ABSTRACT

EIGENVALUES FOR SUMS OF HERMITIAN MATRICES

by

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In this thesis we explore how the eigenvalues of $n \times n$ Hermitian matrices A, B relate to the eigenvalues of their sum $C = A + B$. We mainly focus on inequalities bounding sums of r eigenvalues for C by sums of r eigenvalues for A with r eigenvalues for B, for some r less than n.

We begin by using linear algebra to establish some classical results, including a result by R.C. Thompson that allows us to reformulate the eigenvalue problem in terms of nonempty intersections in the Grassmannian manifold of r-planes in complex n-dimensional space. In particular, every nonempty triple intersection of Schubert varieties in a Grassmannian yields an eigenvalue inequality. Such nonempty intersections correspond to nonzero cup products in the cohomology ring of the Grassmannian, and consequently, to nonzero Littlewood-Richardson coefficients. The Littlewood-Richardson rules provide us with an efficient method of detecting when these coefficients are nonzero, and hence of finding eigenvalue inequalities which necessarily hold for all $n \times n$ Hermitian matrices $A, B, C = A + B$.

For the remainder of this thesis, we turn our attention to particular inequalities of the above form that Alfred Horn conjectured would completely determine the possible eigenvalues of $A, B, C = A + B$. Horn's conjecture, formulated in 1962, was resolved

in the affirmative during the late 1990's in celebrated work of A. Knutson and T. Tao, building on results of A. Klyachko and others. We will develop the connection between Horn's inequalities and the earlier parts of this thesis. In particular, we will see that each Horn inequality corresponds to a nonzero cup product that lies in the top degree cohomology group of the Grassmannian.

An alternate formulation of Horn's Theorem shows that indices yield a Horn inequality if and only if certain associated partitions occur as the eigenvalues for some $r \times r$ Hermitian matrices A, B, $C = A + B$. We will prove that when $r = n - 2$ there are necessarily diagonal $r \times r$ matrices satisfying this condition.

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CHAPTER 1: Introduction

1.1 Preliminaries

In this section we lay out much of our notation and develop some basic results that are needed throughout this thesis.

1.1.1 The vector space $\mathbb{C}^n$

Let $M_n(\mathbb{C})$ denote the ring of $n \times n$ matrices with complex entries. For $A \in M_n(\mathbb{C})$, we let A^* denote the conjugate transpose of A. This notation will also be used for the conjugate transpose of a vector. Thus, the conjugate transpose of a column-vector $x \in \mathbb{C}^n$ is the row-vector x^*.

The standard inner product on $\mathbb{C}^n$ is the map $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined via $\langle x, y \rangle = x^* y =$ "the dot product of x^* and y". We use this map to endow $\mathbb{C}^n$ with the structure of a Hermitian inner product space. Note that $\langle \cdot, \cdot \rangle$ is *conjugate* linear in the *first* variable. Elsewhere one may find Hermitian inner products that are conjugate linear in the second variable.

1.1.2 Hermitian matrices

A matrix $A \in M_n(\mathbb{C})$ is *Hermitian* if it is its own conjugate transpose (i.e., $A = A^*$), or equivalently, if it is self-adjoint with respect to the standard inner product. Thus for every Hermitian matrix $A \in M_n(\mathbb{C})$, we have $\langle Ax, y \rangle = \langle x, A^* y \rangle = \langle x, Ay \rangle$ for all $x, y \in \mathbb{C}^n$. In particular, this implies that

$$\langle x, Ax \rangle = \langle Ax, x \rangle = \overline{\langle x, Ax \rangle} \qquad \forall x \in \mathbb{C}^n$$

and hence that $\langle x, Ax\rangle$ is necessarily a real number. We can therefore regard the set $\text{Herm}(n)$ of $n \times n$ Hermitian matrices as partially ordered under the rule

$$A \leq B \iff \langle x, Ax\rangle \leq \langle x, Bx\rangle \text{ for all } x \in \mathbb{C}^n.$$

The *eigenvalues* of $A \in M_n(\mathbb{C})$ are defined to be the roots of its characteristic polynomial $p_A(t) = \det(A - tI)$. Equivalently, an eigenvalue of $A \in M_n(\mathbb{C})$ is a number $\mu \in \mathbb{C}$ such that $Av = \mu v$ for some nonzero vector $v \in \mathbb{C}^n$. This vector v is called an *eigenvector corresponding to the eigenvalue* μ. The linear subspace spanned by all of μ's corresponding eigenvectors is called μ's *eigenspace*.

Proposition 1.1. *Every $A \in M_n(\mathbb{C})$ has precisely n eigenvalues (taken with multiplicity).*

Proof. It is easily seen that $p_A(t)$ is a degree-n polynomial, and thus $p_A(t)$ has exactly n roots by the fundamental theorem of algebra. Therefore, by definition, A has precisely n eigenvalues (taken with multiplicity). □

Theorem 1.2 (Spectral Theorem for Hermitian Matrices)**.** *Let A be an $n \times n$ Hermitian matrix with (distinct) eigenvalues $\alpha_1, \ldots, \alpha_k$. For each j, let E_j denote the eigenspace of α_j, and let A_j denote the orthogonal projection of $\mathbb{C}^n$ onto E_j. Then the following are true:*

(1) $\mathbb{C}^n = E_1 \oplus \cdots \oplus E_k$ (orthogonal direct sum)

(2) $A = \alpha_1 A_1 + \cdots + \alpha_k A_k$.

The expression in (2) is called the spectral resolution *of A.*

Proof. See [3, p.456]. □

It follows from (1) that there is an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of A. Let $\{u_1, \ldots, u_n\}$ be such a basis with say $Au_j = \lambda_j u_j$. Then the matrix $U = [u_1|\cdots|u_n]$ with columns u_j is *unitary* (i.e. $\langle Uz, Uw\rangle = \langle z, w\rangle$ for all $z, w \in \mathbb{C}^n$) and $U^*AU = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. As U is unitary one has $UU^* = I$ and thus, in particular, we have the following.

Corollary 1.3. *Given any $A \in \operatorname{Herm}(n)$, there is an invertible matrix P and a diagonal matrix D such that $D = PAP^{-1}$.*

Proposition 1.4. *All eigenvalues of a Hermitian matrix are real.*

Proof. If λ is an eigenvalue of $A \in \operatorname{Herm}(n)$ with $Ax = \lambda x$ $(x \neq 0)$, then

$$\lambda\langle x, x\rangle = \langle x, \lambda x\rangle = \langle x, Ax\rangle = \langle Ax, x\rangle = \langle \lambda x, x\rangle = \overline{\lambda}\langle x, x\rangle.$$

Since $x \neq 0$ and the standard inner product on $\mathbb{C}^n$ is positive definite, it follows that $\lambda = \overline{\lambda}$ is real. $\square$

Notation 1.5. Let $\lambda^{\downarrow} : \operatorname{Herm}(n) \to \mathbb{R}^n$ be the map sending each $n \times n$ Hermitian matrix to the n-tuple consisting of its eigenvalues (taken with multiplicity) in weakly-decreasing order. We denote the jth entry of $\lambda^{\downarrow}(A)$ by $\lambda_j^{\downarrow}(A)$.

Example 1.6. If A is the diagonal matrix $\operatorname{diag}(3, 1, 5, 5)$, then $\lambda^{\downarrow}(A) = (5, 5, 3, 1)$, $\lambda_1^{\downarrow}(A) = \lambda_2^{\downarrow}(A) = 5$, $\lambda_3^{\downarrow}(A) = 3$, and $\lambda_4^{\downarrow}(A) = 1$.

