(g) \left.\frac{d^2}{dt^2}\right|_0 \pi(\exp(tX_k))e_j, e_i\rangle \\
&= \langle \pi(g)\Omega_\pi e_j, e_i\rangle \\
&= -k_\pi\langle \pi(g)e_j, e_i\rangle \\
&= -k_\pi\pi_{ij}(g).
\end{aligned}$$

□

2.7 Uniform Continuity on Compact Groups

Recall that a continuous function on a compact metric space is *uniformly* continuous. Later we require a version of uniform continuity for functions on $SU(2)$. In general a continuous function on a *compact* topological group is uniformly continuous in the sense of the following proposition.

Proposition 2.29. *Let G be a compact topological group and $f : G \to \mathbb{C}$ a continuous function. Then for any $\varepsilon > 0$ there exists an open neighborhood V of the identity $e \in G$ such that*

$$\big|f(yx) - f(x)\big| < \varepsilon$$

for all $x \in G$, $y \in V$. Likewise one can find a neighborhood $V' \ni e$ with $\big|f(xy) - f(x)\big| < \varepsilon$ for all $x \in G$, $y \in V'$. One says that f is left and right uniformly continuous.

Proof. Let $f : G \to \mathbb{C}$ be continuous and $\varepsilon > 0$ given. For each fixed $x_\circ \in G$ the mapping

$$F : G \times G \to \mathbb{C}, \qquad F(x, y) := f(yx_\circ) - f(yx)$$

is continuous with $F(x_\circ, e) = f(x_\circ) - f(x_\circ) = 0$. Thus there exist open neighborhoods $U^1_{x_\circ} \ni x_\circ$ and $V^1_{x_\circ} \ni e$ with $|F(x, y)| < \varepsilon/3$ for all $(x, y) \in U^1_{x_\circ} \times V^1_{x_\circ}$. That is $\big|f(yx_\circ) - f(yx)\big| < \varepsilon/3$ for all $x \in U^1_{x_\circ}$ and $y \in V^1_{x_\circ}$. Also, by continuity of f at $x_\circ$, there is a neighborhood $U^2_{x_\circ} \ni x_\circ$ with

$$\big|f(z) - f(x_\circ)\big| < \frac{\varepsilon}{3} \quad \text{for all } z \in U^2_{x_\circ}. \tag{2.5}$$

Let

$$U_{x_\circ} := U^1_{x_\circ} \cap U^2_{x_\circ}, \qquad V_{x_\circ} := V^1_{x_\circ} \cap \big(U^2_{x_\circ} x_\circ^{-1}\big).$$

These are neighborhoods of $x_\circ$ and e respectively for which both

$$\big|f(yx_\circ) - f(yx)\big| < \frac{\varepsilon}{3} \quad \text{for all } x \in U_{x_\circ},\ y \in V_{x_\circ} \text{ and} \tag{2.6}$$

$$\big|f(yx_\circ) - f(x_\circ)\big| < \frac{\varepsilon}{3} \quad \text{for all } y \in V_{x_\circ}. \tag{2.7}$$

As G is compact the open cover $\big\{U_{x_\circ} \ : \ x_\circ \in G\big\}$ has a finite subcover,

$$G = U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n}$$

say. Let V be the neighborhood of e defined via

$$V := V_{x_1} \cap V_{x_2} \cap \cdots \cap V_{x_n}.$$

For any given $x \in G$ we have $x \in U_{x_k}$ for some k. So now for all $y \in V$ one has

$$\big|f(yx)-f(x)\big| \leq \big|f(yx)-f(yx_k)\big|+\big|f(yx_k)-f(x_k)\big|+\big|f(x_k)-f(x)\big| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon$$

as desired. Indeed

- $\big|f(yx) - f(yx_k)\big| < \varepsilon/3$ by (2.6) since $x \in U_{x_k}$ and $y \in V \subset V_{x_k}$.
- $\big|f(yx_k) - f(x_k)\big| < \varepsilon/3$ by (2.7) since $y \in V \subset V_{x_k}$.
- $\big|f(x_k) - f(x)\big| < \varepsilon/3$ by (2.5) since $x \in U_{x_k} \subset U_{x_k}^2$.

This completes the proof. □

CHAPTER 3: Analysis on $U(1)$

Recall that $U(n) = \{g \in M(n, \mathbb{C}) \ : \ g^*g = I\}$. For $n = 1$ this reduces to $U(1) = \{z \in \mathbb{C} \ : \ |z|^2 = 1\} = \{e^{i\theta} \ : \ \theta \in \mathbb{R}\}$. Thus we have that the Lie group $U(1)$ is the unit circle in the complex plane. We first see how the Plancherel Theorem for this group specializes to the classical L^2-theory of Fourier series for 2π-periodic functions on the real line. Then we see under what conditions we have absolute and uniform convergence on $U(1)$.

Given a function $f : U(1) \to \mathbb{C}$ we define $\widetilde{f} : \mathbb{R} \to \mathbb{C}$ as the 2π-periodic function

$$\widetilde{f}(\theta) = f(e^{i\theta}).$$

Using this convention the Haar measure on $U(1)$ is

$$\int\limits_{U(1)} f(x)\, d\mu(x) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \widetilde{f}(\theta) dt.$$

It is easy to see that this is normalized.

3.1 Determination of $\widehat{U(1)}$

We now need to find the set of irreducible unitary representations of $U(1)$, up to equivalence, known as $\widehat{U(1)}$. First note that as $U(1)$ is abelian each irreducible representation (π, V) is necessarily one dimensional. Indeed each operator $\pi(z)$ commutes with $\pi(U(1))$ and Schur's Lemma (2.1) implies that each $\pi(z)$ is a scalar operator, say $\pi(z) = \chi(z)\, id$. This means that any one dimensional subspace of V is $\pi(U(1))$-invariant, but as π is irreducible we must have $\dim(V) = 1$.

If we identify an irreducible representation (π, V), as above, with the mapping

$\chi : U(1) \to \mathbb{C}$ such that $\pi(z) = \chi(z)\, id$ then, in fact, as π is a unitary representation we must have $|\chi(z)| = 1$ for each z. That is

$$\chi : U(1) \to U(1).$$

Moreover as $\pi(z_1 z_2) = \pi(z_1)\pi(z_2)$ we have $\chi(z_1 z_2) = \chi(z_1)\chi(z_2)$. That is, χ is a continuous group homomorphism $U(1) \to U(1)$. Let

$$\chi_m(z) = z^m$$

where $m \in \mathbb{Z}$.

Lemma 3.1. *The mapping χ_m is a representation of $U(1)$.*

Proof. It is clear to see that χ_m is a continuous mapping. Also we have

$$\chi_m(z_1 z_2) = (z_1 z_2)^m = z_1^m z_2^m = \chi_m(z_1)\chi_m(z_2),$$

which shows χ_m is a homomorphism and therefore a representation of $U(1)$. □

Theorem 3.2. $\widehat{U(1)} = \{\chi_m \ : \ m \in \mathbb{Z}\}$.

Proof. We follow the proof in Folland [4, Page 90]. The previous lemma shows that χ_m is a representation of $U(1)$ and thus $\{\chi_m \ : \ m \in \mathbb{Z}\} \subset \widehat{U(1)}$. It remains to show the converse. First note that

$$\widetilde{\chi} : \mathbb{R} \to U(1), \quad \widetilde{\chi}(t) = \chi(e^{i\theta})$$

is a continuous group homomorphism from $(\mathbb{R}, +)$ to $U(1)$. As $\widetilde{\chi}$ is continuous and

$\widetilde{\chi}(0) = 1$ we must have that

$$A := \int_0^{\varepsilon} \widetilde{\chi}(s)\, ds \neq 0$$

provided $\varepsilon > 0$ is sufficiently small. Now observe

$$A\widetilde{\chi}(\theta) = \int_0^{\varepsilon} \widetilde{\chi}(\theta)\widetilde{\chi}(\phi)\, d\phi = \int_0^{\varepsilon} \widetilde{\chi}(\theta + \phi)\, d\phi = \int_{\theta}^{\theta+\varepsilon} \widetilde{\chi}(\phi)\, d\phi.$$

The Fundamental Theorem of Calculus ensures us that $\widetilde{\chi}$ is differentiable with

$$A\widetilde{\chi}'(\theta) = \widetilde{\chi}(\theta + \varepsilon) - \widetilde{\chi}(\theta) = (\widetilde{\chi}(\varepsilon) - 1)\widetilde{\chi}(\theta).$$

Letting

$$\lambda := \frac{\widetilde{\chi}(\varepsilon) - 1}{A}$$

we see that $\widetilde{\chi}$ satisfies the first order differential equation

$$\widetilde{\chi}' = \lambda\widetilde{\chi}$$

with initial condition $\widetilde{\chi}(0) = 1$. Thus $\widetilde{\chi}(\theta) = e^{\lambda\theta}$ and as $|\widetilde{\chi}(t)| = 1$ for each θ then we must have that $\lambda \in i\mathbb{R}$, say $\lambda = im$. Thus now

$$\widetilde{\chi}(\theta) = e^{im\theta} \quad \text{and} \quad \chi(z) = z^m.$$

Finally as $1 = \chi(e^{2\pi i}) = e^{2\pi i m}$ we must have that $m \in \mathbb{Z}$. Hence $\chi = \chi_m$ as desired. $\square$

3.2 Classical Fourier Series: L^2-theory

Let $f : U(1) \to \mathbb{C}$ be an integrable function. Using Equation 2.3 for $\widehat{f}$ gives

$$\widehat{f}(m) := \widehat{f}(\chi_m) = \int\limits_{U(1)} f(z)\chi_m(z^{-1})d\mu(z) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \widetilde{f}(\theta)e^{-im\theta}d\theta.$$

Here notice that the numbers $\widehat{f}(m)$ are the classical Fourier coefficients for the 2π-periodic function $\widetilde{f} : \mathbb{R} \to \mathbb{C}$. Now if we apply the Plancherel Theorem 2.11 we see that for $f \in L^2(U(1))$ that

$$f(z) = \sum_{m=-\infty}^{\infty} \widehat{f}(m)\chi_m(z)$$

converges in the L^2-sense. Equivalently

$$\widetilde{f}(\theta) = \sum_{m=-\infty}^{\infty} \widehat{f}(m)e^{im\theta} \tag{3.1}$$

with convergence in $L^2([-\pi, \pi])$. This is the formula for the classical Plancherel Theorem. Theorem 2.11 part (2) implies

$$\|\widetilde{f}\|_2^2 = \int\limits_{-\pi}^{\pi} |\widetilde{f}(\theta)|^2 \, d\theta = 2\pi \sum_{m=-\infty}^{\infty} |\widehat{f}(m)|^2 \tag{3.2}$$

which is sometimes referred to as *Parseval's Equation* [3, Section 3.3]. This shows that Theorem 2.11 can be thought of as a generalization of the classical L^2-theory of Fourier series.

3.3 Classical Fourier Series: Point-wise Convergence

Theorem 3.3. *If* $f \in C^1(U(1))$ *then*

$$\sum_{m=-\infty}^{\infty} \widehat{f}(m)e^{im\theta}$$

converges absolutely and uniformly to $\widetilde{f}(\theta)$.

Proof. Let $f \in C^1(U(1))$. Thus $\widetilde{f}$ is a 2π-periodic C^1-function $\mathbb{R} \to \mathbb{C}$. We compute

$$\widehat{f'}(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widetilde{f}'(z)e^{-im\theta}\,d\theta$$

via integration by parts using $u = e^{-im\theta}$ and $dv = \widetilde{f}'(\theta)\,d\theta$. Now $du = -ime^{-im\theta}\,d\theta$ and $v = \widetilde{f}(\theta)$ and

$$\widehat{f'}(m) = \frac{1}{2\pi}\left[e^{-im\theta}\widetilde{f}(\theta)\right]_{-\pi}^{\pi} + \frac{im}{2\pi}\int_{-\pi}^{\pi}\widetilde{f}(\theta)e^{-im\theta}\,d\theta.$$

Observe that $e^{-im\theta}\widetilde{f}(\theta)$ is the multiplication of two 2π-periodic functions and therefore we must have that

$$\left[e^{-imt}\widetilde{f}(\theta)\right]_{-\pi}^{\pi} = 0.$$

Thus

$$\widehat{f'}(m) = \frac{im}{2\pi}\int_{-\pi}^{\pi}\widetilde{f}(\theta)e^{-imt}\,dt = im\widehat{f}(m).$$

Now by using Proposition 2.13 we only need to see that

$$\sum_{m=-\infty}^{\infty} |\widehat{f}(m)| < \infty$$

and the rest follows. We calculate

$$\sum_{m=-\infty, m\neq 0}^{\infty} |\widehat{f}(m)| = \sum_{m=-\infty, m\neq 0}^{\infty} \frac{1}{|m|} |\widehat{f'}(m)|.$$

Let ℓ^2 be the space of square summable sequences indexed by $\mathbb{Z} - \{0\}$. Notice that

$$\left(\frac{1}{|m|}\right)_{m=-\infty, m\neq 0}^{\infty} \in \ell^2$$

and

$$\left(|\widehat{f'}(m)|\right)_{m=-\infty, m\neq 0}^{\infty} \in \ell^2$$

since

$$\sum_{m=-\infty}^{\infty} |\widehat{f'}(m)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\widetilde{f'}(x)|^2 dx < \infty$$

by (3.2) for the function $f' \in C(U(1)) \subset L^2(U(1))$. Now by using the Cauchy-Schwarz inequality we have

$$\sum_{m=-\infty, m\neq 0}^{\infty} \frac{1}{|m|} |\widehat{f'}(m)| \leq \left(\sum_{m=-\infty, m\neq 0}^{\infty} \frac{1}{m^2}\right)^{\frac{1}{2}} \left(\sum_{m=-\infty, m\neq 0}^{\infty} |\widehat{f'}(m)|^2\right)^{\frac{1}{2}} < \infty. \quad \square$$

3.4 Abel Summability on $U(1)$

Theorem 3.3 states that the series

$$\sum_{m=-\infty}^{\infty} \widehat{f}(m) \chi_m(z)$$

converges absolutely and uniformly to $f(z)$ when $f \in C^1(U(1))$. For $f \in C(U(1))$ we will look at the series

$$\sum_{m=-\infty}^{\infty} r^{|m|} \widehat{f}(m) \chi_m(z)$$

where $0 < r < 1$.

Lemma 3.4. *For fixed $0 < r < 1$ the series*

$$\sum_{m=-\infty}^{\infty} r^{|m|} \widehat{f}(m) \chi_m(z)$$

converges absolutely and uniformly on $U(1)$.

Proof. Observe

$$|r^{|m|} \widehat{f}(m) \chi_m(z)| = |\widehat{f}(m)| r^{|m|}.$$

Notice $\sum_{m=-\infty}^{\infty} |\widehat{f}(m)| r^{|m|} < \infty$ since $0 < r < 1$ and $\widehat{f}(m)$ is bounded. Indeed as $C(U(1)) \subset L^2(U(1))$ Parseval's Equation (3.2) ensures that $\sum_{m=-\infty}^{\infty} |\widehat{f}(m)|^2$ converges. □

Therefore formally we have that

$$\lim_{r \to 1^-} \sum_{m=-\infty}^{\infty} r^{|m|} \widehat{f}(m) \chi_m(z) = f(z)$$

which is indeed the case according to the following theorem.

Theorem 3.5. *([8, Chapter 4 Theorem 6-3].) For $f \in C(U(1))$,*

$$\lim_{r \to 1^-} \sum_{m=-\infty}^{\infty} r^{|m|} \widehat{f}(m) e^{im\theta} = \widetilde{f}(\theta)$$

for all $\theta \in \mathbb{R}$. Moreover convergence is uniform in θ as $r \to 1^-$.

Definition 3.6. The *Poisson kernel* $P_r(\theta)$ is the 2π-periodic function defined as

$$P_r(\theta) = \sum_{m=-\infty}^{\infty} r^{|m|} e^{im\theta}$$

for $\theta \in \mathbb{R}$ and $0 < r < 1$.

Observe that the above series converges uniformly in θ for fixed $0 < r < 1$. Using the Poisson kernel we can rewrite the series

$$\sum_{m=-\infty}^{\infty} r^{|m|} \widehat{f}(m) \chi_m(z)$$

as a *convolution integral.* We write

$$(g \star f)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta - \phi) h(\phi)\, ds$$

for 2π-periodic functions g, h integrable on $[-\pi, \pi]$. Thus for given $f \in C^1(U(1))$ we have

$$\begin{aligned}
(P_r \star \widetilde{f})(\theta) &= \sum_{m=-\infty}^{\infty} \frac{r^{|m|}}{2\pi} \int_{-\pi}^{\pi} e^{im(\theta-\phi)} \widetilde{f}(\phi)\, d\phi \\
&= \sum_{m=-\infty}^{\infty} r^{|m|} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \widetilde{f}(\phi) e^{-im\phi}\, d\phi \right) e^{im\theta} \\
&= \sum_{m=-\infty}^{\infty} r^{|m|} \widehat{f}(m) e^{im\theta}.
\end{aligned}$$

We can now restate Theorem 3.5.

Theorem 3.7. *For $f \in C(U(1))$,*

$$\lim_{r\to 1^-} (P_r \star \widetilde{f})(\theta) = \widetilde{f}(\theta)$$

converges uniformly in θ as $r \to 1^-$.