The spectral resolution given in part (2) of Theorem 1.2 can be written as follows.

Lemma 1.7. *Let A be an $n \times n$ Hermitian matrix. Then we can write $A = \sum \alpha_j u_j u_j^*$ where $\alpha_j = \lambda_j^{\downarrow}(A)$ and $\{u_1, \ldots, u_n\}$ is an orthonormal basis of eigenvectors of A with $Au_j = \alpha_j u_j$.*

Proof. First, let $S_1 \cup \cdots \cup S_k$ be a partition of $\{1, \ldots, n\}$ such that each eigenspace E_{α_j} is the direct sum $\bigoplus_{\ell \in S_j} \mathbb{C}u_\ell$.

By the Spectral Theorem, we can write $A = \sum \alpha_j A_j$ where A_j is the orthogonal projection onto the eigenspace E_{α_j}. Therefore, we just need to show that $A_j = \sum_{\ell \in S_j} u_\ell u_\ell^*$.

Let $x = c_1 u_1 + \cdots + c_n u_n \in \mathbb{C}^n$. Then

$$A_j(x) = \sum_{\ell \in S_j} c_\ell u_\ell = \sum_{\ell \in S_j} \langle u_\ell, x \rangle u_\ell = \sum_{\ell \in S_j} u_\ell^* x u_\ell = \sum_{\ell \in S_j} u_\ell u_\ell^* x = \left(\sum_{\ell \in S_j} u_\ell u_\ell^* \right) x$$

where the fourth equality uses the fact that $u_\ell^* x$ is a scalar. Since x was arbitrary, this completes the proof. □

Proposition 1.8. *The trace of every $A \in M_n(\mathbb{C})$ equals the sum of its eigenvalues (taken with multiplicity).*

Comment. We need only the case where A is Hermitian for our purposes, so we restrict our attention to this special case. Our proof relies on the following standard results:

(a) Every Hermitian matrix is similar to a diagonal matrix (Corollary 1.3).

(b) Similar matrices have the same eigenvalues, with the same multiplicities.

(c) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any $A, B \in M_n(\mathbb{C})$. □

Proof (Hermitian case). Let $A \in \mathrm{Herm}(n)$. By (a), there is a diagonal matrix D and an invertible matrix P such that $D = PAP^{-1}$. It is clear that the trace of D is equal to the sum of its eigenvalues. Applying (b) and (c), we obtain

$$\sum \lambda_j^{\downarrow}(A) = \mathrm{tr}(D) = \mathrm{tr}(PAP^{-1}) = \mathrm{tr}(AP^{-1}P) = \mathrm{tr}(A)$$

as claimed. □

1.2 Eigenvalue inequalities

Note that the sum $A + B$ of $n \times n$ Hermitian matrices is Hermitian. We ask the following:

How do the eigenvalues of a sum of Hermitian matrices A, B relate to the eigenvalues of A and B?

This is the main problem to be addressed in this thesis.

Perhaps the most obvious result is the following:

Proposition 1.9 (The trace equality)**.** *For $A, B \in \mathrm{Herm}(n)$ one has*

$$\sum_{j=1}^{n} \lambda_j^{\downarrow}(A+B) = \sum_{j=1}^{n} \lambda_j^{\downarrow}(A) + \sum_{j=1}^{n} \lambda_j^{\downarrow}(B).$$

Proof. This follows from Proposition 1.8 in combination with the fact that $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$. □

Apart from the trace equality, there are many less obvious *inequalities* relating sums of eigenvalues for $A + B$ to sums of eigenvalues for A and B. All inequalities that will subsequently appear have the general form given by (IJK) below.

Given integers $1 \leq r \leq n$, let $\mathcal{P}_r^n$ denote the set of r-tuples $(i_1, \ldots, i_r)$ such that $i_j \in \{1, \ldots, n\}$ for each j and $i_1 < \cdots < i_r$.

Definition 1.10. For $1 \leq r \leq n$, let H_r^n denote the set of triples $(I, J, K) \in (\mathcal{P}_r^n)^3$ such that

$$\sum_{k \in K} \lambda_k^{\downarrow}(A+B) \leq \sum_{i \in I} \lambda_i^{\downarrow}(A) + \sum_{j \in J} \lambda_j^{\downarrow}(B) \tag{IJK}$$

holds for all $A, B \in \mathrm{Herm}(n)$.

1.2.1 Classical inequalities

Below are some famous eigenvalue inequalities that are special instances of Inequality (IJK). For $n \times n$ Hermitian matrices A, B, and $C = A + B$, one has

- *(Weyl Inequalities)*
$$\lambda^{\downarrow}_{i+j-1}(C) \leq \lambda^{\downarrow}_{i}(A) + \lambda^{\downarrow}_{j}(B)$$
for all i, j such that $i + j - 1 \leq n$,
- *(Lidskii Inequalities)*
$$\sum_{\ell=1}^{r} \lambda^{\downarrow}_{i_\ell}(C) \leq \sum_{\ell=1}^{r} \lambda^{\downarrow}_{i_\ell}(A) + \sum_{j=1}^{r} \lambda^{\downarrow}_{j}(B)$$
for $1 \leq i_1 < \cdots < i_r \leq n$,
- *(Thompson-Freede Inequalities)*
$$\sum_{\ell=1}^{r} \lambda^{\downarrow}_{i_\ell + j_\ell - \ell}(C) \leq \sum_{j=1}^{r} \lambda^{\downarrow}_{i_\ell}(A) + \sum_{\ell=1}^{r} \lambda^{\downarrow}_{j_\ell}(B)$$
for any indices $1 \leq i_1 < \cdots < i_r \leq n$ and $1 \leq j_1 < \cdots < j_r \leq n$ such that $i_r + j_r - r \leq n$.

These can be found in [2], [19], and elsewhere. We provide proofs of the Weyl and Lidskii inequalities in each of Chapters 2-5. The Thompson-Freede inequalities are proven in both Chapter 4 and Chapter 5.

CHAPTER 2: Eigenvalue Inequalities via Linear Algebra

In this chapter we investigate the eigenvalue problem as its earliest researchers did - being completely dependent on basic linear algebra for any results.

In Section 2.1 we establish the bare minimum needed to provide elementary proofs of the Weyl and Lidskii inequalities. These proofs are then given in Section 2.2. The goal of Section 2.3 is to establish a result by R.C. Thompson which provides a relationship between Inequality (IJK) and the intersections of particular subsets (called Schubert varieties) of the complex Grassmannian $G_r(\mathbb{C}^n)$. This relationship will supply us with a powerful new interpretation of the eigenvalue problem, which we then use to provide additional proofs of the Weyl and Lidskii inequalities in Section 2.4.

General references for this chapter include [1] and [2].

2.1 Some classical results

The initial work on the eigenvalue problem was, as most would expect, extremely reliant on basic linear algebra. In this section we develop two of the most important tools from this period - the minimax principle, stated below in Proposition 2.3, and then Weyl's monotonicity principle as an immediate corollary.

The following lemma will be useful in the subsequent proof of the minimax principle and, together with Weyl's monotonicity principle, gives us a large enough toolkit to provide simple proofs of both the Weyl and Lidskii inequalities.