We conclude this chapter with an explicit formula for the Poisson kernel.

Proposition 3.8.

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos(\theta)+r^2} = 1 + 2\sum_{n=1}^{\infty} r^n \cos(n\theta).$$

Proof. We have

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \sum_{n=0}^{\infty} r^n e^{in\theta} + \sum_{n=1}^{\infty} r^n e^{-in\theta}.$$

These are geometric series that converge for $r < 1$, therefore we observe

$$P_r(\theta) = \frac{1}{1-re^{i\theta}} + \frac{re^{-i\theta}}{1-re^{-i\theta}} = \frac{1-r^2}{(1-re^{i\theta})(1-re^{-i\theta})} = \frac{1-r^2}{1-2r\cos(\theta)+r^2}.$$

If we now go back to the two series and combine them we obtain

$$P_r(\theta) = \sum_{n=0}^{\infty} r^n e^{in\theta} + \sum_{n=1}^{\infty} r^n e^{-in\theta} = 1 + \sum_{n=1}^{\infty} r^n (e^{in\theta} + e^{-in\theta}) = 1 + 2\sum_{n=1}^{\infty} r^n \cos(n\theta). \quad \square$$

CHAPTER 4: Analysis on $SU(2)$

Recall that $SU(n) = \{g \in GL(n, \mathbb{C}) : gg^* = I, det(g) = 1\}$. Thus it is easy to see that $SU(2)$ consists of the following matrices,

$$g = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1.$$

Here we see that $SU(2)$ is homeomorphic to the unit sphere in $\mathbb{C}^2 = \mathbb{R}^4$, therefore it is compact and connected. From Section 1.2 we know that the Lie algebra of $SU(2)$ is

$$\begin{aligned} su(2) &= \{X \in M(2, \mathbb{C}) \ : \ X^* = -X, \ tr(X) = 0\} \\ &= \left\{ \begin{bmatrix} ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & -ix_1 \end{bmatrix} \ : \ x_1, x_2, x_3 \in \mathbb{R} \right\}. \end{aligned}$$

4.1 Haar Measure on $SU(2)$

As $SU(2)$ is a compact Lie group we know that it supports a unique normalized (left) Haar measure μ which is also right invariant. In this section we will give an explicit formula for this Haar measure. Let

$$\begin{aligned} \mathbb{H} &= \left\{ \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \ : \ \alpha, \beta \in \mathbb{C} \right\} \\ &= \left\{ M_x = \begin{bmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{bmatrix} \ : \ x = (x_1, x_2, x_3, x_4) \in \mathbb{R} \right\}, \end{aligned}$$

then it is clear that $\mathbb{H} \simeq \mathbb{R}^4$ via the isomorphism of real vector spaces

$$M : \mathbb{R}^4 \to \mathbb{H}, \quad M(x) = M_x. \tag{4.1}$$

Observe that for $(q = M_x) \in \mathbb{H}$ we have $\det(q) = x_1^2 + x_2^2 + x_3^2 + x_4^2 = \|x\|^2$, the square of the Euclidean norm of $x \in \mathbb{R}^4$. Also $\det(q) = 1$ if and only if $q \in SU(2)$, thus $SU(2) \subset \mathbb{H}$ corresponds to the unit sphere $S^3 \subset \mathbb{R}^4$ via the mapping M. We will obtain a Haar measure on $SU(2)$ via a suitable measure on S^3 using this identification.

Since S^3 is a sub-manifold of $\mathbb{R}^4$, using [6] we can do calculus with differential forms on manifolds. First we define the tangent space to S^3 at $x \in S^3$ to be

$$S_x = \{v_x \ : \ v \in \mathbb{R}^4, \ v \cdot x = 0\}.$$

The standard volume element ω on S^3 is the 3-form defined via

$$\omega_x(u_x, v_x, w_x) = \det(x, u, v, w)$$

for $x \in S^3$ and $u_x, v_x, w_x \in S_x$. A basis $\{u_x, v_x, w_x\}$ for S_x^3 is said to be *positively oriented* if $\omega_x(u_x, v_x, w_x) = \det(x, u, v, w) > 0$. A diffeomorphism $f : S^3 \to S^3$ is *orientation preserving* if $\{(Df)_x(u_x), (Df)_x(v_x), (Df)_x(w_x)\}$ is a positively oriented basis for $S^3_{f(x)}$ whenever $\{u_x, v_x, w_x\}$ is positively oriented.

We can think of the group $SO(4)$ as being the set of orthogonal linear transformations $L : \mathbb{R}^4 \to \mathbb{R}^4$ with $\det(L) = 1$, that is,

$$SO(4) = \{L \in GL(\mathbb{R}^4) \ : \ (Lu) \cdot (Lv) = u \cdot v, \text{for all } u, v \in \mathbb{R}^4 \text{ and} \det(L) = 1\}.$$

In particular if $x \in S^3$ then $\|Lx\| = \|x\| = 1$, so $Lx \in S^3$. That is, each $L \in SO(4)$ preserves S^3.

Proposition 4.1. *Each $L \in SO(4)$ is an orientation preserving diffeomorphism of the sphere $S^3 \subset \mathbb{R}^4$.*

Proof. Since $L : \mathbb{R}^4 \to \mathbb{R}^4$ is in $SO(4)$, it is linear and its derivative $(DL)_x : \mathbb{R}^4_x \to \mathbb{R}^4_{Lx}$ at a point $x \in S^3$ is $(DL)_x(u_x) = (Lu)_{Lx}$. Now notice that

$$(Lu) \cdot (Lx) = u \cdot x = 0$$

since $u_x \in S_x$. Thus the derivative $(DL|_{S^3})_x : S_x \to S_{Lx}$ of $L|_{S^3} : S^3 \to S^3$ is also given by

$$(DL|_{S^3})_x(u_x) = (Lu)_{Lx}.$$

Now suppose that $\{u_x, v_x, w_x\}$ is a positively oriented basis for S_x. Then the corresponding basis for S_{Lx}

$$\{(DL|_{S^3})_x(u_x), (DL|_{S^3})_x(v_x), (DL|_{S^3})_x(w_x)\} = \{(Lu)_{Lx}, (Lv)_{Lx}, (Lw)_{Lx}\},$$

has

$$\det(Lx, Lu, Lv, Lw) = \det(L)\det(x, u, v, w) = det(x, u, v, w) > 0. \quad \square$$

Now that we have shown each $L \in SO(4)$ is orientation preserving we get that

$$\int_{S^3} L^*(\eta) = \int_{S^3} \eta \tag{4.2}$$

for any 3-form on S^3. [6, Page 123].

Lemma 4.2. *ω is $SO(4)$-invariant. That is $L^*(\omega) = \omega$ for each $L \in SO(4)$.*

Proof. Let $L \in SO(4)$, $x \in S^3$ and $u_x, v_x, w_x \in S_x$. Then

$$(L^*\omega)_x(u_x, v_x, w_x) = \omega_{Lx}((DL)_x(u_x), (DL)_x(v_x), (DL)_x(w_x))$$

$$= \omega_{Lx}((Lu)_{Lx}, (Lv)_{Lx}, (Lw)_{Lx})$$
$$= det(Lx, Lu, Lv, Lw)$$
$$= det(L)\omega_x(u_x, v_x, w_x)$$
$$= \omega_x(u_x, v_x, w_x). \qquad \square$$

Let

$$v_3 := \int\limits_{S^3} \omega$$

be the volume of S^3. Now we can define a measure μ on S^3 via

$$\int\limits_{S^3} f(x)\, d\mu(x) = \frac{1}{v_3} \int\limits_{S^3} f\omega, \quad \text{for } f \in C(S^3).$$

It is clear to see that μ is normalized, that is $\int\limits_{S^3} d\mu(x) = 1$.

Lemma 4.3.

$$\int\limits_{S^3} f(Lx)\, d\mu(x) = \int\limits_{S^3} f(x)\, d\mu(x) \textit{ for } L \in SO(4).$$

Proof. By the previous lemma, as $L^*(\omega) = \omega$ we obtain

$$\int\limits_{S^3} f(Lx)\, d\mu(x) = \frac{1}{v_3} \int\limits_{S^3} (f \circ L)\omega = \frac{1}{v_3} \int\limits_{S^3} (f \circ L|_{S^3})(L|_{S^3})^*(\omega)$$
$$= \frac{1}{v_3} \int\limits_{S^3} (L|_{S^3})^*(f\omega).$$

Now by Equation 4.2 we get

$$\frac{1}{v_3} \int\limits_{S^3} (L|_{S^3})^*(f\omega) = \frac{1}{v_3} \int\limits_{S^3} f\omega = \int\limits_{S^3} f(x)\, d\mu(x). \quad \square$$

Identifying $SU(2)$ with S^3 via the mapping $M : \mathbb{R}^4 \to \mathbb{H}$ we can regard the measure μ as living on $SU(2)$, explicitly

$$\int_{SU(2)} f(g)\, d\mu(g) := \int_{S^3} f(M_x)\, d\mu(x), \qquad \text{for } f \in C(SU(2)). \tag{4.3}$$

Lemma 4.4. *μ is a Haar measure for $SU(2)$.*

Proof. We already have that μ is normalized and we simply need to check that it is left invariant on $SU(2)$. Let $k \in SU(2)$ and define a map

$$L_k : SU(2) \to SU(2), \quad L_k(g) = kg.$$

Notice that this map corresponds to the restriction of a transformation, say $\widehat{L}_k \in SO(4)$, to S^3. Indeed, let

$$T_k : \mathbb{H} \to \mathbb{H}, \quad T_k(q) = kq,$$

then L_k is the restriction to $SU(2)$ of T_k. Now, under the isomorphism from Equation 4.1 we have that T_k corresponds to the linear map

$$\widehat{L}_k : \mathbb{R}^4 \to \mathbb{R}^4, \quad \widehat{L}_k = M^{-1} \circ T_k \circ M.$$

We just need to verify that $\widehat{L}_k$ belongs to $SO(4)$.

Let $x, y \in \mathbb{R}^4$ and suppose $\widehat{L}_k(x) = y$. Then we have

$$\begin{aligned} kM_x = M_y &\implies \det(kM_x) = \det(M_y) \\ &\implies \det(k)\det(M_x) = \det(M_y) \end{aligned}$$

$$\implies \det(M_x) = \det(M_y)$$
$$\implies x_1^2 + x_2^2 + x_3^2 + x_4^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2.$$

Thus $\widehat{L}_k$ is an orthogonal transformation and just need to check that the determinant is one. Since $k \to \det(\widehat{L}_k) = \pm 1$ is continuous on $SU(2)$ and $\det(I = \widehat{L}_I) = 1$ then by the connectivity of $SU(2)$ we must have $\det(\widehat{L}_k) = 1$ for each $k \in SU(2)$. □

One can replace the map L_k in the preceding proof by a right multiplication,

$$R_k : SU(2) \to SU(2), \quad R_k(g) = gk.$$

Just as above this corresponds to the restriction to S^3 of a transformation in $SO(4)$. So the proof argument also establishes the right invariance property of the Haar measure mentioned earlier.

Lemma 4.5. *Every matrix $x \in SU(2)$ decomposes as*

$$x = \begin{bmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{bmatrix} \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}$$

where $0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \pi, \quad -\pi \leq \psi \leq \pi.$

Proof. Let $x \in SU(2)$,

$$x = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}$$

say where $\alpha, \beta, \in \mathbb{C}$. Let $\alpha = re^{ia}$ and $\beta = se^{ib}$ where $r, s, a, b \in \mathbb{R}$, $r, s \geq 0$. Since $x \in SU(2)$ then $r^2 + s^2 = 1$. There is a unique $\theta \in [0, \frac{\pi}{2}]$ with $r = \cos\theta$ and $s = \sin\theta$. Also letting $\psi = (a+b)/2$ and $\phi = (a-b)/2$ we can write $a = \psi + \phi$ and $b = \psi - \phi$.

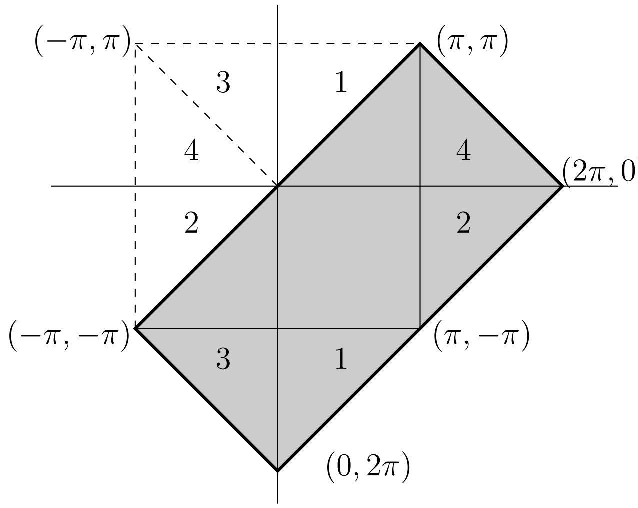

Figure 4.1: (a, b)

So we have

$$x = \begin{bmatrix} re^{i(\psi+\phi)} & se^{i(\psi-\phi)} \\ -se^{-i(\psi-\phi)} & re^{-i(\psi+\phi)} \end{bmatrix} = \begin{bmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{bmatrix} \begin{bmatrix} re^{i\phi} & se^{-i\phi} \\ -se^{i\phi} & re^{-i\phi} \end{bmatrix}$$
$$= \begin{bmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{bmatrix} \begin{bmatrix} r & s \\ -s & r \end{bmatrix} \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}$$
$$= \begin{bmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{bmatrix} \begin{bmatrix} cos(\theta) & sin\theta \\ -sin(\theta) & cos\theta \end{bmatrix} \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}$$

as desired. To complete the proof it remains to shows that we obtain all of $SU(2)$ when angles ϕ, ψ range over the intervals $[0, \pi]$ and $[-\pi, \pi]$ respectively. In Figure 4.1 the shaded rectangle is the image of $[-\pi, \pi] \times [0, \pi]$ under the mapping

$$(\psi, \phi) \mapsto (\psi + \phi, \psi - \phi) = (a, b).$$

As indicated in the figure, for any point $(a, b) \in \mathbb{R}^2$ one has that $(a + 2k\pi, b + 2l\pi)$ lies in the image of $[-\pi, \pi] \times [0, \pi]$ for some integers k, l.

□

The numbers θ, ϕ, ψ are called the *Euler angles* of the matrix x. Using this decomposition we can find an integration formula as follows.

Proposition 4.6. *Let μ be the normalized Haar measure of $SU(2)$ and f an integrable function on $SU(2)$. Then*

$$\int_{SU(2)} f(x)\, d\mu(x) = \frac{1}{2\pi^2} \int_0^{\frac{\pi}{2}} sin2\theta\, d\theta \int_0^{\pi} d\phi \int_{-\pi}^{\pi} f \circ \Phi(\theta, \phi, \psi)\, d\psi$$

where $\Phi : [0, \pi/2] \times [0, \pi] \times [-\pi, \pi] \to SU(2)$ is the map

$$\Phi(\theta, \phi, \psi) = \begin{bmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{bmatrix} \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}.$$

Proof. Let $x \in SU(2)$ where

$$x = \begin{bmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{bmatrix}$$

then using the decomposition above we can write

$$\begin{aligned} x_1 &= cos\,\theta\ cos\,s \\ x_2 &= cos\,\theta\ sin\,s \\ x_3 &= sin\,\theta\ cos\,t \\ x_4 &= sin\,\theta\ sin\,t \end{aligned}$$

where $s = \psi + \phi$ and $t = \psi - \phi$. Now by differentiating

$$\begin{aligned} dx_1 &= -cos\,\theta\ sin\,s\, ds - sin\,\theta\ cos\,s\, d\theta \\ dx_2 &= cos\,\theta\ cos\,s\, ds - sin\,\theta\ sin\,s\, d\theta \\ dx_3 &= -sin\,\theta\ sin\,t\, dt + cos\,\theta\ cos\,t\, d\theta \\ dx_4 &= sin\,\theta\ cos\,t\, dt + cos\,\theta\ sin\,t\, d\theta \end{aligned},$$

also $ds = d\psi + d\phi$ and $dt = d\psi - d\phi$ so we have

$$\begin{array}{l} dx_1 = -sin\,\theta\ cos\,s\,d\theta - cos\,\theta\ sin\,s\,(d\psi + d\phi) \\ dx_2 = -sin\,\theta\ sin\,s\,d\theta + cos\,\theta\ cos\,s\,(d\psi + d\phi) \\ dx_3 = cos\,\theta\ cos\,t\,d\theta - sin\,\theta\ sin\,t\,(d\psi - d\phi) \\ dx_4 = cos\,\theta\ sin\,t\,d\theta + sin\,\theta\ cos\,t\,(d\psi - d\phi) \end{array},$$

and we calculate

$$\det \begin{bmatrix} x_1 & -sin\theta cos(\phi + \psi) & -cos\theta sin(\psi + \phi) & -cos\theta sin(\psi + \phi) \\ x_2 & -sin\theta sin(\psi + \phi) & cos\theta cos(\psi + \phi) & cos\theta cos(\psi + \phi) \\ x_3 & cos\theta cos(\psi - \phi) & -sin\theta sin(\psi - \phi) & sin\theta sin(\psi - \phi) \\ x_4 & cos\theta sin(\psi - \phi) & sin\theta cos(\psi - \phi) & -sin\theta cos(\psi - \phi) \end{bmatrix} = -2cos\theta sin\theta.$$

Thus we have that $|\Phi^*\omega| = 2cos\theta sin\theta\ d\theta \wedge d\phi \wedge d\psi = sin2\theta\, d\theta \wedge d\phi \wedge d\psi$. It remains to find v_3 where

$$v_3 = \int_{S^3} \omega$$

. We calculate

$$\begin{aligned} \int_{S^3} \omega = \int_0^{\frac{\pi}{2}} sin2\theta d\theta \int_0^{\pi} d\phi \int_{-\pi}^{\pi} d\psi d\phi d\theta &= 2\pi^2 \int_0^{\frac{\pi}{2}} sin(2\theta)\, d\theta \\ &= 2\pi^2[-\frac{1}{2}cos(2\theta)]_0^{\frac{\pi}{2}} \\ &= 2\pi^2 \end{aligned}$$

as desired. □

4.2 Central Functions on $SU(2)$

Recall that a function f on a group G is central if and only if for all $g, x \in G$, $f(gxg^{-1}) = f(x)$ from Definition 2.14

Lemma 4.7. *Let $x \in SU(2)$ then there exists $k \in SU(2)$ such that*

$$kxk^{-1} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

where $0 \leq \theta \leq \pi$.