Lemma 2.1. *Let $A \in \mathrm{Herm}(n)$, and let $\{x_1, \ldots, x_n\}$ be an orthonormal basis of eigenvectors of A with $Ax_j = \lambda_j^{\downarrow}(A)x_j$ for each j. Given positive integers $r \leq n$ and*

$1 \le i_1 < \cdots < i_r \le n$, *set* $S = \operatorname{span}\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$. *Then*

$$\max_{\substack{x\in S\\ \|x\|=1}} \langle x, Ax\rangle = \lambda^{\downarrow}_{i_1}(A) \qquad \textit{and} \qquad \min_{\substack{x\in S\\ \|x\|=1}} \langle x, Ax\rangle = \lambda^{\downarrow}_{i_r}(A)$$

Proof. Choose $x \in S$ with $\|x\| = 1$. We have, say, $x = a_1 x_{i_1} + \cdots + a_r x_{i_r} \in S$, and we see that

$$1 = \|x\| = \langle x, x\rangle = \left\langle \sum_{j=1}^{r} a_j x_{i_j}\ ,\ \sum_{j=1}^{r} a_j x_{i_j} \right\rangle = \sum_{j,\ell=1}^{r} \overline{a_j} a_\ell \langle x_{i_j}, x_{i_\ell}\rangle = \sum_{j=1}^{r} |a_j|^2.$$

Consequently,

$$\begin{aligned}
\langle x, Ax\rangle &= \left\langle a_1 x_{i_1} + \cdots + a_r x_{i_r}\ ,\ a_1 \lambda^{\downarrow}_{i_1}(A) x_{i_1} + \cdots + a_r \lambda^{\downarrow}_{i_r}(A) x_{i_r} \right\rangle \\
&= \sum_{j,\ell=1}^{r} \overline{a_j} a_\ell \lambda^{\downarrow}_{i_\ell}(A) \langle x_{i_j}, x_{i_\ell}\rangle = \sum_{j=1}^{r} |a_j|^2 \lambda^{\downarrow}_{i_j}(A) \\
&\le \lambda^{\downarrow}_{i_1}(A) \sum_{j=1}^{r} |a_j|^2 \\
&= \lambda^{\downarrow}_{i_1}(A),
\end{aligned}$$

and it follows that

$$\max_{\substack{x\in S\\ \|x\|=1}} \langle x, Ax\rangle \le \lambda^{\downarrow}_{i_1}(A).$$

Moreover, as $x_{i_1} \in S$ and $\langle x_{i_1}, Ax_{i_1}\rangle = \lambda^{\downarrow}_{i_1}(A)$, we see that also

$$\max_{\substack{x\in S\\ \|x\|=1}} \langle x, Ax\rangle \ge \lambda^{\downarrow}_{i_1}(A),$$

which establishes the first equation. A similar argument works for the second. $\square$

Definition 2.2. The Grassmannian $G_r(\mathbb{C}^n)$ is the set

$$G_r(\mathbb{C}^n) = \{L \ : \quad L \text{ is a subspace of } \mathbb{C}^n \text{ with } \dim_{\mathbb{C}}(L) = r\}$$

of all r dimensional subspaces of $\mathbb{C}^n$.

Proposition 2.3 (Minimax Principle). *If $A \in \mathrm{Herm}(n)$, and $1 \leq \ell \leq n$, then*

$$\lambda_\ell^{\downarrow}(A) = \max_{V \in G_\ell(\mathbb{C}^n)} \min_{\substack{x \in V \\ \|x\|=1}} \langle x, Ax \rangle \tag{2.1}$$

$$= \min_{V \in G_{n+1-\ell}(\mathbb{C}^n)} \max_{\substack{x \in V \\ \|x\|=1}} \langle x, Ax \rangle \tag{2.2}$$

Proof. Similar arguments are needed to prove both equations, so we prove only Equation (2.1).

Let $\{x_1, \ldots, x_n\}$ be an orthonormal basis of eigenvectors for A with $Ax_j = \lambda_j^{\downarrow}(A)x_j$ for each j. Let V be any ℓ-dimensional subspace of $\mathbb{C}^n$, and define $W = \mathrm{span}\{x_\ell, \ldots, x_n\}$. Since $\dim V + \dim W = n + 1$, the intersection $V \cap W$ is nonzero. So if x is any unit vector in this intersection then we have $\langle x, Ax \rangle \in [\lambda_n^{\downarrow}(A), \lambda_\ell^{\downarrow}(A)]$ by Lemma 2.1, and it follows that

$$\min_{\substack{x \in V \\ \|x\|=1}} \langle x, Ax \rangle \leq \min_{\substack{x \in V \cap W \\ \|x\|=1}} \langle x, Ax \rangle \leq \lambda_\ell^{\downarrow}(A).$$

On the other hand, taking $V = \mathrm{span}\{x_1, \ldots, x_\ell\} \in G_\ell(\mathbb{C}^n)$ gives

$$\min_{\substack{x \in V \\ \|x\|=1}} \langle x, Ax \rangle = \lambda_\ell^{\downarrow}(A),$$

completing the proof. □

Recall from the introduction that we defined $A \leq B$ for $A, B \in \mathrm{Herm}(n)$ if and only if $\langle x, Ax \rangle \leq \langle x, Bx \rangle$ for all $x \in \mathbb{C}^n$.

Corollary 2.4 (Weyl's monotonicity principle)**.** *If $A, B \in \mathrm{Herm}(n)$ and $A \leq B$, then $\lambda_j^{\downarrow}(A) \leq \lambda_j^{\downarrow}(B)$ for each $j = 1, \ldots, n$.*

Proof. By the minimax principle,

$$\lambda_j^{\downarrow}(A) = \max_{V \in G_j(\mathbb{C}^n)} \min_{\substack{x \in V \\ \|x\|=1}} \langle x, Ax \rangle \leq \max_{V \in G_j(\mathbb{C}^n)} \min_{\substack{x \in V \\ \|x\|=1}} \langle x, Bx \rangle = \lambda_j^{\downarrow}(B). \qquad \square$$

2.2 Applications

We are now already able to give our first proofs of the Weyl and Lidskii inequalities, which were mentioned in the introduction.

Proposition 2.5 (Weyl inequalities)**.** *If $A, B \in \mathrm{Herm}(n)$, then*

$$\lambda_{i+j-1}^{\downarrow}(A + B) \leq \lambda_i^{\downarrow}(A) + \lambda_j^{\downarrow}(B)$$

for all $i, j \in \mathbb{N}$ such that $i + j - 1 \leq n$.

Proof. Let $\{u_1, \ldots, u_m\}$, $\{v_1, \ldots, v_n\}$, $\{w_1, \ldots, w_n\}$ be orthonomal bases for $\mathbb{C}^n$ consisting of eigenvectors for A, B, and $A + B$, respectively, with

$$Au_i = \lambda_i^{\downarrow}(A)u_i, \quad Bv_i = \lambda_i^{\downarrow}(B)v_i, \quad (A + B)w_i = \lambda_i^{\downarrow}(A + B)w_i.$$

Let $1 \leq i, j \leq n$ be integers such that $k := i + j - 1 \leq n$. Define

$$U = \mathrm{span}\{u_i, \ldots, u_n\}, \quad V = \mathrm{span}\{v_j, \ldots, v_n\}, \quad W = \mathrm{span}\{w_1, \ldots, w_k\}.$$

As $\dim(U) + \dim(V) + \dim(W) = 2n + 1$, it follows that $U \cap V \cap W$ is nonzero. Indeed,

$$\dim(U \cap V \cap W) = \dim(U) + \dim(V \cap W) - \dim(U + (V \cap W))$$

$$\begin{aligned} &= \dim(U) + \dim(V) + \dim(W) - \dim(V+W) - \dim(U + (V \cap W)) \\ &\geq \dim(U) + \dim(V) + \dim(W) - 2n \\ &= 1. \end{aligned}$$