Proof. Let $x \in SU(2)$ be given. By the spectral theorem for normal operators [5] there exists $g \in U(2)$ such that

$$g^{-1}xg = \begin{bmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{bmatrix}$$

for some $\theta_1, \theta_2 \in \mathbb{R}$. Suppose $det(g) = e^{ia}$ where $a \in \mathbb{R}$. Now define $k = e^{-i\frac{a}{2}}g$. Then

$$det(k) = det(e^{-i\frac{a}{2}}g) = (e^{-i\frac{a}{2}})^2 det(g) = e^{-ia}e^{ia} = 1$$

and thus $k \in SU(2)$. Also $k^{-1} = (e^{-i\frac{a}{2}}g)^{-1} = e^{i\frac{a}{2}}g^{-1}$ then we see

$$k^{-1}k = e^{i\frac{a}{2}}e^{-i\frac{a}{2}}gg^{-1} = I.$$

Now

$$kxk^{-1} = e^{-i\frac{a}{2}}e^{i\frac{a}{2}}gxg^{-1} = \begin{bmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{bmatrix}.$$

Now since $k, x \in SU(2)$ we get $kxk^{-1} \in SU(2)$ and hence $e^{i(\theta_1+\theta_2)} = \det(kxk^{-1}) = 1$. We conclude that for $x \in SU(2)$ there exists $k \in SU(2)$ and $\theta \in (-\pi, \pi]$ with

$$kxk^{-1} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}.$$

Also note that for

$$k = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \in SU(2)$$

we have

$$k \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} k^{-1} = \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}.$$

Thus we can ensure that $\theta \in [0, \pi]$. □

Lemma 4.8. *If f and g are two central functions on the group $SU(2)$ then $f = g$ if and only if*

$$f\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\right) = g\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\right), \quad \textit{for all} \quad \theta \in [0, \pi].$$

Proof. Suppose

$$f\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\right) = g\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\right) \quad \text{for all} \quad \theta \in [0, \pi]$$

and let $x \in SU(2)$ be given. Then by the previous Lemma there exists $k \in SU(2)$ such that

$$kxk^{-1} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \quad \text{with} \quad 0 \le \theta \le \pi.$$

Then we get

$$g(x) = g(kxk^{-1}) = g\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\right) = f\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\right) = f(kxk^{-1}) = f(x). \quad \square$$

This Lemma allows us to prove the following important Proposition.

Proposition 4.9. *Given $F : [-1, 1] \to \mathbb{C}$ then $f : SU(2) \to \mathbb{C}$ defined as*

$$f(x) = F\left(\frac{1}{2}tr(x)\right)$$

is a central function on $SU(2)$. Moreover, every central function $f : SU(2) \to \mathbb{C}$ is obtained in this way for some F.

Proof. Let $F : [-1,1] \to \mathbb{C}$ be given then define $f(x) = F(\frac{1}{2}tr(x))$ for $x \in SU(2)$. Let $k \in SU(2)$ then

$$f(kxk^{-1}) = F\left(\frac{1}{2}tr(kxk^{-1}\right) = F\left(\frac{1}{2}tr(x)\right) = f(x)$$

and thus f is central in $SU(2)$.

Now, let $f : SU(2) \to \mathbb{C}$ be a central function and define $F : [-1,1] \to \mathbb{C}$ via

$$F(t) = f\left(\begin{bmatrix} e^{icos^{-1}t} & 0 \\ 0 & e^{-icos^{-1}t} \end{bmatrix}\right).$$

Then claim that $f(x) = F(\frac{1}{2}tr(x))$ for all $x \in SU(2)$. Indeed for any $\theta \in [0,\pi]$ we have that for $x = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$,

$$F\left(\frac{1}{2}tr(x)\right) = F(\cos\theta) = f\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\right) = f(x).$$

As both f and $x \mapsto F(tr(x)/2)$ are central the previous lemma implies that $f(x) = F(tr(x)/2)$ for every $x \in SU(2)$. $\square$

4.3 Irreducible Representations of $SU(2)$

First we will study the irreducible representations of $sl(2,\mathbb{C})$, which will help us study su(2). Recall from Chapter 1 that $sl(2,\mathbb{C}) = \{M(2,\mathbb{C}) \ : \ tr(X) = 0\}$. The following proposition shows that $sl(2,\mathbb{C})$ is the complexification of $su(2)$.

Proposition 4.10. *If $Z \in sl(2,\mathbb{C})$ then there exists unique $X, Y \in su(2)$ such that*

$$Z = X + iY.$$

Proof. Let $Z \in sl(2,\mathbb{C})$ and set

$$X = \frac{1}{2}(Z - Z^*), \quad Y = -\frac{i}{2}(Z + Z^*).$$

Then we will see that $X, Y \in su(2)$. Indeed,

$$X^* = \frac{1}{2}(Z - Z^*)^* = \frac{1}{2}(Z^* - Z) = -\frac{1}{2}(Z - Z^*) = -X$$

and also

$$tr(X) = \frac{1}{2}(tr(Z) - tr(Z^*)) = \frac{1}{2}(0 - 0) = 0$$

therefore $X \in su(2)$. Likewise

$$Y^* = (-\frac{i}{2}(Z + Z^*))^* = \frac{i}{2}(Z^* + Z) = \frac{i}{2}(Z + Z^*) = -Y$$

and

$$tr(Y) = -\frac{i}{2}(tr(Z) + tr(Z^*)) = -\frac{i}{2}(0 + 0) = 0$$

and thus $Y \in su(2)$. Next see that

$$\begin{aligned} X + iY = \frac{1}{2}(Z - Z^*) + i(-\frac{i}{2}(Z + Z^*)) &= \frac{1}{2}Z - \frac{1}{2}Z^* + \frac{1}{2}Z + \frac{1}{2}Z^* \\ &= Z. \end{aligned}$$

It remains to show that this decomposition is unique. Suppose $X, X', Y, Y' \in su(2)$

and $X + iY = X' + iY' = Z$. Then we have $X - X' = i(Y' - Y)$ by rearranging. Here see $X - X' \in su(2)$ hence skew-hermitian but $Y' - Y \in su(2)$ so $i(Y' - Y)$ is hermitian. The only matrix which is both skew-hermitian and hermitian is the zero matrix. Thus $X = X'$ and $Y = Y'$ completing the proof of uniqueness. □

Definition 4.11. Let $\mathcal{P}_m$ be the the space of two variable polynomials with complex coefficients that are homogeneous of degree m, namely

$$\mathcal{P}_m := \operatorname{Span}\big(\{f_j(u,v) = u^j v^{m-j} \ : \ 0 \leq j \leq m\}\big).$$

As $\{f_j(u,v) = u^j v^{m-j} \ : \ 0 \leq j \leq m\}$ is a basis for $\mathcal{P}_m$ we have $\dim(\mathcal{P}_m) = m+1$. Now define a map

$$\pi_m : SL(2,\mathbb{C}) \to GL(\mathcal{P}_m)$$

via the rule

$$(\pi_m(g)f)(u,v) = f(au + cv, bu + dv)$$

where

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Another way to view this is $(\pi_m(g)f)(u,v) = f((u,v)g)$.

Proposition 4.12. *$\pi_m : SL(2,\mathbb{C}) \to GL(\mathcal{P}_m)$ is a continuous group homomorphism, i.e. π_m is a representation of the group $SL(2,\mathbb{C})$ on the vector space $\mathcal{P}_m$.*

Proof. First note that one has

$$\pi_m(g)(f_1 + f_2) = \pi_m(g)f_1 + \pi_m(g)f_2, \quad \pi_m(g)(cf) = c\pi_m(g)f$$

for $g \in SL(2,\mathbb{C})$, $f, f_1, f_2 \in \mathcal{P}_m$ and $c \in \mathbb{C}$. Thus indeed $\pi_m(g) \in GL(\mathcal{P}_m)$. We have

$$\begin{aligned}\pi_m(g_1g_2)f(u,v) = f((u,v)g_1g_2) &= f(((u,v)g_1)g_2)\\ &= (\pi_m(g_2)f)((u,v)g_1)\\ &= (\pi_m(g_1)(\pi_m(g_2)f))(u,v)\\ &= ((\pi_m(g_1)\circ\pi_m(g_2))f)(u,v).\end{aligned}$$

Thus π_m is a group homomorphism. □

To study the derived representation $d\pi_m : sl(2,\mathbb{C}) \to gl(\mathcal{P}_m)$ we need the following basis for $sl(2,\mathbb{C})$,

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We will also need to know the commutator relations of these basis matrices. It is easy to calculate,

$$[H,E] = 2E,\ [H,F] = -2F,\ [E,F] = H.$$

Recall that for $X \in sl(2,\mathbb{C})$ one has

$$d\pi_m(X) = \frac{d}{dt}\pi_m(\exp(tX))\Big|_{t=0}.$$

Understanding the derived representation will require knowing $\exp(tH), \exp(tE), \exp(tF)$.

$$\begin{aligned}\exp(tH) = I + tH + \frac{t^2}{2!}H^2 + \frac{t^3}{3!}H^3 + ... &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix} + \begin{bmatrix} \frac{t^2}{2} & 0 \\ 0 & \frac{t^2}{2} \end{bmatrix} + ...\\ &= \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}\end{aligned}$$

$$\exp(tE) = I + tE + \frac{t^2}{2!}E^2 + \frac{t^3}{3!}E^3 + ... = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + ...$$
$$= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$\exp(tF) = I + tF + \frac{t^2}{2!}F^2 + \frac{t^3}{3!}F^3 + ... = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + ...$$
$$= \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}.$$

The derived representations are now obtained as follows

$$\begin{aligned} d\pi_m(H)f(u,v) = \left.\frac{d}{dt}\right|_0 \pi_m(\exp(tH))f(u,v) &= \left.\frac{d}{dt}\right|_0 f\left((u,v)\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}\right) \\ &= \left.\frac{d}{dt}\right|_0 f(e^t u, e^{-t} v) \\ &= \left(e^t u\frac{\partial f}{\partial u} - e^{-t} v\frac{\partial f}{\partial v}\right)\Big|_0 \\ &= u\frac{\partial f}{\partial u} - v\frac{\partial f}{\partial v} \end{aligned}$$

$$\begin{aligned} d\pi_m(E)f(u,v) = \left.\frac{d}{dt}\right|_0 \pi_m(\exp(tE))f(u,v) &= \left.\frac{d}{dt}\right|_0 f\left((u,v)\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right) \\ &= \left.\frac{d}{dt}\right|_0 f(u, ut+v) \\ &= u\frac{\partial f}{\partial v} \end{aligned}$$

$$\begin{aligned} d\pi_m(F)f(u,v) = \left.\frac{d}{dt}\right|_0 \pi_m(\exp(tF)f(u,v) &= \left.\frac{d}{dt}\right|_0 f\left((u,v)\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}\right) \\ &= \left.\frac{d}{dt}\right|_0 f(u+tv, v) \\ &= v\frac{\partial f}{\partial u}. \end{aligned}$$

Applying these operators to the basis $\mathcal{B} = \{f_j = u^j v^{m-j} \ : \ 0 \leq j \leq m\}$ for $\mathcal{P}_m$ gives

$$\begin{aligned} d\pi_m(H)f_j &= (u\tfrac{\partial f_j}{\partial u} - v\tfrac{\partial f_j}{\partial v}) = ju^jv^{m-j} - (m-j)u^jv^{m-j} = (2j-m)f_j, \\ d\pi_m(E)f_j &= u\tfrac{\partial f_j}{\partial v} = (m-j)u^{j+1}v^{m-(j+1)} = (m-j)f_{j+1}, \\ d\pi_m(F)f_j &= v\tfrac{\partial f_j}{\partial u} = ju^{j-1}v^{m-j+1} = jf_{j-1}. \end{aligned} \tag{4.4}$$

It follows that the matrices for $d\pi_m(H), d\pi_m(E)$, and $d\pi_m(F)$ with respect to $\mathcal{B}$ are

$$[d\pi_m(H)]_{\mathcal{B}} = \begin{bmatrix} -m & 0 & \cdots & 0 & 0 \\ 0 & -m+2 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & m-2 & 0 \\ 0 & 0 & \cdots & 0 & m \end{bmatrix},$$
$$[d\pi_m(E)]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ m & 0 & \cdots & 0 & 0 \\ 0 & m-1 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$
$$[d\pi_m(F)]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \ddots & m \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Proposition 4.13. *The representation $d\pi_m$ is irreducible for each m.*

Proof. Let $W \subseteq \mathcal{P}_m$ be a non-zero invariant subspace. We will show that $W = \mathcal{P}_m$. Indeed, since W is a complex vector space its characteristic polynomial for $d\pi_m(H) : W \to W$ must have at least one root. So $d\pi_m(H) : W \to W$ has an eigenvector. As $d\pi_m(H)$ has one dimensional eigenspaces, spanned by the f_j's, we conclude that W contains f_j for some $j = 1, ..., m$. Now by acting on f_j with $d\pi_m(E)$ and $d\pi_m(F)$ it is clear we can conclude $f_j \in W$ for all $j = 1, ..., m$. Now since $\mathcal{B} = \{f_j\} \subset W$ then $W = \mathcal{P}_m$. □

Theorem 4.14. *Every irreducible finite dimensional $\mathbb{C}$-linear representation of $sl(2, \mathbb{C})$ is equivalent to one of the representations $d\pi_m$.*

Proof. We follow the proof given in [2, Section 7.5]. Let ρ be an irreducible representation of $sl(2,\mathbb{C})$ on a finite dimensional complex vector space V. Since V is a complex vector space, $\rho(H)$ admits at least one eigenvalue of minimum real part, say $\lambda_\circ$. Let $\phi_\circ$ be an associated eigenvector. So

$$\rho(H)\phi_\circ = \lambda_\circ\phi_\circ.$$

First we will show $\phi_1 \equiv \rho(E)\phi_0$, if non-zero, is an eigenvector of $\rho(H)$. Indeed,

$$\begin{aligned}\rho(H)\phi_1 = \rho(H)\rho(E)\phi_0 &= \rho(E)\rho(H)\phi_0 + \rho(H)\rho(E)\phi_0 - \rho(E)\rho(H)\phi_0\\ &= \rho(E)\rho(H)\phi_0 + \rho([H,E])\phi_0\\ &= \lambda_\circ\rho(E)\phi_0 + 2\rho(E)\phi_0\\ &= (\lambda_\circ + 2)\phi_1. \end{aligned} \tag{4.5}$$

Letting $\phi_k = \rho(E)^k\phi_0$ for $k = 1, 2, \ldots$ we have

$$\rho(H)\phi_k = (\lambda_\circ + 2k)\phi_k.$$

by Equation 4.5 and induction. If these vectors are non-zero, then since each one is an eigenvector of $\rho(H)$ with distinct eigenvalues they are linearly independent.

Since ϕ_0 is non-zero then there exists $m \in \mathbb{N}$ with $\phi_k \neq 0$ for all $k \leq m$ and $\phi_{m+1} = 0$. This is true since V is finite dimensional. Now let W be the subspace of V spanned by the set $\{\phi_0, \ldots, \phi_k\}$. So far we have

$$\begin{aligned}\rho(H)\phi_k &= (\lambda_\circ + 2k)\phi_k\\ \rho(E)\phi_k &= \phi_{k+1}\end{aligned}$$

Therefore W is invariant under $\rho(H)$ and $\rho(E)$.

We will show that W is also invariant under $\rho(F)$. As E, F and H span $sl(2,\mathbb{C})$ it follows that W is invariant under $\rho(sl(2,\mathbb{C}))$. As (ρ, V) is irreducible this shows that $W = V$.