We can therefore choose a unit vector x from this intersection. Lemma 2.1 tells us that $\langle x, Ax\rangle \in [\lambda_n^{\downarrow}(A), \lambda_i^{\downarrow}(A)]$, $\langle x, Bx\rangle \in [\lambda_n^{\downarrow}(B), \lambda_j^{\downarrow}(B)]$, and $\langle x, (A+B)x\rangle \in [\lambda_k^{\downarrow}(A+B), \lambda_1^{\downarrow}(A+B)]$. Thus,

$$\begin{aligned} \lambda_k^{\downarrow}(A+B) &\leq \langle x, (A+B)x\rangle \\ &= \langle x, Ax\rangle + \langle x, Bx\rangle \\ &\leq \lambda_i^{\downarrow}(A) + \lambda_j^{\downarrow}(B) \end{aligned}$$

as was needed. □

We now turn our attention to the Lidskii inequalities. The following lemma will be useful in their proof.

Lemma 2.6. *Suppose* $A, B \in \mathrm{Herm}(n)$*, and that we are given integers* $1 \leq r \leq n$ *and* $1 \leq i_1 < \cdots < i_r \leq n$*. If*

$$\sum_{j=1}^{r} \left[\lambda_{i_j}^{\downarrow}(A+B) - \lambda_{i_j}^{\downarrow}(A)\right] \leq \sum_{j=1}^{r} \lambda_j^{\downarrow}(B), \tag{2.3}$$

then it follows that

$$\sum_{j=1}^{r} \left[\lambda_{i_j}^{\downarrow}(A+B+\lambda_r^{\downarrow}(B)I) - \lambda_{i_j}^{\downarrow}(A)\right] \leq \sum_{j=1}^{r} \lambda_j^{\downarrow}(B + \lambda_r^{\downarrow}(B)I).$$

Proof. Suppose that Inequality (2.3) holds. Then

$$\begin{aligned}\sum_{j=1}^{r}\left[\lambda^{\downarrow}_{i_j}(A+B+\lambda^{\downarrow}_{r}(B)I)-\lambda^{\downarrow}_{i_j}(A)\right] &= \sum_{j=1}^{r}\left[\lambda^{\downarrow}_{i_j}(A+B)+\lambda^{\downarrow}_{r}(B)-\lambda^{\downarrow}_{i_j}(A)\right]\\ &= r\lambda^{\downarrow}_{r}(B)+\sum_{j=1}^{r}\left[\lambda^{\downarrow}_{i_j}(A+B)-\lambda^{\downarrow}_{i_j}(A)\right]\\ &\le r\lambda^{\downarrow}_{r}(B)+\sum_{j=1}^{r}\lambda^{\downarrow}_{j}(B)\\ &= \sum_{j=1}^{r}\left[\lambda^{\downarrow}_{j}(B)+\lambda^{\downarrow}_{r}(B)\right]\\ &= \sum_{j=1}^{r}\left[\lambda^{\downarrow}_{j}(B+\lambda^{\downarrow}_{r}(B)I)\right].\end{aligned}$$

Here the first and last equalities follow from the fact that μ is an eigenvalue of a matrix M if and only if $\mu + c$ is an eigenvalue of $M + cI$. □

We now follow the proof of the Lidskii inequalities given in [16]. It should be noted that of the many published elementary proofs of the Lidskii inequalities, this one is perhaps the simplest.[1]

Proposition 2.7 (Lidskii inequalities)**.** *Let* $A, B \in \mathrm{Herm}(n)$*. Given integers* $1 \le r \le n$ *and* $1 \le i_1 < \cdots < i_r \le n$*, one has*

$$\sum_{j=1}^{r}\lambda^{\downarrow}_{i_j}(A+B) \le \sum_{j=1}^{r}\lambda^{\downarrow}_{i_j}(A)+\sum_{j=1}^{r}\lambda^{\downarrow}_{j}(B).$$

Proof. We need to prove that

$$\sum_{j=1}^{r}\left[\lambda^{\downarrow}_{i_j}(A+B)-\lambda^{\downarrow}_{i_j}(A)\right] \le \sum_{j=1}^{r}\lambda^{\downarrow}_{j}(B). \tag{2.4}$$

[1] The subsequent proofs given in this thesis for Lidskii's inequalities are much shorter but rely on more machinery.

Replacing B by $B - \lambda_r^{\downarrow}(B)I$, we can assume, by the previous lemma, that $\lambda_r^{\downarrow}(B) = 0$ and hence also $\lambda_j^{\downarrow}(B) \geq 0$ for $j = 1, \ldots, r$.

Write $B = B_+ - B_-$, where B_+ and B_- are the positive and negative parts of B. That is, if B has spectral resolution $B = \sum \beta_j v_j v_j^*$, then

$$B_+ = \sum \max(\beta_j, 0) v_j v_j^* \quad \text{and} \quad B_- = \sum \max(-\beta_j, 0) v_j v_j^*.$$

So for all x,

$$\langle x, (A+B)x\rangle = \langle x, (A+B_+-B_-)x\rangle = \langle x, (A+B_+)x\rangle - \langle x, B_- x\rangle \leq \langle x, (A+B_+)x\rangle,$$

implying that $A + B \leq A + B_+$. So by Weyl's monotonicity principle, $\lambda_{i_j}^{\downarrow}(A+B) \leq \lambda_{i_j}^{\downarrow}(A+B_+)$ for all j, and thus

$$\sum_{j=1}^{r} \left[\lambda_{i_j}^{\downarrow}(A+B) - \lambda_{i_j}^{\downarrow}(A)\right] \leq \sum_{j=1}^{r} \left[\lambda_{i_j}^{\downarrow}(A+B_+) - \lambda_{i_j}^{\downarrow}(A)\right]. \tag{2.5}$$

Weyl's monotonicity principle also gives us that $\lambda_j^{\downarrow}(A+B_+) \geq \lambda_j^{\downarrow}(A)$ for each j, from which it follows that

$$\text{RHS}(2.5) \leq \sum_{j=1}^{n} \left[\lambda_j^{\downarrow}(A+B_+) - \lambda_j^{\downarrow}(A)\right] = \text{tr}(A+B_+) - \text{tr}(A) = \text{tr}(B_+). \tag{2.6}$$

The first equality here follows from Proposition 1.8. Now, $\text{tr}(B_+)$ is the sum of its eigenvalues, which is the sum of the nonnegative eigenvalues of B. As here $\lambda_r^{\downarrow}(B) = 0$ by assumption this gives

$$\text{tr}(B_+) = \sum_{j=1}^{r} \lambda_j^{\downarrow}(B). \tag{2.7}$$

Inequality (2.4) now follows from (2.5), (2.6), and (2.7). $\square$

The original proof of the Thompson-Freede inequalities is *almost*[2] accessible with what we have developed so far. However, this proof is quite cumbersome, so we simply refer the reader to [19] for its proof.

2.3 Eigenvalue inequalities via intersections in the Grassmannian

In view of Lemma 2.1 and the previous proof of the Weyl inequalities, it seems as though the eigenvalue problem can likely be formulated in terms of intersections of certain subsets of the Grassmannian $G_r(\mathbb{C}^n)$. This is indeed the case, and in a rather remarkable way: a result by R.C. Thompson (Theorem 2.18) shows that one has an (IJK) eigenvalue inequality for $A, B \in \mathrm{Herm}(n)$ whenever a particular intersection in the Grassmannian is nontrivial.

The next few results are prerequisites for stating and proving this correspondence between eigenvalue inequalities and intersections in the Grassmannian.

Lemma 2.8. *Let $A \in \mathrm{Herm}(n)$, and let $r \leq n$ be a positive integer. Given integers $1 \leq i_1 < \cdots < i_r \leq n$, there is a sequence $V_{i_1} \subset \cdots \subset V_{i_r}$ of subspaces of $\mathbb{C}^n$ such that $\dim(V_{i_j}) = i_j$ and*

$$\sum_{j=1}^{r} \langle x_{i_j}, Ax_{i_j} \rangle \geq \sum_{j=1}^{r} \lambda^{\downarrow}_{i_j}(A). \tag{2.8}$$

for every orthonormal set $\{x_{i_1}, \ldots, x_{i_r} : x_{i_j} \in V_{i_j}\}$.

Proof. Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis of eigenvectors of A with $Au_j = \lambda^{\downarrow}_j(A)u_j$ for each j, and let $V_{i_j} = \mathrm{span}\{u_1, u_2, \ldots, u_{i_j}\}$. Given any orthonormal set $\{x_{i_1}, \ldots, x_{i_r} : x_{i_j} \in V_{i_j}\}$, we have

$$\langle x_{i_j}, Ax_{i_j} \rangle \geq \min_{\substack{x \in V_{i_j} \\ \|x\|=1}} \langle x, Ax \rangle = \lambda^{\downarrow}_{i_j}(A),$$

[2] The only prerequisite that we have not developed is "Cauchy's interlacing theorem", which is a straightforward consequence of the minimax principle.

by Lemma 2.1, and hence

$$\sum_{j=1}^{r} \langle x_{i_j}, Ax_{i_j} \rangle \geq \sum_{j=1}^{r} \lambda_{i_j}^{\downarrow}(A). \qquad \square$$

Definition 2.9. Let $A \in M_n(\mathbb{C})$, and let L be a k-dimensional subspace of $\mathbb{C}^n$. Then the *compression of* A *along* L is the operator $A_L : L \to L$ defined by $A_L = P_L A|_L$. Here $A|_L : L \to \mathbb{C}^n$ is the restriction of A to L, and P_L is the orthogonal projection of $\mathbb{C}^n$ onto L.

Lemma 2.10. *Let* $L \in G_r(\mathbb{C}^n)$, *and* $A \in \mathrm{Herm}(n)$. *Given any orthonormal basis* $\mathcal{B} = \{x_1, \ldots, x_r\}$ *for* L, *we have*

$$tr(A_L) = \sum_{j=1}^{r} \langle x_j, Ax_j \rangle.$$

Proof. Recall that the trace of a linear operator on a finite-dimensional vector space is the trace of its matrix with respect to a basis. This does not depend on the choice of basis since matrices with respect to different bases are similar. As $\mathcal{B}$ is an orthonormal basis for L, we have

$$A_L(x_j) = \sum_{i=1}^{r} \langle x_i, A_L(x_j) \rangle x_i$$

for each $j = 1, \ldots, r$. Thus the matrix for A_L with respect to $\mathcal{B}$ has entries

$$c_{i,j} = \langle x_i, A_L(x_j) \rangle \quad (1 \leq i, j \leq r).$$

So as

$$\langle x_i, A_L(x_j) \rangle = \langle x_i, PAx_j \rangle = \langle P^* x_i, Ax_j \rangle = \langle Px_i, Ax_j \rangle = \langle x_i, Ax_j \rangle,$$

we see that

$$\mathrm{tr}(A_L) = \sum_{j=1}^{r} c_{j,j} = \sum_{j=1}^{r} \langle x_j, Ax_j \rangle$$

as claimed. □

Proposition 2.11 (Hersch-Zwahlen)**.** *Given $A \in \mathrm{Herm}(n)$, let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of eigenvectors for A with $Av_i = \lambda_i^{\downarrow}(A)v_i$. Set $V_m = \mathrm{span}\{v_1, \ldots, v_m\}$. Then given any sequence $1 \leq i_1 < \cdots < i_r \leq n$, we have*

$$\sum_{j=1}^{r} \lambda_{i_j}^{\downarrow}(A) = \min_{L \in G_r(\mathbb{C}^n)} \{tr(A_L) : \dim(L \cap V_{i_j}) \geq j \ \ for \ \ j = 1, \ldots, r\}. \tag{2.9}$$

Proof. Choose $L \in G_r(\mathbb{C}^n)$ with $\dim(L \cap V_{i_j}) \geq j$ for $j = 1, \ldots, r$. Since $\dim(L \cap V_{i_1}) \geq 1$, there is some unit vector $x_1 \in L \cap V_{i_1}$ and

$$\langle x_1, Ax_1 \rangle \geq \min_{\substack{x \in V_{i_1} \\ \|x\|=1}} \langle x, Ax \rangle = \lambda_{i_1}^{\downarrow}(A),$$

by Lemma 2.1. In general (for $j = 1, \ldots, r$), since $\dim(L \cap V_{i_j}) \geq j$ there is some unit vector $x_j \in L \cap V_{i_j}$ that is orthogonal to $x_1, \ldots, x_{j-1}$, and we have

$$\langle x_j, Ax_j \rangle \geq \min_{\substack{x \in V_{i_j} \\ \|x\|=1}} \langle x, Ax \rangle = \lambda_{i_j}^{\downarrow}(A). \tag{2.10}$$

Carrying out this process for $j = 1, \ldots, r$, we obtain an orthonormal basis $\{x_1, \ldots, x_r\}$ for L, and Equation (2.10) together with Lemma 2.10 give

$$\sum_{j=1}^{r} \lambda_{i_j}^{\downarrow}(A) \leq \sum_{j=1}^{r} \langle x_j, Ax_j \rangle = \mathrm{tr}(A_L).$$

This establishes that "$\leq$" holds in Equation (2.9). On the other hand, taking $L =$

$\mathrm{span}\{v_{i_1}, \ldots, v_{i_r}\}$ gives us

$$\mathrm{tr}(A_L) = \sum_{j=1}^{r} \langle v_{i_j}, Av_{i_j} \rangle = \sum_{j=1}^{r} \lambda_{i_j}^{\downarrow}(A)$$

since $\{v_{i_1}, \ldots, v_{i_r}\}$ is an orthonormal basis for L. This completes the proof. $\square$

Definition 2.12. A *complete flag* is a nested sequence of subspaces

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n$$

with $\dim_{\mathbb{C}} V_j = j$ for each j. Whenever the term "flag" is used, one should assume that we are referring to a complete flag.

Definition 2.13. Let $A \in \mathrm{Herm}(n)$, and let $\{u_1, \ldots, u_n\}$ be an orthonormal basis of eigenvectors of A such that $Au_j = \lambda_j^{\downarrow}(A)u_j$ for each j. Setting $U_j = \mathrm{span}\{u_1, \ldots, u_j\}$ for each j, we obtain a complete flag

$$\mathcal{F}_A : \{0\} = U_0 \subset U_1 \subset \cdots \subset U_{n-1} \subset U_n = \mathbb{C}^n$$

which we call an *eigenflag* for A. Defining $U'_j = U_{n-j}^{\perp} = \mathrm{span}\{u_n, \ldots, u_{n-j+1}\}$, we obtain another flag

$$\mathcal{F}'_A : \{0\} = U'_0 \subset U'_1 \subset \cdots \subset U'_{n-1} \subset U'_n = \mathbb{C}^n$$

called the *complementary eigenflag* for A.