First we need to see that $\rho(F)\phi_0 = 0$. Indeed

$$\begin{aligned}\rho(H)\rho(F)\phi_0 &= \rho(F)\rho(H)\phi_0 + \rho(H)\rho(F)\phi_0 - \rho(F)\rho(H)\phi_0\\ &= \rho(F)\rho(H)\phi_0 + \rho([H,F])\phi_0\\ &= \lambda_\circ\rho(F)\phi_0 - 2\rho(F)\phi_0\\ &= (\lambda_\circ - 2)\rho(F)\phi_0.\end{aligned}$$

Thus if $\rho(F)\phi_\circ \neq 0$ then it is an eigenvector for $\rho(H)$ with eigenvalue $\lambda_\circ - 2$. As $\lambda_\circ$ is an eigenvalue with smallest possible real part, this is not possible. So $\rho(F)\phi_0 = 0$ as claimed.

Now we will prove inductively that $\rho(F)\phi_k = -k(\lambda_\circ + k - 1)\phi_{k-1}$. For $k = 1$ we compute

$$\begin{aligned}\rho(F)\phi_1 = \rho(F)\rho(E)\phi_0 &= \rho(E)\rho(F)\phi_0 + \rho(F)\rho(E)\phi_0 - \rho(E)\rho(F)\phi_0\\ &= \rho(E)\rho(F)\phi_0 + \rho([F,E])\phi_0\\ &= -\rho(H)\phi_0\\ &= -\lambda_\circ\phi_0.\end{aligned}$$

Next assume inductively that $\rho(F)\phi_k = -k(\lambda_\circ + k - 1)\phi_{k-1}$. We will show that $\rho(F)\phi_{k+1} = -(k+1)(\lambda_\circ + k)\phi_k$, completing the induction. Indeed

$$\begin{aligned}\rho(F)\phi_{k+1} &= \rho(F)\rho(E)\phi_k = \rho(E)\rho(F)\phi_k + \rho([F,E])\phi_k \\ &= -k(\lambda_\circ + k - 1)\rho(E)\phi_{k-1} - \rho(H)\phi_k \\ &= (-k(\lambda_\circ + k - 1) - (\lambda_\circ + 2k))\phi_k \\ &= -(k\lambda_\circ + k^2 + k + \lambda_\circ)\phi_k \\ &= -(k+1)(\lambda_\circ + k)\phi_k.\end{aligned}$$

Thus, in particular, W is invariant under $\rho(F)$ and $W = V$ follows as explained above.

Next we will show $\lambda_\circ = -m$. Notice that

$$tr(\rho(H)) = tr[\rho(E), \rho(F)] = tr(\rho(E)\rho(F) - \rho(F)\rho(E)) = 0$$

since $tr(AB) = tr(BA)$ and trace is linear. But,

$$\begin{aligned}tr(\rho(H)) = \lambda_\circ + (\lambda_\circ + 2) + \cdots + (\lambda_\circ + 2m) &= (m+1)\lambda_\circ + 2(1 + 2 + \cdots + m) \\ &= (m+1)\lambda_\circ + m(m+1) \\ &= (m+1)(\lambda_\circ + m)\end{aligned}$$

and hence $\lambda_\circ + m = 0$.

We have established these relations:

$$\begin{aligned}\rho(H)\phi_k &= (2k - m)\phi_k \\ \rho(E)\phi_k &= \phi_{k+1} \\ \rho(F)\phi_k &= k(m - k + 1)\phi_{k-1}.\end{aligned}$$

Now we claim that ρ is equivalent to $d\pi_m$ via the intertwining operator $A : V \to \mathcal{P}_m$

given on the basis vectors ϕ_k by

$$A\phi_k = \frac{m!}{(m-k)!} f_k.$$

We must verify that

1. $A \circ \rho(H) = d\pi_m(H) \circ A$,

2. $A \circ \rho(E) = d\pi_m(E) \circ A$, and

3. $A \circ \rho(F) = d\pi_m(F) \circ A$.

Indeed

1. $A\rho(H)\phi_k = A(2k-m)\phi_k = (2k-m)\frac{m!}{(m-k)!} f_k$ and also
   $d\pi_m(H)A\phi_k = \frac{m!}{(m-k)!} d\pi_m(H) f_k = (2k-m)\frac{m!}{(m-k)!} f_k$,

2. $A\rho(E)\phi_k = A\phi_{k+1} = \frac{m!}{(m-(k+1))!} f_{k+1}$ and also
   $d\pi_m(E)A\phi_k = \frac{m!}{(m-k)!} d\pi_m(E) f_k = \frac{m!}{(m-k)!}(m-k) f_{k+1} = \frac{m!}{(m-(k+1))!} f_{k+1}$,

3. $A\rho(F)\phi_k = k(m-k+1)A\phi_{k-1} = k(m-k+1)\frac{m!}{(m-k+1)!} f_{k-1}$ and also
   $d\pi_m(F)A\phi_k = \frac{m!}{(m-k)!} d\pi_m(F) f_k = \frac{m!}{(m-k)!} k f_{k-1} = k(m-k+1)\frac{m!}{(m-k+1)!} f_{k-1}$.

Thus we have that ρ is equivalent to $d\pi_m$. □

Theorem 4.15. *Each finite dimensional irreducible representation of $SU(2)$ on a complex vector space V is equivalent to $\pi_m|_{SU(2)}$ for some m.*

Proof. Let $\pi : SU(2) \to GL(V)$ be an irreducible finite dimensional complex representation of $SU(2)$. Recall that $Z \in sl(2, \mathbb{C})$ can be written uniquely as $Z = X + iY$ where $X, Y \in su(2)$. Now using Lemma 1.8 the derived representation,

$$d\pi : su(2) \to gl(V),$$

is also irreducible as $SU(2)$ is connected. We define a complex linear mapping $\widetilde{d\pi} : sl(2,\mathbb{C}) \to gl(V)$ via

$$\widetilde{d\pi}(X+iY) = d\pi(X) + id\pi(Y).$$

We will show that this is a Lie algebra representation. Indeed for $Z_1 = X_1 + iY_1$ and $Z_2 = X_2 + iY_2$ in $sl(2,\mathbb{C})$ (where $X_1, X_2, Y_1, Y_2 \in su(2)$) we compute

$$\begin{aligned}
\widetilde{d\pi}([Z_1, Z_2]) &= \widetilde{d_\pi}([X_1 + iY_1, X_2 + iX_2]) \\
&= \widetilde{d_\pi}([X_1, X_2] - [Y_1, Y_2] + i([X_1, Y_2] + [Y_1, X_2])) \\
&= d\pi([X_1, X_2] - [Y_1, Y_2]) + id\pi([X_1, Y_2] + [Y_1, X_2]) \\
&= [d\pi(X_1), d\pi(X_2)] - [d\pi(Y_1), d\pi(Y_2)] + i[d\pi(X_1), d\pi(Y_2)] + i[d\pi(Y_1), d\pi(X_2)] \\
&= [d\pi(X_1) + id\pi(Y_1), d\pi(X_2) + id\pi(Y_2)] \\
&= [\widetilde{d\pi}(Z_1), \widetilde{d\pi}(Z_2)].
\end{aligned}$$

Also, as $d\pi(su(2)) \subset \widetilde{d\pi}(sl(2,\mathbb{C}))$ and $d\pi$ is irreducible it follows that $\widetilde{d\pi}$ is irreducible. Now we know $\widetilde{d\pi}$ is equivalent to one of the representations $d\pi_m : sl(2,\mathbb{C}) \to gl(\mathcal{P}_m)$. Therefore there exists an intertwining operator $T : \mathcal{P}_m \to V$ for which

$$\widetilde{d\pi}(Z) = Td\pi_m(Z)T^{-1}, \quad \text{for all } Z \in sl(2,\mathbb{C}).$$

In particular for $X \in su(2)$ we have $d\pi(X) = Td\pi_m(X)T^{-1}$. This gives us that $d\pi \simeq d\pi_m|_{su(2)}$. Now as $SU(2)$ is connected it follows from Lemma 1.7 that $\pi \simeq \pi_m|_{SU(2)}$. $\square$

Recall that the *character* of a finite dimensional representation π is defined as $\chi_\pi(g) = tr(\pi(g))$ from Definition 2.16.

Lemma 4.16. *Each character is a central function*

Proof. Let G be a linear Lie group and $\pi : G \to GL(V)$ be a representation of G on V. For $x, g \in G$ we have

$$\begin{aligned}\chi_\pi(gxg^{-1}) = tr(\pi(gxg^{-1})) = tr(\pi(g)\pi(x)\pi(g^{-1})) &= tr(\pi(g^{-1})\pi(g)\pi(x)) \\ &= tr(\pi(x)) = \chi_\pi(x).\end{aligned}$$ □

Recall that every matrix in $SU(2)$ is conjugate to a matrix of the form $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$ where $\theta \in [0, \pi]$.

Proposition 4.17.

$$\chi_{\pi_m}\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\right) = \frac{\sin((m+1)\theta)}{\sin(\theta)}.$$

Proof. First notice that

$$\begin{aligned}\pi_m\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\right) f_j(u,v) &= f_j\left((u,v)\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\right)\right) \\ &= f_j(e^{i\theta}u, e^{-i\theta}v) \\ &= e^{ij\theta}u^j e^{-i\theta(m-j)}v^{m-j} \\ &= e^{i\theta(2j-m)}u^j v^{m-j} \\ &= e^{i\theta(2j-m)} f_j(u,v).\end{aligned}$$

Thus the matrix for $\pi_m\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\right)$ with respect to the basis $\mathcal{B} = \{f_j : 0 \le j \le m\}$ for $\mathcal{P}_m$ is

$$\left[\pi_m\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix} e^{-im\theta} & 0 & \cdots & 0 \\ 0 & e^{-i(2-m)\theta} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{im\theta} \end{bmatrix}.$$

In other words, the eigenvalues of $\pi_m\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{bmatrix}\right)$ are $e^{i(2j-m)\theta}$ where $j = 0, ..., m$. Thus we have

$$\begin{aligned}\chi_{\pi_m}\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{bmatrix}\right) = tr\left(\pi_m\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{bmatrix}\right)\right) &= e^{-im\theta}(1 + e^{2i\theta} + \cdots + e^{2im\theta}) \\ &= e^{-im\theta}\left(\frac{e^{2i(m+1)\theta} - 1}{e^{2i\theta} - 1}\right) \\ &= \frac{e^{i(m+1)\theta} - e^{-i(m+1)\theta}}{e^{i\theta} - e^{-i\theta}} \\ &= \frac{sin((m+1)\theta)}{sin(\theta)}. \qquad \square\end{aligned}$$

We will now look at some specific examples of hermitian inner products on $\mathcal{P}_m$ which make π_m unitary. Recall that $\mathcal{P}_m$ is the space of homogeneous polynomials in two variables of degree m. Consider the inner product on $\mathcal{P}_m$ defined via

$$\langle p, q\rangle = \frac{1}{\pi^2}\int\limits_{\mathbb{C}^2} p(u,v)\overline{q(u,v)}e^{-(|u|^2+|v|^2)}\lambda(du)\lambda(dv)$$

where λ is the Lebesque measure on $\mathbb{C}$. This is known as the *Fock inner product.*

Proposition 4.18. *π_m is a* unitary *representation of $SU(2)$ on $(\mathcal{P}_m, \langle\cdot,\cdot\rangle)$.*

Proof. Let $g \in SU(2)$ then

$$\begin{aligned}\langle \pi_m(g)p, \pi_m(g)q\rangle &= \frac{1}{\pi^2}\int\limits_{\mathbb{C}^2} \pi_m(g)p(u,v)\overline{\pi_m(g)q(u,v)}e^{-(|u|^2+|v|^2)}\lambda(du)\lambda(dv) \\ &= \frac{1}{\pi^2}\int\limits_{\mathbb{C}^2} p((u,v)g)\overline{q((u,v)g)}e^{-|(u,v)|^2}\lambda(du)\lambda(dv).\end{aligned}$$

Here we have that $|\cdot|^2$ is the square of the usual norm on $\mathbb{C}^2$. Also, notice that

$|(u,v)|^2 = |(u,v)g|^2$ since $g \in SU(2)$. Indeed,

$$\begin{aligned}|(u,v)g|^2 &= (u,v)g(\overline{(u,v)g})^\intercal \\ &= (u,v)g\overline{g}^\intercal\overline{(u,v)}^\intercal \\ &= (u,v)\overline{(u,v)}^\intercal = |(u,v)|^2.\end{aligned}$$

Therefore the integral becomes

$$\begin{aligned}&\frac{1}{\pi^2}\int\limits_{\mathbb{C}^2} p((u,v)g)\overline{q((u,v)g)}e^{-|(u,v)|^2}\lambda(du)\lambda(dv) \\ &= \frac{1}{\pi^2}\int\limits_{\mathbb{C}^2} p((u,v)g)\overline{q((u,v)g)}e^{-|(u,v)g|^2}\lambda(du)\lambda(dv).\end{aligned}$$

Now if we use change of variable:

$$T_g(u,v) = (u,v)g, \quad T_g : \mathbb{C}^2 \to \mathbb{C}^2$$

then $\det(T_g'(u,v)) = \det(g^\intercal) = 1$ and the integral becomes

$$\frac{1}{\pi^2}\int\limits_{\mathbb{C}^2} p(u,v)\overline{q(u,v)}e^{-|(u,v)|^2}\lambda(du)\lambda(dv) = \langle p, q\rangle. \quad \square$$

Proposition 4.19. $\{f_j \ : \ 0 \le j \le m\}$ *is an* orthogonal *basis for* $(\mathcal{P}_m, \langle\cdot,\cdot\rangle)$ *with* $\|f_j\|^2 = j!(m-j)!$. *Equivalently*

$$\left\{\sqrt{\frac{1}{j!(m-j)!}}f_j \ : \ j = 1, ..., m\right\}$$

is an orthonormal basis.

Proof. We must compute

$$\begin{aligned}\langle f_j, f_k\rangle &= \frac{1}{\pi^2}\int\limits_{\mathbb{C}^2} f_j(u,v)\overline{f_k(u,v)}e^{-|(u,v)|^2}\lambda(du)\lambda(dv)\\ &= \frac{1}{\pi^2}\int\limits_{\mathbb{C}^2} u^j v^{m-j}\overline{u}^k\overline{v}^{m-k}e^{-|(u,v)|^2}\lambda(du)\lambda(dv)\\ &= \frac{1}{\pi^2}\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^{2\pi}\int\limits_0^{2\pi} r^{j+k}e^{i\theta j}e^{-i\theta k}s^{2m-k-j}e^{i\phi(m-j)}e^{-i\phi(m-k)}e^{-r^2-s^2}rs\,d\theta d\phi dr ds\end{aligned}$$

where we let $u = re^{i\theta}$ and $v = se^{i\phi}$ and continue in polar coordinates. Suppose that $0 \le j < k \le m$ then the integral above becomes

$$\frac{1}{\pi^2}\int\limits_0^\infty\int\limits_0^{2\pi} r^{k+j}e^{i\theta(j-k)}e^{-r^2}r\,d\theta dr\int\limits_0^\infty\int\limits_0^{2\pi} s^{2m-k-j}e^{i\phi(k-j)}e^{-s^2}s\,d\phi ds.$$

If we focus on the integral with respect to theta we see that

$$\int\limits_0^{2\pi} e^{i\theta(j-k)}\,d\theta = 0.$$

Therefore we have that $\langle f_j, f_k\rangle = 0$ if $j \neq k$. Suppose that $j = k$ then

$$\begin{aligned}\|f_j\|^2 &= \frac{1}{\pi^2}\int\limits_0^\infty\int\limits_0^{2\pi} r^{2j}e^{-r^2}r dr d\theta\int\limits_0^\infty\int\limits_0^{2\pi} s^{2(m-j)}e^{-s^2}s\,ds d\phi\\ &= \int\limits_0^\infty r^{2j}e^{-r^2}2r\,dr\int\limits_0^\infty s^{2(m-j)}e^{-s^2}2s\,ds.\end{aligned}$$

Now we do a change of variable and let $t = r^2$ and $p = s^2$ and we get

$$\begin{aligned}\|f_j\|^2 &= \int_0^\infty r^{2j} e^{-r^2} 2r\, dr \int_0^\infty s^{2(m-j)} e^{-s^2} 2s\, ds \\ &= \int_0^\infty t^j e^{-t}\, dt \int_0^\infty p^{m-j} e^{-p}\, dp \\ &= j!(m-j)!. \end{aligned}$$

□

We will now consider another inner product on $\mathcal{P}_m$, namely

$$\langle p, q\rangle' = \int_{SU(2)} p(\alpha,\beta)\overline{q(\alpha,\beta)}\, d\mu(g)$$

where

$$g = \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix}.$$

Proposition 4.20. *π_m is a* unitary *representation of $SU(2)$ on $(\mathcal{P}_m, \langle \cdot, \cdot \rangle')$.*

Proof. First see that $(\alpha, \beta) = (1, 0)g_{\alpha,\beta}$. Now let $g_\circ \in SU(2)$ then

$$\begin{aligned}\langle \pi_m(g_\circ)p, \pi_m(g_\circ)q\rangle' &= \int_{SU(2)} \pi_m(g_\circ)p(\alpha,\beta)\overline{\pi_m(g_\circ)q(\alpha,\beta)} d\mu(g_{\alpha,\beta}) \\ &= \int_{SU(2)} p((1,0)g_{\alpha,\beta}g_\circ)\overline{q((1,0)g_{\alpha,\beta}g_\circ)} d\mu(g_{\alpha,\beta}).\end{aligned}$$

Thus by the right invariance of the Haar measure we have that the integral reduces to $\langle p, q\rangle'$. □

Proposition 4.21. $\{f_j \ : \ 0 \leq j \leq m\}$ *is an* orthogonal *basis for* $(\mathcal{P}_m, \langle\cdot,\cdot\rangle')$ *with* $\|f_j\|^2 = \frac{j!(m-j)!}{(m+1)!}$. *Equivalently*

$$\left\{\sqrt{\frac{(m+1)!}{j!(m-j)!}} f_j \ : \ j = 1, ..., m\right\}$$

is an orthonormal basis.

Comparing Propositions 4.19 and 4.21 we see that $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$ are related via

$$\langle\cdot,\cdot\rangle' = \frac{1}{(m+1)!}\langle\cdot,\cdot\rangle.$$

Proof. We simply need to find $\langle f_j, f_k\rangle'$.

$$\begin{aligned}\langle f_j, f_k\rangle' &= \int\limits_{SU(2)} f_j(\alpha,\beta)\overline{f_k(\alpha,\beta)}\, d\mu(g) \\ &= \int\limits_{SU(2)} \alpha^j\beta^{m-j}\overline{\alpha}^k\overline{\beta}^{m-k}\, d\mu(g).\end{aligned}$$

Now by using the Euler angle formula (Proposition 4.6), where $\alpha = e^{i(\psi+\phi)}\cos(\theta)$ and $\beta = e^{i(\psi-\phi)}\sin(\theta)$, to represent the Haar measure the integral becomes

$$\begin{aligned}&\int\limits_{SU(2)} \alpha^j\beta^{m-j}\overline{\alpha}^k\overline{\beta}^{m-k}\, d\mu(g) \\ &= \int\limits_{SU(2)} e^{i(j-k)(\psi+\phi)}\cos^{j+k}(\theta)e^{-i(j-k)(\psi-\phi)}\sin^{2m-j-k}(\theta)\, d\mu(g) \\ &= \frac{1}{2\pi^2}\int\limits_0^{\frac{\pi}{2}}\int\limits_0^{\pi}\int\limits_{-\pi}^{\pi} e^{i(j-k)\phi}\cos^{j+k}(\theta)e^{i(j-k)\phi}\sin^{2m-j-k}(\theta)2\cos(\theta)\sin(\theta)\, d\psi d\phi d\theta.\end{aligned}$$

Notice if $j \neq k$ then the integral with respect to ϕ becomes

$$\int_0^{\pi} e^{2i(j-k)\phi}\, d\phi = 0.$$

Thus it remains to find $\langle f_j, f_j\rangle'$. We have

$$\begin{aligned}
\langle f_j, f_l\rangle' &= \frac{1}{2\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_{-\pi}^{\pi} \cos^{2j}(\theta)\sin^{(2m-2j)}(\theta)2\cos(\theta)\sin(\theta)\, d\psi d\phi d\theta \\
&= \frac{1}{2\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_{-\pi}^{\pi} \cos^{2j}(\theta)\sin^{2(m-j)}(\theta)2\cos(\theta)\sin(\theta)\, d\psi d\phi d\theta \\
&= \int_0^{\frac{\pi}{2}} \cos^{2j}(\theta)\sin^{2(m-j)}(\theta)2\cos(\theta)\sin(\theta)\, d\theta \\
&= \int_0^{\frac{\pi}{2}} (1-\sin^2(\theta))^j \sin^{2(m-j)}(\theta)2\cos(\theta)\sin(\theta)\, d\psi d\phi d\theta.
\end{aligned}$$

Let $t = \sin^2(\theta)$ then $dt = 2\sin(\theta)\cos(\theta)$ then the integral becomes

$$\int_0^1 (1-t)^j t^{m-j}\, dt.$$

Now we let $u = (1-t)^j$ and $dv = t^{m-j}\, dt$ then we get $du = -j(1-t)^{j-1}\, dt$ and $v = \frac{1}{m-j+1}t^{m-j+1}$ and perform integration by parts

$$\left[(1-t)^j \frac{1}{m-j+1}t^{m-j+1}\right]_0^1 + \int_0^1 \frac{1}{m-j+1}t^{m-j+1} j(1-t)^{j-1}\, dt.$$

Now by performing integration by parts j times we have

$$\frac{j(j-1)\cdots 1}{(m-j+1)(m-j+2)\cdots(m)}\int_0^1 t^m\,dt = \frac{j!}{(m-j+1)\cdots(m+1)} = \frac{j!(m-j)!}{(m+1)!}.$$

Thus we have that

$$\langle f_j, f_j\rangle' = \frac{j!(m-j)!}{(m+1)!}. \quad \square$$

4.4 An Alternate Model for π_m

Let V_m denote the vector space of complex polynomials in one variable of degree at most m. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $SL(2,\mathbb{C})$ define $\sigma_m(g) : V_m \to V_m$ via

$$(\sigma_m(g)p)(x) = (bx+d)^m p\left(\frac{ax+c}{bx+d}\right)$$

for $p \in V_m$. In particular for $p(x) = x^j$ $(0 \leq j \leq m)$ we have $(\sigma_m(g)p)(x) = (ax+c)^j(bx+d)^{m-j}$. Note that the mapping $\sigma_m(g)$ does send V_m to V_m and is clearly linear in p.

Proposition 4.22. *σ_m is a representation of $SL(2,\mathbb{C})$.*

Proof. As the mapping σ_m sends V_m to V_m then it is just a mapping of polynomials and therefore continuous. Therefore we just need to check that σ_m is a homomorphism. Let $g, g' \in SL(2,\mathbb{C})$ where $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $g' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$. Then

$$gg' = \begin{bmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{bmatrix}.$$

Thus we have

$$
\begin{aligned}
(\sigma_m(gg')p)(x) &= ((ab'+bd')x+(cb'+dd'))^m p\left(\frac{(aa'+bc')x+(ca'+dc')}{(ab'+bd')x+(cb'+dd')}\right)\\
&= ((b'(ax+c)+d'(bx+d))^m p\left(\frac{a'(ax+c)+c'(bx+d)}{b'(ax+c)+d'(bx+d)}\right)\\
&= \left((bx+d)\left(b'\left(\frac{ax+c}{bx+d}\right)+d'\right)\right)^m p\left(\frac{a'(\frac{ax+c}{bx+d})+c'}{b'(\frac{ax+c}{bx+d})+d'}\right)\\
&= \sigma_m(g)\sigma_m(g')p(x).
\end{aligned}
$$

□

Lemma 4.23. $\sigma_m \simeq \pi_m$.

Proof. Given $f \in \mathcal{P}_m$ we write

$$
\begin{aligned}
f(u,v) = a_\circ v^m + a_1 uv^{m-1} + \ldots + a_m u^m &= v^m(a_\circ + a_1(u/v) + \ldots + a_m(u/v)^m)\\
&= v^m p(u/v)
\end{aligned}
$$

where $a_\circ, \ldots, a_m \in \mathbb{C}$ and $p \in V_m$. We define an isomprphism $A : V_m \to \mathcal{P}_m$ via

$$Ap(u,v) = v^m p(u/v).$$

Recall that, for $f \in \mathcal{P}_m$ we have $\pi_m(g)f(u,v) = f(au+cv, bu+dv)$ where $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now as $Ap(u,v) \in \mathcal{P}_m$ then

$$\pi_m(g)(Ap)(u,v) = (Ap)(au+cv, bu+dv) = (bu+dv)^m p((au+cv)/(bu+dv)).$$

On the other hand

$$A(\sigma_m(g)p)(u,v) = v^m(\sigma_m(g)p)(u/v) = v^m(bu/v+d)^m p((au/v+c)/(bu/v+d))$$

$$= (bu + dv)^m p((au + cv)/(bu + dv)).$$

Thus we have that $A\pi_m = \sigma_m A$, that is A intertwines π_m and σ_m. $\square$

Since we have $\pi_m \simeq \sigma_m$ then we also get that $\sigma_m|_{SU(2)}$ is a representation of $SU(2)$ on V_m equivalent to $\pi_m|_{SU(2)}$. Let $g \in SU(2)$ where

$$g = \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix}, \quad \alpha, \beta \in \mathbb{C}$$

we have

$$(\sigma_m(g)p)(z) = (\beta z + \overline{\alpha})^m p\left(\frac{\alpha z - \overline{\beta}}{\beta z + \overline{\alpha}}\right).$$

Define an inner product on V_m via

$$(p|q) = \frac{m+1}{\pi} \int\limits_{\mathbb{C}} p(z)\overline{q(z)}(1 + |z|^2)^{-m-2} d\lambda(z).$$

Proposition 4.24. *$\{p_j : 0 \leq j \leq m\}$ is an orthogonal basis for $(V_m, (\cdot|\cdot))$ with $\|p_j\|^2 = \frac{j!(m-j)!}{m!} = \binom{m}{j}^{-1}$.*

Proof.

$$(p_j|p_k) = \frac{m+1}{\pi} \int\limits_{\mathbb{C}} p_j(z)\overline{p_k(z)}(1+|z|^2)^{-m-2} d\lambda(z) = \frac{m+1}{\pi} \int\limits_{\mathbb{C}} z^j \overline{z}^k (1+|z|^2)^{-m-2} d\lambda(z)$$

Now let $z = re^{i\theta}$. Now $dz = r\, dr d\theta$ and

$$(p_j|p_k) = \frac{m+1}{\pi} \int\limits_0^\infty \int\limits_0^{2\pi} r^{j+k} e^{i(j-k)\theta} \frac{r}{(1+r^2)^{m+2}}\, d\theta dr.$$

If we look at the integral with respect to θ we see

$$\int_0^{2\pi} e^{i(j-k)\theta}\, d\theta = 0$$

as before when $j \neq k$. Moreover for $j = k$ we have

$$\|p_j\|^2 = \frac{m+1}{\pi} \int_0^{\infty} \int_0^{2\pi} r^{2j} \frac{r}{(1+r^2)^{m+2}}\, d\theta dr = (m+1) \int_0^{\infty} \frac{r^{2j}}{(1+r^2)^{m+2}} 2r\, dr.$$

Now if we let $s = r^2$ then $ds = 2r\, dr$ and the integral becomes

$$(m+1) \int_0^{\infty} s^j (1+s)^{-m-2}\, ds.$$

Now using integration by parts we let $u = s^j$ and $dv = (1+s)^{-m-2}\, ds$. We get $du = js^{j-1}\, ds$ and $v = \frac{1}{-m-1}(1+s)^{-m-1}$ so that the integral becomes

$$(m+1)\left[\left[s^j \frac{1}{-m-1}(1+s)^{-m-1} \right]_0^{\infty} - \int_0^{\infty} \frac{1}{-m-1}(1+s)^{-m-1} j s^{j-1}\, ds \right]$$
$$= j \int_0^{\infty} (1+s)^{-m-1} s^{j-1}\, ds.$$

If we continue using integration by parts j times we get

$$\frac{j(j-1)\cdots 1}{(m)(m-1)\cdots(m+2-j)} \int_0^{\infty} (1+s)^{-(m+2-j)}\, ds = \frac{j!}{(m)(m-1)\cdots(m+1-j)}$$
$$= \frac{j!(m-j)!}{m!}. \qquad \square$$

Proposition 4.25. *The inner product $(\cdot|\cdot)$ is σ_m-invariant on V_m.*

Proof. We have already shown that the Fock inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{P}_m$ is π_m-invariant from Proposition 4.18. We have also seen the isomorphism $A : V_m \to \mathcal{P}_m$, which intertwines σ_m with π_m.

Now as $\langle f_j, f_k \rangle = \delta_{jk} k!(m-k)!$ and $\{f_j\}$ is a basis for $\mathcal{P}_m$ we see that

$$(p|q) = \frac{1}{m!}\langle Ap, Aq \rangle$$

holds for all $p, q \in V_m$ in view of Proposition 4.24. Thus for $g \in SU(2)$ we have

$$\begin{aligned}(\sigma_m(g)p|\sigma_m(g)q) = \frac{1}{m!}\langle A\sigma_m(g)p, A\sigma_m(g)q \rangle &= \frac{1}{m!}\langle \pi_m(g)Ap, \pi_m(g)Aq \rangle \\ &= \frac{1}{m!}\langle Ap, Aq \rangle = (p|q).\end{aligned}$$

□

4.5 Reproducing Kernels for $\mathcal{P}_m$ and V_m

Let V be a finite dimensional space of complex valued functions on a set S and $(\cdot|\cdot)$ a hermitian inner product on V. Let $\mathcal{B} = \{\phi_0, \phi_1, ..., \phi_m\}$ be any orthonormal basis for $(V, (\cdot|\cdot))$ where $\dim(V) = m + 1$. We define a function $K : S \times S \to \mathbb{C}$ via

$$K(z, w) = \sum_{j=0}^{m} \phi_j(z)\overline{\phi_j(w)} = K_w(z)$$

for $z, w \in S$.

Proposition 4.26. *$f(w) = (f|K_w)$ for any $f \in V$ and $w \in S$.*

Proof. Since $\{\phi_j\}$ is an orthonormal basis we have for $f \in V$ that

$$f = \sum_{j=0}^{m}(f|\phi_j)\phi_j.$$

Here see that

$$(f|K_w) = (f|\sum_{j=0}^{m} \overline{\phi_j(w)}\phi_j) = \sum_{j=0}^{m} \phi_j(w)(p|\phi_j) = f(w). \quad \square$$

Proposition 4.27. *$K(z,w)$ does not depend on choice of orthonormal basis.*

Proof. Suppose $\{\psi_j\}$ is another orthonormal basis for $(V, (\cdot|\cdot))$ and set

$$K'(z,w) = \sum_{j=0}^{m} \psi_j(z)\overline{\psi_j(w)}.$$

Now by using the previous proposition

$$K'(z,w) = K'_w(z) = (K'_w|K_z) = \overline{(K_z|K'_w)} = \overline{K_z(w)} = \overline{K(w,z)} = K(z,w). \quad \square$$

Below we give formulas for the reproducing kernels for $(V_m, (\cdot|\cdot))$ and for $(\mathcal{P}_m, \langle\cdot,\cdot\rangle)$. For these examples we have set $S = \mathbb{C}$ and $S = \mathbb{C}^2$ respectively.

Proposition 4.28. *For $V = V_m$ and $z, w \in \mathbb{C}$ we have*

$$K(z,w) = (1 + z\overline{w})^m.$$

Proof. Letting $\phi_j = \sqrt{\frac{m!}{j!(m-j)!}}\, p_j$, Proposition 4.24 shows that $\{\phi_j\}$ is an orthonormal basis for $(V_m, (\cdot|\cdot))$. Then

$$\begin{aligned} K(z,w) = \sum_{j=0}^{m} \phi_j(z)\overline{\phi_j(w)} = \sum_{j=0}^{m} \frac{m!}{j!(m-j)!} z^j \overline{w}^j &= \sum_{j=0}^{m} \frac{m!}{j!(m-j)!} (z\overline{w})^j \\ &= \sum_{j=0}^{m} \binom{m}{j} (z\overline{w})^j \\ &= (1 + z\overline{w})^m. \end{aligned} \quad \square$$

Proposition 4.29. *For $V = \mathcal{P}_m$ and $z, w \in \mathbb{C}^2$ we have*

$$K(z,w) = \frac{1}{m!}(z_1\overline{w}_1 + z_2\overline{w}_2)^m.$$

Proof. Proposition 4.19 shows that

$$\{\phi_j\} = \left\{\sqrt{\frac{1}{j!(m-j)!}}f_j\right\}$$

is an orthonormal basis for $(\mathcal{P}_m, \langle\cdot,\cdot\rangle)$. Then we have

$$\begin{aligned} K(z,w) = \sum_{j=0}^{m} \frac{1}{j!(m-j)!} z_1^j z_2^{m-j} \overline{w}_1^j \overline{w}_2^{m-j} &= \sum_{j=0}^{m} \frac{1}{j!(m-j)!}(z_1\overline{w}_1)^j (z_2\overline{w}_2)^{m-j} \\ &= \frac{1}{m!}\sum_{j=0}^{m}\binom{m}{j}(z_1\overline{w}_1)^j(z_2\overline{w}_2)^{m-j} \\ &= \frac{1}{m!}(z_1\overline{w}_1 + z_2\overline{w}_2)^m. \end{aligned}$$

□

4.6 Laplace Operator on $SU(2)$

We now let $G = SU(2)$ and $\mathfrak{g} = su(2)$ and recall that for $X \in \mathfrak{g}$ we have $X^* = -X$ and $tr(X) = 0$. Then define an inner product on $\mathfrak{g}$ via

$$\langle X, Y\rangle = \frac{1}{2}tr(XY^*) = -\frac{1}{2}tr(XY).$$

Here we have that $\langle\cdot,\cdot\rangle : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ since for $X, Y \in \mathfrak{g}$ we have

$$\overline{tr(XY)} = tr(\overline{XY}) = tr((X^*)^\intercal (Y^*)^\intercal) = tr((-X)^\intercal(-Y)^\intercal)$$
$$= tr(X^\intercal Y^\intercal)$$
$$= tr((YX)^\intercal)$$
$$= tr(YX) = tr(XY).$$

Clearly this inner product is bilinear, since the trace function is linear. Also we have $\langle X, X\rangle \geq 0$. Indeed, using the spectral theorem we know that X is conjugate to a diagonal matrix with diagonal entries ia and $-ia$ where $a \in \mathbb{R}$. Then we get

$$\langle X, X\rangle = -\frac{1}{2} tr(X^2) = -\frac{(-a^2 - a^2)}{2} = a^2 > 0.$$

Also it is easy to see that for

$$X_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, X_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, X_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

we have $\{X_1, X_2, X_3\}$ is an orthonormal basis for $\mathfrak{g}$ with respect to $\langle \cdot, \cdot \rangle$.