Note that the eigenflag for A is unique if and only if all n eigenvalues of A are distinct. Indeed, if all of its eigenvalues are distinct, then the eigenspace of each $\lambda_j^{\downarrow}(A)$ has dimension 1, and so all of A's eigenvectors corresponding to $\lambda_j^{\downarrow}(A)$ are scalar multiples of each other. Uniqueness of $\mathcal{F}_A$ is therefore clear. On the other hand, if say $\lambda^{\downarrow}_{\ell}(A) = \lambda^{\downarrow}_{\ell+1}(A)$ then one obviously has $\text{span}\{u_1, \dots, u_\ell\} \neq \text{span}\{u_1, \dots, u_{\ell-1}, u_{\ell+1}\}$ by the orthogonality and linear independence of the u_i.

Lemma 2.14. *If $\mathcal{F}_A$ is an eigenflag for $A \in \text{Herm}(n)$ then the complementary eigenflag $\mathcal{F}'_A$ is an eigenflag for the matrix $-A$.*

Proof. Let $\{u_1, \dots, u_n\}$ be an orthonormal basis of eigenvectors for A with $Au_j = \lambda^{\downarrow}_j(A)u_j$ for each j. Set $U_j = \text{span}\{u_1, \dots, u_j\}$ and $U'_j = \text{span}\{u_n, \dots, u_{n-j+1}\}$ to obtain the eigenflag $\mathcal{F}_A : \{0\} = U_0 \subset U_1 \subset \cdots \subset U_{n-1} \subset U_n = \mathbb{C}^n$ and complementary eigenflag $\mathcal{F}'_A : \{0\} = U'_0 \subset U'_1 \subset \cdots \subset U'_{n-1} \subset U'_n = \mathbb{C}^n$ for the matrix A. As $-Au_j = -\lambda^{\downarrow}(A)u_j$ for each j, we see that $-A$ has eigenvalues $\lambda^{\downarrow}(-A) = \{-\lambda^{\downarrow}_n(A) \geq \cdots \geq -\lambda^{\downarrow}_1(A)\}$. So

$$\lambda^{\downarrow}_j(-A) = -\lambda^{\downarrow}_{n-j+1}(A) \tag{2.11}$$

for each j, and $\{u_n, u_{n-1}, \dots, u_1\}$ is an orthonormal basis of eigenvectors for $-A$ with $-Au_{n-j+1} = \lambda^{\downarrow}_j(-A)u_{n-j+1}$. Thus $\mathcal{F}'_A$ is an eigenflag for the matrix $-A$. □

Definition 2.15. Let $\mathcal{F} : \{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n$ be a complete flag, $r \leq n$ a positive integer, and $I = (i_1 < \cdots < i_r) \in \mathcal{P}^n_r$. The subset

$$S(I, \mathcal{F}) = \left\{L \in G_r(\mathbb{C}^n) : \dim(L \cap V_{i_j}) \geq j \text{ for } 1 \leq j \leq r\right\}$$

of the Grassmannian $G_r(\mathbb{C}^n)$ is called a *Schubert variety.*

This new notation can be used to recast Proposition 2.11 as follows.

Proposition 2.16. *For $A \in \mathrm{Herm}(n)$ and $I \in \mathcal{P}_r^n$ one has*

$$\sum_{i\in I} \lambda_i^{\downarrow}(A) = \min\bigl\{ tr(A_L) \ : \ L \in S(I, \mathcal{F}_A)\bigr\}, \tag{2.12}$$

where $\mathcal{F}_A$ denotes an eigenflag for A.

Definition 2.17. Given $I \in \mathcal{P}_r^n$, we define its *complementary indices* to be the r-tuple $I' := (i_r' < \cdots < i_1') \in \mathcal{P}_r^n$ where $\ell' := n - \ell + 1$ for values $\ell \in \{1, \ldots, n\}$.

The following result by R.C. Thompson establishes the previously mentioned correspondence between (IJK) inequalities and intersections in the Grassmannian.

Theorem 2.18. *Given $I, J, K \in \mathcal{P}_r^n$ and Hermitian matrices A, B, and $C = A + B$ in $M_n(\mathbb{C})$,*

$$\sum_{k\in K} \lambda_k^{\downarrow}(C) \le \sum_{i\in I} \lambda_i^{\downarrow}(A) + \sum_{j\in J} \lambda_j^{\downarrow}(B) \tag{2.13}$$

holds whenever

$$S(I', \mathcal{F}_A') \cap S(J', \mathcal{F}_B') \cap S(K, \mathcal{F}_C) \neq \emptyset$$

for some eigenflag $\mathcal{F}_C$ for C and complementary eigenflags $\mathcal{F}_A', \mathcal{F}_B'$ for A, B.

Proof. We follow the proof given in [2]. See also [10, Lemma 4.2]. Suppose that $S(I', \mathcal{F}_A') \cap S(J', \mathcal{F}_B') \cap S(K, \mathcal{F}_C) \neq \emptyset$ and choose $L_\circ \in S(I', \mathcal{F}_A') \cap S(J', \mathcal{F}_B') \cap S(K, \mathcal{F}_C)$. As

$$-A - B + C = O$$

we have

$$0 = \mathrm{tr}(-A_{L_\circ}) + \mathrm{tr}(-B_{L_\circ}) + \mathrm{tr}(C_{L_\circ}).$$

Lemma 2.14 shows that $\mathcal{F}'_A$ is an eigenflag for $-A$ and hence

$$\sum_{i\in I}\lambda^{\downarrow}_{i'}(-A)=\min\big\{\operatorname{tr}(-A_L)\ :\ L\in S(I',\mathcal{F}'_A)\big\}$$

by Equation (2.12). As $L_\circ \in S(I',\mathcal{F}'_A)$ this yields

$$\sum_{i\in I}\lambda^{\downarrow}_{i'}(-A)\le \operatorname{tr}(-A_{L_\circ}).$$

Likewise,

$$\sum_{j\in J}\lambda^{\downarrow}_{j'}(-B)\le \operatorname{tr}(-B_{L_\circ})\quad\text{and}\quad \sum_{k\in K}\lambda^{\downarrow}_{k}(C)\le \operatorname{tr}(C_{L_\circ}).$$

Thus,

$$0=\operatorname{tr}(-A_{L_\circ})+\operatorname{tr}(-B_{L_\circ})+\operatorname{tr}(C_{L_\circ})\ge \sum_{i\in I}\lambda^{\downarrow}_{i'}(-A)+\sum_{j\in J}\lambda^{\downarrow}_{j'}(-B)+\sum_{k\in K}\lambda^{\downarrow}_{k}(C)$$

and hence

$$\sum_{k\in K}\lambda^{\downarrow}_{k}(C)\le -\sum_{i\in I}\lambda^{\downarrow}_{i'}(-A)-\sum_{j\in J}\lambda^{\downarrow}_{j'}(-B).$$

But Equation (2.11) gives

$$\sum_{i\in I}\lambda^{\downarrow}_{i'}(-A)=\sum_{\ell=1}^{r}\lambda^{\downarrow}_{n-i_\ell+1}(-A)=\sum_{\ell=1}^{r}-\lambda^{\downarrow}_{i_\ell}(A)=-\sum_{i\in I}\lambda^{\downarrow}_{i}(A)$$

and similarly,

$$\sum_{j\in J}\lambda^{\downarrow}_{j'}(-B)=-\sum_{j\in J}\lambda^{\downarrow}_{j}(B).$$

Therefore,

$$\sum_{k\in K}\lambda^{\downarrow}_{k}(C)\le \sum_{i\in I}\lambda^{\downarrow}_{i}(A)+\sum_{j\in J}\lambda^{\downarrow}_{j}(B)$$

as stated. $\square$

Corollary 2.19. *Given $I, J, K \in \mathcal{P}_r^n$, suppose that*

$$S(I', \mathcal{F}_1) \cap S(J', \mathcal{F}_2) \cap S(K, \mathcal{F}_3) \neq \emptyset$$

for all complete flags $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ in $\mathbb{C}^n$. Then Inequality (IJK) holds for all $n \times n$ Hermitian matices A, B, $C = A + B$. That is, $(I, J, K) \in H_r^n$.