Proposition 4.30. *Let π be a representation of $SU(2)$ on a finite dimension complex vector space V. The Casimer operator $\Omega_\pi : V \to V$ (see Section 2.6) satisfies*

$$-\Omega_\pi = d\pi(H)^2 + 2d\pi(H) + 4d\pi(F)d\pi(E).$$

Proof. Recall the matrices

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

in $sl(2,\mathbb{C})$. Notice that

$$iH = X_1, \quad E - F = X_2, \quad i(E+F) = X_3.$$

Thus we have

$$\begin{aligned} d\pi(X_1)^2 &= d\pi(iH)^2 = -d\pi(H)^2 \\ d\pi(X_2)^2 &= d\pi(E-F)^2 = d\pi(E)^2 + d\pi(F)^2 - d\pi(E)d\pi(F) - d\pi(F)d\pi(E) \\ d\pi(X_3)^2 &= d\pi(i(E+F))^2 = -d\pi(E)^2 - d\pi(F)^2 - d\pi(E)d\pi(F) - d\pi(F)d\pi(E) \end{aligned}$$

Thus

$$\Omega_\pi = d\pi(X_1)^2 + d\pi(X_2)^2 + d\pi(X_3)^2 = -d\pi(H)^2 - 2d\pi(E)d\pi(F) - 2d\pi(F)d\pi(E).$$

Now using the relation $[E,F] = H$ gives

$$\begin{aligned} \Omega_\pi &= -d\pi(H)^2 - 2d\pi(H) + 2d\pi([E,F]) - 2d\pi(E)d\pi(F) - 2d\pi(F)d\pi(E) \\ &= -d\pi(H)^2 - 2d\pi(H) - 4d\pi(F)d\pi(E). \end{aligned}$$

Finally

$$-\Omega_\pi = d\pi(H)^2 + 2d\pi(H) + 4d\pi(F)d\pi(E). \quad \square$$

For $G = SU(2)$ we know that every irreducible representation is equivalent to π_m for some $m \geq 0$. Also we have shown that $\Omega_{\pi_m} = -k_{\pi_m}\, id$. We will write $\Omega_m = \Omega_{\pi_m}$ and $k_m = -k_{\pi_m}$.

Proposition 4.31. *$k_m = m(m+2)$.*

Proof. For $f \in \mathcal{P}_m$ we get $\Omega_m f = -k_m f$. We now let $f_m(u,v) = u^m$ and recall that

$d\pi(H)f_m = mf_m$ and $d\pi(E)f_m = 0$. Thus by the previous proposition

$$-\Omega_m f_m = (m^2 + 2m)f_m = m(m+2)f_m. \quad \square$$

Proposition 2.28 shows that each $f \in \mathcal{M}_{\pi_m}$ is an eigenfunction of the Laplace operator with

$$\Delta f = -m(m+2)f. \tag{4.6}$$

Now suppose that f is a central function and recall that a central function is determined by its values on diagonal matrices

$$a(\theta) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}.$$

Proposition 4.32. *If f is a central function then*

$$(\Delta f)_\circ = L f_\circ$$

where $f_\circ(\theta) = f(a(\theta))$ and

$$\begin{aligned} Lf_\circ &= \frac{d^2 f_\circ}{d\theta^2} + 2cot(\theta)\frac{df_\circ}{d\theta} \\ &= \frac{1}{sin^2(\theta)}\frac{d}{d\theta}\left(sin^2(\theta)\frac{df_\circ}{d\theta}\right) \\ &= \frac{1}{sin(\theta)}\left(\frac{d^2}{d\theta^2} + 1\right) sin(\theta) f_\circ. \end{aligned}$$

Proof. It is a simple calculation to check the above formulas for $Lf_\circ$ are equal. From Proposition 4.9 we know that $f(x) = F(\frac{1}{2}tr(x))$ for some $F : [-1,1] \to \mathbb{C}$. Also

$\Delta = \rho(X_1)^2 + \rho(X_2)^2 + \rho(X_3)^2$. Thus

$$\begin{aligned}
(\rho(X_1)^2 f)(a(\theta)) &= \frac{d^2}{dt^2}\bigg|_0 f\left(a(\theta)\exp\left(t\begin{bmatrix} i & 0 \\ 0 & -i\end{bmatrix}\right)\right) \\
&= \frac{d^2}{dt^2}\bigg|_0 f\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{bmatrix}\begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it}\end{bmatrix}\right) \\
&= \frac{d^2}{dt^2}\bigg|_0 f(a(\theta+t)) \\
&= \frac{d^2}{dt^2}\bigg|_0 F(cos(\theta+t)) \\
&= \frac{d^2}{d\theta^2} F(cos(\theta)) \\
&= \frac{d}{d\theta} F'(cos(\theta))(-sin(\theta)) \\
&= -cos(\theta)F'(cos(\theta)) + F''(cos(\theta))sin^2(\theta).
\end{aligned}$$

Also,

$$\begin{aligned}
(\rho(X_2)^2 f)(a(\theta)) &= \frac{d^2}{dt^2}\bigg|_0 f\left(a(\theta)\exp\left(t\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}\right)\right) \\
&= \frac{d^2}{dt^2}\bigg|_0 f\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{bmatrix}\begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t)\end{bmatrix}\right) \\
&= \frac{d^2}{dt^2}\bigg|_0 f\begin{bmatrix} cos(t)e^{i\theta} & sin(t)e^{i\theta} \\ -e^{-i\theta}sin(t) & e^{-i\theta}cos(t)\end{bmatrix} \\
&= \frac{d^2}{dt^2}\bigg|_0 F(cos(\theta)cos(t)) \\
&= \frac{d^2}{dt^2}\bigg|_0 F'(cos(\theta)cos(t))cos(\theta)(-sin(t)) \\
&= [F''(cos(\theta)cos(t))cos^2(\theta)sin^2(t) + F'(cos(\theta)cos(t))cos(\theta)(-cos(t))]_{t=0} \\
&= -F'(cos(\theta))cos(\theta).
\end{aligned}$$

Finally

$$\begin{aligned}
\rho(X_3)^2 f(a(\theta)) &= \frac{d^2}{dt^2}\bigg|_0 f\left(a(\theta)\exp\begin{bmatrix} 0 & it \\ it & 0 \end{bmatrix}\right) \\
&= \frac{d^2}{dt^2}\bigg|_0 f\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\begin{bmatrix} \cos(t) & i\sin(t) \\ i\sin(t) & \cos(t) \end{bmatrix}\right) \\
&= \frac{d^2}{dt^2}\bigg|_0 f\begin{bmatrix} e^{i\theta}cos(t) & e^{i\theta}isin(t) \\ e^{-i\theta}isin(t) & e^{-i\theta}cos(t) \end{bmatrix} \\
&= \frac{d^2}{dt^2}\bigg|_0 F(cos(t)cos(\theta)) \\
&= -F'(cos(\theta))cos(\theta).
\end{aligned}$$

So we observe

$$(\Delta f)_\circ(a(\theta)) = F''(cos(\theta))sin^2(\theta) - 3F'(cos(\theta))cos(\theta).$$

Also notice that

$$\begin{aligned}
f_\circ(\theta) &= F(cos(\theta)) \\
f_\circ'(\theta) &= F'(cos(\theta))(-sin(\theta)) \\
f_\circ''(\theta) &= F''(cos(\theta))(sin^2(\theta)) - F'(cos(\theta))cos(\theta)
\end{aligned}$$

and finally

$$(\Delta f)_\circ(a(\theta)) = f_\circ''(\theta) + 2\frac{cos(\theta)}{sin(\theta)}f_\circ'(\theta) = \frac{d^2 f_\circ}{d\theta^2} + 2cot(\theta)\frac{df_\circ}{d\theta} = Lf_\circ$$

as claimed. □

4.7 Fourier Series on $SU(2)$

Recall that if f is an integrable function on a linear Lie group G and $(\pi, V_\pi) \in \widehat{G}$ then $\widehat{f}(\pi)$ is the operator

$$\widehat{f}(\pi) = \int_G f(g)\pi(g^{-1})\, d\mu(g)$$

on V_π. This is the *operator valued Fourier transform.*

Now for $G = SU(2)$ the irreducible unitary representation π_m has dimension $d_{\pi_m} = \dim(\mathcal{P}_m) = m+1$. The Plancherel Theorem (2.11) (1) and ensures that for $f \in L^2(SU(2))$

$$f(x) = \sum_{m=0}^{\infty} (m+1) tr(\widehat{f}(\pi_m)\pi_m(x))$$

where the series converges in the L^2-sense. Part (2) of the Plancherel Theorem gives

$$\|f(x)\|_2^2 = \int_G |f(x)|^2\, d\mu(x) = \sum_{m=0}^{\infty} (m+1)|||\widehat{f}(\pi_m)|||^2.$$

Proposition 4.33. *If $f \in C^2(SU(2)) \subset L^2(SU(2))$ then*

$$\widehat{\Delta f}(\pi_m) = -m(m+2)\widehat{f}(\pi_m).$$

Proof. For $u, v \in \mathcal{P}_m$ Lemma 2.10 shows

$$\langle \widehat{\Delta f}(\pi_m)v, u\rangle_{\pi_m} = \langle \Delta f, \phi_{\pi_m,u,v}\rangle_2$$

where $\phi_{\pi_m,u,v}(g) = \langle \pi_m(g)u, v\rangle_\pi$. Now as Δ is self-adjoint we have

$$\langle \Delta f, \phi_{\pi_m,u,v}\rangle_2 = \langle f, \Delta\phi_{\pi_m,u,v}\rangle_2.$$

Now we know that $\phi_{\pi_m,u,v} \in \mathcal{M}_{\pi_m}$ and $\Delta|_{\mathcal{M}_{\pi_m}} = -m(m+2)\, id$. Thus

$$\langle f, \Delta\phi_{\pi_m,u,v}\rangle_2 = -m(m+2)\langle \widehat{f}(\pi_m)v, u\rangle_{\pi_m}.$$

Since this works for all $u, v \in \mathcal{P}_m$, we conclude that $\widehat{\Delta f}(\pi_m) = -m(m+2)\widehat{f}(\pi_m)$. $\square$

Theorem 4.34. *If $f \in C^2(SU(2))$ then*

$$\sum_{m=0}^{\infty}(m+1)^{\frac{3}{2}}|||\widehat{f}(\pi_m)||| < \infty$$

and the series

$$\sum_{m=0}^{\infty}(m+1)tr(\widehat{f}(\pi_m)\pi_m(g))$$

converges absolutely and uniformly on $SU(2)$ to $f(g)$.

Proof. Using Proposition 2.13 and the fact that $f \in C^2(SU(2)) \subset C(SU(2))$ we just need to show that

$$\sum_{m=0}^{\infty}(m+1)^{\frac{3}{2}}|||\widehat{f}(\pi_m)||| < \infty$$

and the rest follows. First by the last proposition we can write

$$\sum_{m=1}^{\infty}(m+1)^{\frac{3}{2}}|||\widehat{f}(\pi_m)||| = \sum_{m=1}^{\infty}\frac{(m+1)^{\frac{3}{2}}}{m(m+2)}|||\widehat{\Delta f}(\pi_m)|||.$$

As $\Delta f \in C(SU(2)) \in L^2(SU(2))$ part (2) of the Plancherel Thm (2.11) gives

$$\sum_{m=0}^{\infty}\left(\sqrt{m+1}|||\widehat{\Delta f}(\pi_m)|||\right)^2 = \sum_{m=0}^{\infty}(m+1)|||\widehat{\Delta f}(\pi_m)|||^2 = \|\Delta f\|_2^2 < \infty.$$

We let $\ell_2(\mathbb{N})$ be the space of square summable sequences. Then we have that

$$\left(a_m := \sqrt{m+1}|||\widehat{f}(\pi_m)|||\right)_{m=1}^{\infty} \in \ell_2(\mathbb{N})$$

and

$$\left(b_m := \frac{m+1}{m(m+2)}\right)_{m=1}^{\infty} \in \ell_2(\mathbb{N}).$$

Then by the Cauchy-Schwartz inequality we see that

$$\sum_{m=1}^{\infty} \frac{(m+1)^{\frac{3}{2}}}{m(m+2)}|||\widehat{\Delta f}(\pi_m)||| = |\sum_{m=1}^{\infty} a_m \overline{b_m}| \leq \left(\sum_{m=1}^{\infty} |a_m|^2\right)^{\frac{1}{2}} \left(\sum_{m=1}^{\infty} (b_m)^2\right)^{\frac{1}{2}} < \infty. \quad \square$$

The following result concerns convergence of Fourier series for *central* functions on $SU(2)$. This follows immediately from Theorems 2.18 and 4.34.

Corollary 4.35. *For $f \in L^2(SU(2))^{Ad}$,*

$$f(g) = \sum_{m=0}^{\infty} \langle f, \chi_m \rangle \chi_m(g)$$

converges in the L^2-sense. Moreover if f is continuous and central and

$$\sum_{m=0}^{\infty} (m+1)|\langle f, \chi_m \rangle_2| < \infty$$

then the series converges absolutely and uniformly to f on G. In particular this is the case when $f \in C^2(SU(2))^{Ad}$.

4.8 Poisson Kernel on $SU(2)$

Let $0 < r < 1$ and define $P_r : SU(2) \to \mathbb{C}$ as

$$P_r(x) = (1 - r^2) \det(I - rx)^{-2}.$$

P_r is known as the *Poisson kernel*. The following Lemma ensures that P_r is well-defined.

Lemma 4.36. $\det(I - rx) \neq 0$ *for* $x \in SU(2)$ *and* $r \in (0, 1)$.

Proof. If $\det(I - rx) = 0$ then also $\det(\frac{1}{r}I - x) = 0$. That is, $1/r$ is an eigenvalue of the matrix x. But as $x \in SU(2)$ is a unitary matrix we know that its eigenvalues lie on the unit circle. As $1/r > 1$ it follows that $1/r$ is not an eigenvalue for x. □

Proposition 4.37. P_r *is a central function, i.e*

$$P_r(x) = P_r(gxg^{-1}), \quad \textit{for all } x, g \in SU(2).$$

Proof. For $x, g \in SU(2)$ we have

$$\begin{aligned} P_r(gxg^{-1}) = (1 - r^2) \det(I - rgxg^{-1})^{-2} &= (1 - r^2) \det(g(I - rx)g^{-1})^{-2} \\ &= (1 - r^2) \det(g^{-1}g(I - rx))^{-2} \\ &= (1 - r^2) \det(I - rx)^{-2} \\ &= P_r(x). \end{aligned}$$

□

Since central functions are determined by their values at

$$a(\theta) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

we can find an easier expression for P_r.

$$\det(I - ra(\theta)) = \det\left(\begin{bmatrix} 1 - re^{i\theta} & 0 \\ 0 & 1 - re^{-i\theta} \end{bmatrix}\right) = (1 - re^{i\theta})(1 - re^{-i\theta})$$
$$= 1 - r(e^{i\theta} + e^{-i\theta}) + r^2$$
$$= 1 - 2r\cos(\theta) + r^2.$$

Thus we can write

$$P_r(a(\theta)) = \frac{1 - r^2}{(1 - 2r\cos(\theta) + r^2)^2}.$$

This shows that the Poisson Kernel is a smooth function and thus

$$\sum_{m=0}^{\infty} \langle P_r, \chi_m \rangle \chi_m(x)$$

converges uniformly to $P_r(x)$ by Corollary 4.35.

Proposition 4.38. $\langle P_r, \chi_m \rangle = (m+1)r^m$. *Thus the series*

$$\sum_{m=0}^{\infty} (m+1) r^m \chi_m(x)$$

converges absolutely and uniformly to $P_r(x)$ *in* $x \in SU(2)$.

Proof. Recall from Proposition 3.8 the formula for the Poisson kernel on $U(1)$,

$$\frac{1 - r^2}{1 - 2r\cos(\theta) + r^2} = 1 + 2\sum_{m=1}^{\infty} r^m \cos(m\theta).$$

If we now differentiate term-wise in θ we get

$$-(1 - r^2)(1 - 2r\cos(\theta) + r^2)^{-2}(2r\sin(\theta)) = 2\sum_{m=1}^{\infty} mr^m(-\sin(m\theta))$$

and as $|-mr^m\sin(m\theta)| \leq mr^m$ with $\sum mr^m < \infty$ (since $r < 1$) the M-test shows that the resulting series converges uniformly, justifying our term-wise differentiation. Finally we observe

$$\begin{aligned} P_r(a(\theta)) = \frac{1-r^2}{(1-2r\cos(\theta)+r^2)^2} &= \sum_{m=1}^{\infty} \frac{1}{r\sin(\theta)} mr^m \sin(m\theta) \\ &= \sum_{m=0}^{\infty} (m+1)r^m \frac{\sin((m+1)\theta)}{\sin(\theta)} \\ &= \sum_{m=0}^{\infty} (m+1)r^m \chi_m(a(\theta)) \end{aligned}$$

using Proposition 4.17. □

Proposition 4.39. *The Poisson kernel carries three important properties*

- $P_r(x) \geq 0$
- $\int_{SU(2)} P_r(x)\, d\mu(x) = 1$
- *For any neighborhood V containing I one has* $\lim_{r\to 1^-} \int_V P_r(x)\, d\mu(x) = 1$
- *For $f \in C(SU(2))$,* $\lim_{r\to 1^-}(P_r * f)(x) = f(x)$

Proof. 1.) Since

$$P_r(a(\theta)) = \frac{1-r^2}{(1-2r\cos(\theta)+r^2)^2}$$

the first property is clear as $r < 1$.