2.4 Applications of Theorem 2.18 and Corollary 2.19

We now use the ideas from the previous section to give additional proofs of both Weyl's and Lidskii's inequalities.

Example 2.20 (The Weyl inequalities)**.** We show that if $1 \leq i, j \leq n$ and $i+j-1 \leq n$, then

$$S((n+1-i), \mathcal{F}_1) \cap S((n+1-j), \mathcal{F}_2) \cap S((i+j-1), \mathcal{F}_3) \neq \emptyset$$

for any complete flags $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ in $\mathbb{C}^n$. The Weyl inequalities then follow from Corollary 2.19.

So, let $\mathcal{F}_1 : U_1 \subset \cdots \subset U_n$, $\mathcal{F}_2 : V_1 \subset \cdots \subset V_n$, and $\mathcal{F}_3 : W_1 \subset \cdots \subset W_n$ be complete flags. We just need to prove that there is necessarily a 1-dimensional subspace Λ contained in the intersection $U_{n+1-i} \cap V_{n+1-j} \cap W_{i+j-1}$. That such a subspace always exists follows immediately from the following calculation:

$$\begin{aligned}\dim(U_{n+1-i} \cap V_{n+1-j} \cap W_{i+j-1}) \geq\ & \dim(U_{n+1-i}) + \dim(V_{n+1-j}) + \dim(W_{i+j-1}) - 2n \\ &= (n+1-i) + (n+1-j) + (i+j-1) - 2n \\ &= 1.\end{aligned}$$

□

The following two propositions are needed in the proof of Lidskii's inequality in

Example 2.23. Since these ideas come up later in a different context (Grassmann cohomology), we present them here separately for future reference.

Proposition 2.21. *For $J = (1, 2, \ldots, r) \in \mathcal{P}_r^n$, one has $S(J', \mathcal{F}) = G_r(\mathbb{C}^n)$ for all complete flags $\mathcal{F} : V_1 \subset \cdots \subset V_n$ of $\mathbb{C}^n$.*

Proof. Since $J' = (n + 1 - r, \ldots, n) = (n - r + \ell)_{\ell=1}^r$, we have

$$S(J', \mathcal{F}) = \{\Lambda \in G_r(\mathbb{C}^n) : \dim(\Lambda \cap V_{n-r+\ell}) \geq \ell \text{ for } \ell = 1, \ldots, r\}.$$

If Λ is any r-dimensional subspace of $\mathbb{C}^n$, then

$$\begin{aligned} \dim(\Lambda \cap V_{n-r+\ell}) &\geq \dim(\Lambda) + \dim(V_{n-r+\ell}) - n \\ &= r + (n - r + \ell) - n \\ &= \ell, \end{aligned}$$

and we see that $\Lambda \in S(J', \mathcal{F})$. Hence $S(J', \mathcal{F}) = G_r(\mathbb{C}^n)$. □

Proposition 2.22. *If $I = (i_1 < \cdots < i_r) \in \mathcal{P}_r^n$ and $\mathcal{F}_1, \mathcal{F}_2$ are complete flags for $\mathbb{C}^n$, then*

$$S(I, \mathcal{F}_1) \cap S(I', \mathcal{F}_2) \neq \emptyset.$$

Proof. Lemma 3.11 together with Theorem A.14, proved later in this thesis, show that it suffices to check that $S(I, \mathcal{F}) \cap S(I', \mathcal{F}') \neq \emptyset$ for some complete flag $\mathcal{F}$ and its complementary flag $\mathcal{F}'$. Write say $\mathcal{F} : \{0\} \subset U_1 \subset \cdots \subset U_n = \mathbb{C}^n$ where $U_j = \text{span}\{u_1, \ldots, u_j\}$ for each j. The complementary flag is then $\mathcal{F}' : \{0\} \subset U_1' \subset \cdots \subset U_n' = \mathbb{C}^n$ where $U_j' = \text{span}\{u_n, \ldots, u_{n-j+1}\}$. Also, the complementary indices

I' for $I = (i_1, \ldots, i_r)$ are

$$I' = (\widetilde{i}_1, \ldots, \widetilde{i}_r) = (n - i_r + 1, \ldots, n - i_1 + 1) = \big(n + 1 - i_{r-j+1}\big)_{j=1}^{r}.$$

Thus,

$$S(I, \mathcal{F}) = \{\Lambda \in G_r(\mathbb{C}^n) : \dim(\Lambda \cap U_{i_j}) \geq j \text{ for } j = 1, \ldots, r\},$$
$$S(I', \mathcal{F}') = \{\Lambda \in G_r(\mathbb{C}^n) : \dim(\Lambda \cap U'_{\widetilde{i}_j}) \geq j \text{ for } j = 1, \ldots, r\}$$

where

$$U'_{\widetilde{i}_j} = \operatorname{span}\{u_n, \ldots, u_{n-\widetilde{i}_j+1}\} = \operatorname{span}\{u_n, \ldots, u_{i_{r-j+1}}\}.$$

Now, the subspace $L = \operatorname{span}\{u_{i_\ell} : \ell = 1, \ldots, r\}$ lies in $G_r(\mathbb{C}^n)$, and we have both $\dim(L \cap U_{i_j}) = j$ and $\dim(L \cap U'_{\widetilde{i}_j}) = j$, the latter equation following from the observation that

$$\begin{aligned} L \cap U'_{\widetilde{i}_j} &= \operatorname{span}\{u_{i_1}, \ldots, u_{i_r}\} \cap \operatorname{span}\{u_n, \ldots, u_{i_{r-j+1}}\} \\ &= \operatorname{span}\{u_{i_\ell} : r - j + 1 \leq \ell \leq r\} \end{aligned}$$

which has dimension $r + 1 - (r - j + 1) = j$. Hence $L \in S(I, \mathcal{F}) \cap S(I', \mathcal{F}')$. $\square$

Example 2.23 (The Lidskii inequalities)**.** Let I and $J = (1, \ldots, r)$ belong to $\mathcal{P}_r^n$. By Corollary 2.19, the Lidskii inequality

$$\sum_{i \in I} \lambda_i^{\downarrow}(C) \leq \sum_{i \in I} \lambda_i^{\downarrow}(A) + \sum_{j \in J} \lambda_j^{\downarrow}(B)$$

holds for all $A, B, C = A + B \in \operatorname{Herm}(n)$ if $S(I', \mathcal{F}_1) \cap S(J', \mathcal{F}_2) \cap S(I, \mathcal{F}_3) \neq \emptyset$ for all complete flags $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ for $\mathbb{C}^n$. Applying Propositions 2.21 and 2.22, we find that

$$\begin{aligned} S(I',\mathcal{F}_1) \cap S(J',\mathcal{F}_2) \cap S(I,\mathcal{F}_3) &= S(I',\mathcal{F}_1) \cap G_r(\mathbb{C}^n) \cap S(I,\mathcal{F}_3) \\ &= S(I',\mathcal{F}_1) \cap S(I,\mathcal{F}_3) \\ &\neq \emptyset \end{aligned}$$

as was needed. □

CHAPTER 3: Grassmann Cohomology and Schubert Calculus

We saw in the previous chapter that Inequality (IJK) holds for $I, J, K \in \mathcal{P}_r^n$ and all $A, B, C = A+B \in \text{Herm}(n)$ whenever the Schubert varieties $S(I', \mathcal{F}_1)$, $S(J', \mathcal{F}_2)$, and $S(K, \mathcal{F}_3)$ intersect for all complete flags $\mathcal{F}_j$, $j = 1, 2, 3$.