2.) As $P_r(x) = \sum_{m=0}^{\infty}(m+1)r^m\chi_m(x)$ converges uniformly we can integrate term-wise. We obtain

$$\int_{SU(2)} P_r(x)\, d\mu(x) = \sum_{m=0}^{\infty}(m+1)r^m \int_{SU(2)} \chi_m(x)\, d\mu(x)$$

where $\int_{SU(2)} \chi_m(x)\, d\mu(x) = \langle \chi_m, \chi_0 \rangle_2 = \delta_{m,0}$ from Theorem (2.3). Thus our integral becomes

$$\sum_{m=0}^{\infty}(m+1)r^m \int\limits_{SU(2)} \chi_m(x)\,d\mu(x) = 1r^0 = 1.$$

3.) Let $I \in V$ be given and pick a $f \in C^2(SU(2))$ with $0 \leq f \leq 1, f(I) = 1, f|_{V^c} = 0$, and $f(x^{-1}) = f(x)$ for all $x \in SU(2)$. Then we have

$$\begin{aligned}
\langle f, \chi_m\rangle_2 &= \int\limits_{SU(2)} f(x)\overline{\chi_m}(x)\,d\mu(x) \\
&= \int\limits_{SU(2)} f(x)\chi_m(x)\,d\mu(x) \quad \text{(as } \chi_m \text{ is real-valued)} \\
&= \int\limits_{SU(2)} f(x^{-1})\chi_m(x^{-1})\,d\mu(x) \quad \text{(by right-invariance of } d\mu\text{)} \\
&= \int\limits_{SU(2)} f(x)\chi_m(x^{-1})\,d\mu(x) \quad \text{(as } f(x) = f(x^{-1})\text{)} \\
&= tr\left(\int\limits_{SU(2)} f(x)\pi_m(x^{-1})\,d\mu(x)\right) \\
&= tr(\widehat{f}(m)).
\end{aligned}$$

As $f \in C^2(SU(2))$ then we know

$$\begin{aligned}
1 = f(I) = \sum_{m=0}^{\infty}(m+1)tr(\widehat{f}(m)\pi_m(I)) &= \sum_{m=0}^{\infty}(m+1)tr(\widehat{f}(m)) \\
&= \sum_{m=0}^{\infty}(m+1)\langle f, \chi_m\rangle_2.
\end{aligned}$$

But as $P_r = \sum\limits_{m=0}^{\infty}(m+1)r^m\chi_m$ and both P_r and f are in $L^2(SU(2))$ we can perform the following inner product term-wise

$$\langle f, P_r \rangle_2 = \sum_{m=0}^{\infty} (m+1) r^m \langle f, \chi_m \rangle_2$$

and observe that

$$\lim_{r \to 1^-} \langle f, P_r \rangle_2 = \sum_{m=0}^{\infty} (m+1) \langle f, \chi_m \rangle_2 = 1$$

from above. As both f and P_r are non-negative and real-valued then

$$\begin{aligned}
\langle f, P_r \rangle_2 &= \int\limits_{SU(2)} f(x) P_r(x) \, d\mu(x) \\
&= \int\limits_V f(x) P_r(x) \, d\mu(x) \\
&\leq \int\limits_V P_r(x) \, d\mu(x) \quad (\text{as } 0 \leq f \leq 1) \\
&\leq \int\limits_{SU(2)} P_r(x) \, d\mu(x) \quad (\text{as } P_r \geq 0) \\
&= 1 \quad (\text{from the second property of the Poisson Kernel}).
\end{aligned}$$

Thus we have $\langle f, P_r \rangle_2 \leq \int\limits_V P_r(x) \, d\mu(x) \leq 1$ and now as $\lim_{r \to 1^-} \langle f, P_r \rangle_2 = 1$ then by the Squeeze Theorem we must have

$$\lim_{r \to 1^-} \int\limits_V P_r(x) \, d\mu(x) = 1.$$

4.) The fourth property will be proved at the end of the chapter. □

These properties prove that the Poisson kernel is an *Approximate identity*.

4.9 Abel Summability of Fourier Series on $SU(2)$

In this section we obtain Abel summability of Fourier series for continuous functions on $SU(2)$, analogous to that in the $U(1)$ setting as described in Section 3.4. *Convolution* of integrable functions on $SU(2)$ is defined via

$$(f \star g)(x) = \int_{SU(2)} f(y)g(y^{-1}x)\, d\mu(y) = \int_{SU(2)} f(xy^{-1})g(y)\, d\mu(y).$$

The second integral is obtained from the first via the change of variable $y \to xy^{-1}$. As $d\mu$ is right-invariant these integrals agree.

For $f : SU(2) \to \mathbb{C}$ continuous and $0 < r < 1$ fixed we consider

$$\begin{aligned} P_r(xy^{-1})f(y) &= \sum_{m=0}^{\infty}(m+1)r^m \chi_m(xy^{-1})f(y) \\ &= \sum_{m=0}^{\infty}(m+1)r^m tr(\pi_m(xy^{-1}))f(y) \\ &= \sum_{m=0}^{\infty}(m+1)r^m tr\Big(f(y)\pi_m(y^{-1})\pi_m(x)\Big). \end{aligned}$$

Lemma 4.40. *The series*

$$\sum_{m=0}^{\infty}(m+1)r^m tr\Big(f(y)\pi_m(y^{-1})\pi_m(x)\Big)$$

converges uniformly in (x, y) *to* $P_r(xy^{-1})f(y)$.

Proof. Recall that for $x \in SU(2)$ that $|||\pi_m(x)||| = \sqrt{d_{\pi_m}} = \sqrt{m+1}$ (Lemma 2.12) and calculate

$$|(m+1)r^m tr(\pi_m(xy^{-1})f(y))| \leq (m+1)r^m \sup(|f|)\, |||\pi_m(xy^{-1})||| = (m+1)^{\frac{3}{2}} r^m \sup(|f|).$$

Since $\sum_{m=0}^{\infty}(m+1)^{\frac{3}{2}}r^m \sup(|f|)$ converges, as $0<r<1$, the M-test yields the result. □

Proposition 4.41. *The series*

$$\sum_{m=0}^{\infty}(m+1)r^m tr(\widehat{f}(\pi_m)\pi_m(x))$$

converges absolutely and uniformly to $(P_r * f)(x)$ *in* $x \in SU(2)$*.*

Proof. Recall that $(P_r * f)(x) = \int_{SU(2)} P_r(xy^{-1})f(y)d\mu(y)$. We substitute the series expression for $P_r(xy^{-1})f(y)$, given in Lemma 4.40 and use the uniform convergence to justify term-wise integration. First we compute

$$\begin{aligned}(P_r * f)(x) &= \sum_{m=0}^{\infty}(m+1)r^m \int_{SU(2)} tr(f(y)\pi_m(y^{-1})\pi_m(x))d\mu(y) \\ &= \sum_{m=0}^{\infty}(m+1)r^m tr\left(\left(\int_{SU(2)} f(y)\pi_m(y^{-1})d\mu(y)\right)\circ\pi_m(x)\right) \\ &= \sum_{m=0}^{\infty}(m+1)r^m tr(\widehat{f}(\pi_m)\pi_m(x)).\end{aligned}$$

To see that this series converges absolutely and uniformly we use the M-test and compute

$$|(m+1)r^m tr(\widehat{f}(\pi_m)\pi_m(x))| \leq (m+1)r^m |||\widehat{f}(\pi_m)|||\,|||\pi_m(x)|||.$$

As f is continuous and $SU(2)$ compact we have $f \in L^2(SU(2))$. The Plancherel Theorem (2.11) ensures that $\sum_{m=0}^{\infty}|||\widehat{f}(\pi_m)|||^2 < \infty$, so, in particular, $(|||\widehat{f}(\pi_m)|||)_m$ is a bounded sequence, $|||\widehat{f}(\pi_m)||| \leq C$ say. Lemma 2.12 gives $|||\pi_m(x)||| = \sqrt{d_{\pi_m}} =$

$\sqrt{m+1}$. So now

$$|(m+1)r^m tr(\widehat{f}(\pi_m)\pi_m(x))| \leq C(m+1)^{3/2}r^m$$

and $\sum_m (m+1)^{3/2}r^m$ converges as $0 < r < 1$. $\square$

Proposition 4.42. *For* $f \in C(SU(2))$

$$\lim_{r\to 1^-}(P_r * f)(x) = \lim_{r\to 1^-}\sum_{m=0}^{\infty}(m+1)r^m tr(\widehat{f}(\pi_m)\pi_m(x)) = f(x)$$

with uniform convergence in $x \in SU(2)$.

Proof. Let $0 < r < 1$ be fixed and let

$$f_r(x) = (P_r * f)(x) = \int\limits_{SU(2)} P_r(y)f(y^{-1}x)d\mu(y).$$

Let $\varepsilon > 0$ be given. As $SU(2)$ is compact and f is continuous Proposition 2.29 ensures that there is a neighborhood V around I with

$$|f(yx) - f(x)| < \varepsilon, \quad \text{for all } x \in SU(2),\ y \in V.$$

Thus also $U := V^{-1} = \{y^{-1} \ : \ y \in V\}$ is a neighborhood of I with

$$|f(y^{-1}x) - f(x)| < \varepsilon, \quad \text{for all } x \in SU(2),\ y \in U.$$

Now using the properties of the Poisson kernel we calculate

$$|f_r(x) - f(x)|$$

$$= \left| \int\limits_{SU(2)} P_r(y)(f(y^{-1}x) - f(x))d\mu(y) \right|$$
$$\leq \int\limits_{SU(2)} P_r(y)|f(y^{-1}x) - f(x)|d\mu(y)$$
$$= \int\limits_{U} P_r(y)|f(y^{-1}x) - f(x)|d\mu(y) + \int\limits_{U^C} P_r(y)|f(y^{-1}x) - f(x)|d\mu(y)$$
$$\leq \varepsilon \int\limits_{U} P_r(y)d\mu(y) + 2\sup(|f|) \int\limits_{U^C} P_r(y)d\mu(y)$$
$$\leq \varepsilon \int\limits_{SU(2)} P_r(y)d\mu(y) + 2\sup(|f|) \left(1 - \int\limits_{U} P_r(y)d\mu(y)\right)$$
$$\leq \varepsilon + 2\sup(|f|) \left(1 - \int\limits_{U} P_r(y)d\mu(y)\right).$$

If we now take the previous calculation and take the limit as $r \to 1^-$ we get

$$\lim_{r \to 1^-} \left(\varepsilon + 2\sup(|f|) \left(1 - \int\limits_{U} P_r(y)d\mu(y)\right)\right) = \varepsilon.$$

Thus we get

$$\lim_{r \to 1^-} |f_r(x) - f(x)| \leq \varepsilon$$

for every $\varepsilon > 0$ and hence

$$\lim_{r \to 1^-} f_r(x) = f(x).$$

Moreover as our above estimate is independent of x, convergence is uniform in x. $\square$

CHAPTER 5: Heat Equation on $U(1)$ and $SU(2)$

In this chapter we study the heat equation on $SU(2)$. For purposes of motivation and comparison we first observe the heat equation on the linear Lie group $U(1)$. This is a very classical example since $U(1)$ can be thought of as the unit circle in $\mathbb{C}$. In fact Fourier (1768-1830) applied his methods to the study of heat conduction in a thin circular loop of wire.

5.1 Heat Equation on $U(1)$

The heat equation on $\mathbb{R}$ can be written

$$\frac{\partial u}{\partial t}(t,\theta) = \frac{\partial^2 u}{\partial \theta^2}(t,\theta).$$

Recall that for $f : U(1) \to \mathbb{C}$ we defined $\widetilde{f} : \mathbb{R} \to \mathbb{C}$ as the 2π - periodic function

$$\widetilde{f}(\theta) = f(e^{i\theta})$$

from Chapter 3. The *Cauchy problem* for the heat equation on $U(1)$ is the following.

- Given a continuous function $f : U(1) \to \mathbb{C}$ find a continuous function $u(t,\theta)$ on $[0,\infty) \times \mathbb{R}$ which is 2π-periodic in the θ variable, $u \in C^2((0,\infty) \times \mathbb{R})$ such that

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \theta^2}, \qquad u(0,\theta) = \widetilde{f}(\theta).$$

First notice that the functions

$$e^{-m^2 t} e^{im\theta}$$

are solutions to the heat equation. We let the *heat kernel* be defined on $(0,\infty)\times\mathbb{R}$ via

$$h(t,\theta)=\sum_{m\in\mathbb{Z}} e^{-m^2t}e^{im\theta}.$$

Also recall that Equation 3.1 asserts that for $f\in C^1(U(1))$ one has

$$\widetilde{f}(\theta)=\sum_{m\in\mathbb{Z}}\widehat{f}(m)e^{-im\theta}$$

converges uniformly. If we now write

$$u(t,\theta)=\sum_{m\in\mathbb{Z}}\widehat{f}(m)e^{-m^2t}e^{im\theta}$$

then for $t>0$ it is easy to check that u is a solution to the heat equation. Also if we let $t=0$ we get

$$u(0,\theta)=\sum_{m\in\mathbb{Z}}\widehat{f}(m)e^{im\theta}=\widetilde{f}(\theta)$$

and thus $u(t,\theta)$ solves the Cauchy problem. Moreover, for $t>0$ the function $u(t,\theta)$ can be expressed as the convolution of $\widetilde{f}$ with the heat kernel in the θ variable. Indeed for $f\in C^1(U(1))$ one can justify the following formal calculation:

$$\begin{aligned}
u(t,\theta)&=\sum_{m\in\mathbb{Z}}\widehat{f}(m)e^{-m^2t}e^{im\theta}\\
&=\sum_{m\in\mathbb{Z}}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-im\phi}\widetilde{f}(\phi)\,d\phi\right)e^{-m^2t}e^{im\theta}\\
&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{m\in\mathbb{Z}}e^{-m^2t}e^{im(\theta-\phi)}\widetilde{f}(\phi)\,d\phi\\
&=\frac{1}{2\pi}\int_{-\pi}^{\pi}h(t,\theta-\phi)\widetilde{f}(\phi)\,d\phi.
\end{aligned}$$

It is true more generally that the above procedure produces a solution to the Cauchy problem for continuous boundary data. More precisely one has the following. A proof can be found in [7, Page 100].

Theorem 5.1. *For $f \in C(U(1))$,*

$$u(t,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(t,\theta-\phi)\widetilde{f}(\phi)d\phi$$

solves the heat equation on $(0,\infty)\times\mathbb{R}$ and satisfies $\lim_{t\to 0^+} u(t,\theta) = \widetilde{f}(\theta)$. This is the unique solution to the Cauchy problem for boundary data f.

5.2 Heat Equation on $SU(2)$

The heat equation on $SU(2)$ reads $\frac{\partial u}{\partial t} = \Delta u$ where Δ is the Laplace operator from Section 4.6. The *Cauchy problem* for the heat equation is the following.

- Given a continuous function $f : SU(2) \to \mathbb{C}$ find a continuous function $u(t,x) : [0,\infty)\times SU(2) \to \mathbb{C}$ with u C^2 on $(0,\infty)\times SU(2))$ and

$$\frac{\partial u}{\partial t} = \Delta u, \qquad u(0,x) = f(x).$$

We will prove the following fundamental result.

Theorem 5.2. *Given any continuous function $f \in C(SU(2))$ there exists a unique solution u to the Cauchy problem with* initial function f.

Corollary 5.4 below establishes the uniqueness of solutions to the Cauchy problem. Key to this is the following lemma.

Lemma 5.3. *Let* $u : [0, \infty) \times SU(2) \to \mathbb{R}$ *be continuous, real-valued, and* C^2 *on* $(0, \infty) \times SU(2)$ *with* $\frac{\partial u}{\partial t} = \Delta u$*. Suppose that* $u(0, x) \geq 0$ *for all* $x \in SU(2)$*. Then* $u \geq 0$ *on* $[0, \infty) \times SU(2)$*. Likewise if* $u(0, x) \leq 0$ *for all* $x \in SU(2)$ *then* $u \leq 0$ *on* $[0, \infty) \times SU(2)$*.*

Proof. It is suffice to show that for fixed $T_\circ > 0$ we have $u \geq 0$ on the compact set $[0, T_\circ] \times SU(2)$. Let $\varepsilon > 0$ be given and observe the continuous function

$$u_\varepsilon(t, x) = u(t, x) + \varepsilon t.$$

Now suppose that $u(t, x)$ achieves its absolute minimum on the set $[0, \infty] \times SU(2)$ at $(t_\circ, x_\circ)$. Suppose that $t_\circ > 0$ so that $(t_\circ, x_\circ) \in (0, \infty) \times SU(2)$. Now as $\frac{\partial u}{\partial t} = \Delta u$ on $(0, \infty) \times SU(2)$ we see that

$$\frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = \varepsilon > 0 \tag{5.1}$$

and as u_ε achieves its minimum in $[0, T_\circ] \times SU(2)$ at $(t_\circ, x_\circ)$ we must have

$$\frac{\partial u_\varepsilon}{\partial t}(t_\circ, x_\circ) \leq 0. \tag{5.2}$$

In fact we have $\frac{\partial u_\varepsilon}{\partial t}(t_\circ, x_\circ) = 0$ when $t_\circ < T_\circ$. Recall that for the basis $\{X_1, X_2, X_3\}$ of $SU(2)$ we have

$$\begin{aligned} &\Delta u_\varepsilon(t_\circ, x_\circ) \\ &= \frac{d^2}{ds^2}\bigg|_0 u_\varepsilon(t_\circ, x_\circ \exp(sX_1)) + \frac{d^2}{ds^2}\bigg|_0 u_\varepsilon(t_\circ, x_\circ \exp(sX_2)) + \frac{d^2}{ds^2}\bigg|_0 u_\varepsilon(t_\circ, x_\circ \exp(sX_3)). \end{aligned}$$

Since u_ε achieves its absolute minimum at $(t_\circ, x_\circ)$ then each of these second derivatives must be non-negative and thus

$$\Delta u_\varepsilon(t_\circ, x_\circ) \geq 0. \tag{5.3}$$

Using Equation 5.2 and 5.3 we have

$$\frac{\partial u_\varepsilon}{\partial t}(t_\circ, x_\circ) - \Delta u_\varepsilon(t_\circ, x_\circ) \leq 0.$$

But this contradicts Equation 5.1 as $(t_\circ, x_\circ)$ belongs to $(0, \infty) \times SU(2)$. Thus we must have that $t_\circ = 0$ and from this it follows that

$$u_\varepsilon(t_\circ, x_\circ) = u_\varepsilon(0, x_\circ) = u(0, x_\circ) \geq 0$$

and hence that

$$u(t, x) + \varepsilon t \geq 0$$

holds for all $(t, x) \in [0, T_\circ] \times SU(2)$. As $\varepsilon > 0$ is arbitrary this implies that $u \geq 0$ on $[0, T_\circ] \times SU(2)$ as desired. □

Corollary 5.4. *For given initial data $f \in C(SU(2))$ there is at most one solution to the Cauchy problem.*

Proof. Let u_1 and u_2 both solve the Cauchy problem for boundary data $f \in C(SU(2))$. Then $u = u_1 - u_2$ is continuous on $[0, \infty) \times SU(2)$ and C^2 on $(0, \infty) \times SU(2)$ and satisfies

$$\frac{\partial u}{\partial t} = \Delta u, \qquad u(0, x) = 0.$$

Applying the previous lemma to the real and imaginary parts of u then we see that $u \geq 0$ and $u \leq 0$ i.e. $u = 0$ on $[0, \infty) \times SU(2)$. That is $u_1 = u_2$. □

To complete the proof for Theorem 5.2 it remains to establish existence of a solution to the Cauchy problem. Recall for $v \in \mathcal{M}_{\pi_m}$ that

$$\Delta v(x) = -m(m+2)v(x)$$

for all $x \in SU(2)$ (Equation 4.6). Thus it is easy to see that

$$u(t,x) = e^{-m(m+2)t}v(x)$$

is a solution to the heat equation. We will study Fourier series of the general form

$$u(t,x) = \sum_{m=0}^{\infty} e^{-m(m+2)t} v_{\pi_m}(x), \quad \text{where } v_{\pi_m} \in \mathcal{M}_{\pi_m}.$$

Now let us define the *heat kernel* as

$$H(t,x) = \sum_{m=0}^{\infty} (m+1)e^{-m(m+2)t}\chi_m(x)$$

where $(t,x) \in (0,\infty) \times SU(2)$.

Proposition 5.5. $H(t,x) = \sum_{m=0}^{\infty}(m+1)e^{-m(m+2)t}\chi_m(x)$ *converges absolutely and uniformly on* $[t_\circ,\infty) \times SU(2)$ *for each fixed* $t_\circ > 0$.

Proof. Let $t_\circ > 0$ be given. Notice

$$|\chi_m(x)| = |tr(\pi_m(x))|$$

is the modulus of the sum of the eigenvalues for $\pi_m(x)$. As $\pi_m(x)$ is unitary each eigenvalue has modulus 1. It follows that $|\chi_m(x)| \leq m+1$. So we get

$$|(m+1)e^{-m(m+2)t}\chi_m(x)| \leq (m+1)^2 e^{-m(m+2)t_\circ}$$

for all $(x,t) \in [t_\circ, \infty) \times SU(2)$. Thus by the M-test the series for $H(t,x)$ converges absolutely and uniformly on $[t_\circ, \infty) \times SU(2)$. □

Lemma 5.6. *Let $f \in C(G)$ then*

$$\sum_{m=0}^{\infty}(m+1)e^{-m(m+2)t}\chi_m(xy^{-1})f(y)$$

converges absolutely and uniformly to $H(t,xy^{-1})f(y)$ in $y \in SU(2)$ for all fixed $(t,x) \in (0,\infty) \times SU(2)$.

Proof. We calculate

$$|(m+1)e^{-m(m+2)t}\chi_m(xy^{-1})f(y)| \leq (m+1)^2 e^{-m(m+2)t} sup(|f|).$$

Since $SU(2)$ is compact and f continuous then f is bounded, and thus by the M-test we have that the series converges absolutely and uniformly in $y \in SU(2)$. □

Now for given $f \in C(G)$ we define a function u on $(0,\infty) \times SU(2)$ as the convolution in the spacial coordinate of H and f. This amounts to

$$u(t,x) = \int_{SU(2)} H(t,xy^{-1})f(y)d\mu(y).$$

By the previous Lemma we have that the series for $H(t,xy^{-1})f(y)$ converges uniformly in y and thus we can integrate term-wise. So

$$\begin{aligned} u(t,x) &= \int\limits_{SU(2)} \sum_{m=0}^{\infty} (m+1)e^{-m(m+2)t}\chi_m(xy^{-1})f(y)d\mu(y) \\ &= \sum_{m=0}^{\infty}(m+1)e^{-m(m+2)t} \int\limits_{SU(2)} \chi_m(xy^{-1})f(y)d\mu(y). \end{aligned}$$

If we now focus on the integral we see

$$\begin{aligned} \int\limits_{SU(2)} \chi_m(xy^{-1})f(y)d\mu(y) &= \int\limits_{SU(2)} tr(\pi_m(xy^{-1}))f(y)d\mu(y) \\ &= \int\limits_{SU(2)} tr(\pi_m(x)\pi_m(y^{-1}))f(y)d\mu(y) \\ &= \int\limits_{SU(2)} tr(f(y)\pi_m(y^{-1})\pi_m(x))d\mu(y) \\ &= tr\left(\left(\int\limits_{SU(2)} f(y)\pi_m(y^{-1})d\mu(y)\right) \circ \pi_m(x)\right) \\ &= tr(\widehat{f}(m)\pi_m(x)). \end{aligned}$$

Combining these calculations we obtain

$$u(t,x) = \sum_{m=0}^{\infty}(m+1)e^{-m(m+2)t} tr(\widehat{f}(m)\pi_m(x))$$

with point-wise convergence on $(0,\infty) \times SU(2)$.

Proposition 5.7. *For $f \in C(SU(2))$,*

$$\sum_{m=0}^{\infty}(m+1)e^{-m(m+2)t}tr(\widehat{f}(m)\pi_m(x))$$

converges to $u(t,x)$ absolutely and uniformly on $[t_\circ,\infty)\times SU(2)$ for $t_\circ > 0$.

Proof. Let $t_\circ > 0$ be fixed. We calculate

$$|tr(\widehat{f}(m)\pi_m(x)| \leq |||\widehat{f}(m)|||\ |||\pi_m(x)|||$$

and now from Lemma 2.12 we have that $|||\pi_m(x)||| \leq \sqrt{m+1}$ and as $f \in C(SU(2)) \subset L^2(SU(2))$ the Plancherel Theorem (2.11) gives us $\left(|||\widehat{f}(m)|||\right)_{m=0}^{\infty}$ is bounded, $|||\widehat{f}(m)||| \leq C$ say. Thus we see

$$|(m+1)e^{-m(m+2)t}tr(\widehat{f}(m)\pi_m(x))| \leq (m+1)^{\frac{3}{2}}Ce^{-m(m+2)t_\circ}$$

and so by the M-test the above series converges absolutely and uniformly on $[t_\circ,\infty)\times SU(2)$. □

Lemma 5.8. *If $f \in C^2(SU(2))$ then u above solves the Cauchy problem for boundary data f.*

Proof. Applying Δ term-wise to the series expression for u yields

$$-\sum_{m=0}^{\infty} m(m+1)(m+2)e^{-m(m+2)t}tr(\widehat{f}(m)\pi_m(x)).$$

as $tr(\widehat{f}(m)\pi_m(x)) \in \mathcal{M}_{\pi_m}$. Now for fixed $t_\circ > 0$ we have

$$|m(m+1)(m+2)e^{-m(m+2)t}tr(\widehat{f}(m)\pi_m(x)| \leq Cm(m+1)^{\frac{3}{2}}(m+2)e^{-m(m+2)t_\circ}$$

for all $(t,x) \in [t_\circ, \infty) \times SU(2)$, just as in the proof of the previous proposition. Thus by the M-test our term-wise application of Δ is justified. Applying $\frac{\partial}{\partial t}$ to the series expression for u will give the same result and thus

$$\frac{\partial u}{\partial t} = \Delta u$$

on $(0,\infty) \times SU(2)$.

If we now set $t = 0$ then the series expression for u becomes

$$\sum_{m=0}^{\infty} (m+1) tr(\widehat{f}(m)\pi_m(x))$$

which converges uniformly to f on $SU(2)$ by Theorem 4.34 as f is a C^2 function. $\square$

Proposition 5.9. *The heat kernel H has the following properties*

1. $H(t,x) \geq 0$ *for all* $(t,x) \in (0,\infty) \times SU(2)$.
2. $\int\limits_{SU(2)} H(t,x)\, d\mu(x) = 1$.
3. *For all open neighborhoods V of I,* $\lim_{t\to 0} \int\limits_{V} H(t,x)\, d\mu(x) = 1$.

Proof.

1. It is enough to show that

$$\int\limits_{SU(2)} H(t,y) f(y) d\mu(y) \geq 0$$

for every $f \in C^2(SU(2))$ where $f \geq 0$. Lemma 5.8 shows that for such an f we have that

$$u(t,x) = \int\limits_{SU(2)} H(t, xy^{-1}) f(y^{-1}) d\mu(y) = \int\limits_{SU(2)} H(t, xy) f(y) d\mu(y)$$

solves the Cauchy problem for boundary data $\check{f} \in C^2$ where $\check{f}(x) = f(x^{-1})$. Lemma 5.3 shows us that $u \geq 0$ on $[0, \infty) \times SU(2)$ and thus, in particular,

$$u(t, I) = \int\limits_{SU(2)} H(t, y) f(y) d\mu(y) \geq 0$$

as desired.

2. As $\sum\limits_{m=0}^{\infty} (m+1) e^{-m(m+2)t} \chi_m(x)$ converges uniformly to $H(t,x)$ we can integrate term-wise

$$\int\limits_{SU(2)} H(t,x)\, d\mu(x) = \sum_{m=0}^{\infty} (m+1) e^{-m(m+2)t} \int\limits_{SU(2)} \chi_m(x)\, d\mu(x).$$

The integral $\int\limits_{SU(2)} \chi_m(x)\, d\mu(x) = \langle \chi_m, \chi_0 \rangle_2 = \delta_{m,0}$ by Theorem (2.3). Therefore

$$\int\limits_{SU(2)} H(t,x)\, d\mu(x) = (0+1)e^0 = 1.$$

3. Let V be an open neighborhood of I. We can find a $f \in C^2(SU(2))$ with $0 \leq f(x) \leq 1, f(I) = 1, f|_{V^C} = 0$, and $f(x^{-1}) = f(x)$ for all $x \in V$. Now since $f \in C^2(SU(2))$ the function $u(t,x) = \sum\limits_{m=0}^{\infty} (m+1) e^{-m(m+2)t} tr(\widehat{f}(m) \pi_m(x))$ satisfies

$$\lim_{t \to 0^+} u(t, I) = f(I) = 1$$

by Theorem 4.34. Now we calculate

$$\begin{aligned} 1 &= \lim_{t\to 0^+} \int_{SU(2)} H(t,y^{-1})f(y)d\mu(y) \\ &= \lim_{t\to 0^+} \int_{SU(2)} H(t,y)f(y)d\mu(y) \qquad (f=\check{f}) \end{aligned}$$

and

$$\begin{aligned} \int_{SU(2)} H(t,y)f(y)d\mu(y) &= \int_V H(t,y)f(y)d\mu(y), \qquad (\text{since } f|_{V^C}=0) \\ &\leq \int_V H(t,y)d\mu(y), \qquad (\text{as } 0\leq f\leq 1) \\ &\leq \int_{SU(2)} H(t,y)d\mu(y) = 1 \qquad (\text{using properties 1 and 2}). \end{aligned}$$

Finally we have $1 \leq \lim_{t\to 0^+} \int_V H(t,y)f(y)d\mu(y) \leq 1$ and thus

$$\lim_{t\to 0^+} \int_V H(t,y)f(y)d\mu(y) = 1. \quad \Box$$

The following now completes the proof for Theorem 5.2.

Theorem 5.10. *Let $f \in C(SU(2))$ be given. Then*

$$u(t,x) = \int_{SU(2)} H(t,xy^{-1})f(y)d\mu(y)$$

solves the Cauchy problem.

Proof. Since the series for $u(t,x)$ converges uniformly we can differentiate term-wise in t if the resulting series converges uniformly. Observe that formally

$$\frac{\partial u}{\partial t}(t,x) = \sum_{m=0}^{\infty} -m(m+2)(m+1)e^{-m(m+2)t} tr(\widehat{f}(m)\pi_m(x)).$$

Now for fixed $t_\circ > 0$ and all $(t,x) \in [t_\circ, \infty) \times SU(2)$

$$|-m(m+2)(m+1)e^{-m(m+2)t} tr(\widehat{f}(m)\pi_m(x))| \leq |m(m+1)^2(m+2)Ce^{-m(m+2)t_\circ}|$$

where $|||\widehat{f}(m)||| \leq C$ as before. Thus by the M-test the resulting series converges, justifying our term-wise differentiation. Next as $tr(\widehat{f}(m)\pi_m(x)) \in \mathcal{M}_\pi$ and $\Delta|_{\mathcal{M}_\pi} = -m(m+2)\, Id$ we have

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x)$$

on $(0,\infty) \times SU(2)$.

It remains to show $\lim_{t\to 0^+} u(t,x) = f(x)$. Let $\varepsilon > 0$ be given. As $f \in C(SU(2))$ and $SU(2)$ compact Proposition 2.29 yields an open neighborhood V with

$$|f(yx) - f(x)| < \varepsilon, \quad \text{for all } x \in SU(2),\ y \in V.$$

Thus also $U := V^{-1} = \{y^{-1} \ :\ y \in V\}$ is a neighborhood of I with

$$|f(y^{-1}x) - f(x)| < \varepsilon, \quad \text{for all } x \in SU(2),\ y \in U.$$

Now using the properties of the heat kernel we calculate

$$|u(t,x) - f(x)| = \left| \int_{SU(2)} H(t, xy^{-1}) f(y) d\mu(y) - f(x) \right|$$

$$
\begin{aligned}
&= \left| \int\limits_{SU(2)} H(t,y) f(y^{-1}x) d\mu(y) - f(x) \right| \qquad \text{(as } d\mu \text{ is right-invariant)} \\
&= \left| \int\limits_{SU(2)} H(t,y)(f(y^{-1}x) - f(x)) d\mu(y) \right| \qquad \left(\text{as } \int\limits_{SU(2)} H(t,y) d\mu(y) = 1\right) \\
&\leq \int\limits_{SU(2)} H(t,y) |f(y^{-1}x) - f(x)| d\mu(y) \\
&= \int\limits_{U} H(t,y) |f(y^{-1}x) - f(x)| d\mu(y) + \int\limits_{U^C} H(t,y) |f(y^{-1}x) - f(x)| d\mu(y) \\
&\leq \varepsilon \int\limits_{U} H(t,y) d\mu(y) + 2sup(|f|) \int\limits_{U^C} H(t,y) d\mu(y) \\
&\leq \varepsilon \int\limits_{SU(2)} H(t,y) d\mu(y) + 2sup(|f|) \int\limits_{U^C} H(t,y) d\mu(y) \\
&= \varepsilon + 2sup(|f|) \left(1 - \int\limits_{U} H(t,y) d\mu(y) \right).
\end{aligned}
$$

We now take the $\lim_{t\to 0^+}$ then by the third property of the heat kernel we have

$$
\lim_{t\to 0^+} \left(\varepsilon + 2sup(|f|) \left(1 - \int\limits_{U} H(t,y) d\mu(y) \right) \right) = \varepsilon.
$$

Therefore $\lim_{t\to 0^+} |u(t,x) - f(x)| \leq \varepsilon$ for all $\varepsilon > 0$ and finally

$$
\lim_{t\to 0^+} u(t,x) = f(x). \quad \square
$$

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