In this chapter we will see that the Grassmannian $G_r(\mathbb{C}^n)$ can be regarded as a finite-cellular CW complex whose cells are the Schubert cells $C(I, \mathcal{F})$ with $I \in \mathcal{P}_r^n$. This allows us to compute its cellular (co)homology, hence its singular (co)homology, which turns out to be freely generated by the (co)homology classes of the Schubert varieties $S(I, \mathcal{F})$ with $I \in \mathcal{P}_r^n$.

Since the cohomology is a graded ring under cup product, we are at this point able to compute the product of two or more Schubert varieties' cohomology classes. It turns out that such a product is intimately related to the intersection of the relevant Schubert varieties themselves. Indeed, this cup product is nonzero if and only if the corresponding intersection of Schubert varieties is nonempty. Thus, at the end of this chapter we will have established a meaningful connection between (IJK) inequalities ($I, J, K \in \mathcal{P}_r^n$) and the cohomology ring of the Grassmannian $G_r(\mathbb{C}^n)$.

This chapter requires a basic understanding of singular homology theory, finite-cellular CW complexes, and Poincaré duality. General references for this material include [8] and [18].

It should be noted that the coefficient ring for all (co)homology in this thesis is assumed to be $\mathbb{Z}$.

3.1 The Grassmannian $G_r(\mathbb{C}^n)$

Recall that for integers $0 \leq r \leq n$, the complex Grassmannian $G_r(\mathbb{C}^n)$ is defined as the set

$$G_r(\mathbb{C}^n) = \{L \ : \ L \text{ is a subspace of } \mathbb{C}^n \text{ with } \dim_{\mathbb{C}}(L) = r\}.$$

We give $G_r(\mathbb{C}^n)$ a topology as follows. Let $V_r^0(\mathbb{C}^n) \subset (\mathbb{C}^n)^r$ denote the set of all *orthonormal r-frames*, i.e., sequences $(u_1, \ldots, u_r)$ of pair-wise orthogonal unit vectors $u_j \in \mathbb{C}^n$. The projection mapping

$$\pi : V_r^0(\mathbb{C}^n) \to G_r(\mathbb{C}^n), \quad \pi(u_1, \ldots, u_r) = \text{span}\{u_1, \ldots, u_r\}$$

is clearly surjective since every subspace has an orthonormal basis. We give $V_r^0(\mathbb{C}^n)$ the subspace topology from $(\mathbb{C}^n)^r$ and $G_r(\mathbb{C}^n)$ the quotient topology from $V_r^0(\mathbb{C}^n)$. Thus, by definition, a subset U of $G_r(\mathbb{C}^n)$ is open if and only if $\pi^{-1}(U)$ is open in $V_r^0(\mathbb{C}^n)$.

Proposition 3.1. *The set $V_r^0(\mathbb{C}^n)$ of orthonormal r-frames is a closed, bounded, and connected subset of the r-fold product $\mathbb{C}^n \times \cdots \times \mathbb{C}^n$.*

Proof. Boundedness of $V_r^0(\mathbb{C}^n)$ is obvious. Here we will verify that $V_r^0(\mathbb{C}^n)$ is closed and connected.

$V_r^0(\mathbb{C}^n)$ *is closed*: Let $N \subset (\mathbb{C}^n)^r$ be the set of all normalized r-tuples, and $O \subset (\mathbb{C}^n)^r$ the set of all orthogonal r-tuples. Then

$$V_r^0(\mathbb{C}^n) = N \cap O$$

and so it suffices to show that both N and O are closed in $(\mathbb{C}^n)^r$. We do this by exhibiting both sets as the inverse image of a closed set under a continuous function.

Define $f : (\mathbb{C}^n)^r \to \mathbb{R}^r$ and $g : (\mathbb{C}^n)^r \to \mathbb{R}$ via

$$f(v_1, \ldots, v_r) = \|v_1\| \times \cdots \times \|v_r\|$$

and

$$g(v_1, \ldots, v_r) = \sum_{\ell=1}^{r} \sum_{j \neq \ell} |\langle v_j, v_\ell \rangle|^2.$$

Then f and g are obviously continuous, and thus $N = f^{-1}(1, \ldots, 1)$ and $O = g^{-1}(0)$ are both closed in $(\mathbb{C}^n)^r$.

$V_r^0(\mathbb{C}^n)$ *is connected*: For convenience, we identify each r-tuple $(x_1, \ldots, x_r)$ in $(\mathbb{C}^n)^r$ with the $n \times r$ matrix $[x_1|\cdots|x_r]$ whose j^{th} column is x_j.

Our proof assumes the following standard facts: (1) the general linear group $GL_n(\mathbb{C})$ is path-connected, and (2) the Gram-Schmidt function $\mathcal{G} : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ sending any basis for $\mathbb{C}^n$ to an orthonormal basis is a continuous map.

Let $(u_1, \ldots, u_r) \in V_r^0(\mathbb{C}^n)$ be given. It suffices to show that there is a path in $V_r^0(\mathbb{C}^n)$ from $(u_1, \ldots, u_r)$ to $(e_1, \ldots, e_r)$, where e_j is the j^{th} standard basis vector.

First, extend $(u_1, \ldots, u_r)$ to an orthonormal basis $(u_1, \ldots, u_n)$ for $\mathbb{C}^n$. Since $GL_n(\mathbb{C})$ is path-connected, there is a continuous map $f : [0,1] \to GL_n(\mathbb{C})$ with $f(0) = [u_1|\cdots|u_n]$ and $f(1) = [e_1|\cdots|e_n]$. As the Gram-Schmidt function $\mathcal{G}$ is also continuous, $\mathcal{G} \circ f([0,1])$ is a path in $V_n^0(\mathbb{C}^n)$ from $(u_1, \ldots, u_n)$ to $(e_1, \ldots, e_n)$. By the continuity of the projection map

$$P_r : V_n^0(\mathbb{C}^n) \to V_r^0(\mathbb{C}^n), \quad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_r),$$

we see that $P_r \circ \mathcal{G} \circ f([0,1])$ is a path in $V_r^0(\mathbb{C}^n)$ from $(u_1, \ldots, u_r)$ to $(e_1, \ldots, e_r)$, as was needed. □

Since $V_r^0(\mathbb{C}^n)$ is a compact, connected subset of the r-fold product $\mathbb{C}^n \times \cdots \times \mathbb{C}^n$, this next corollary follows from the continuity of the quotient map $\pi : V_r^0(\mathbb{C}^n) \to G_r(\mathbb{C}^n)$.

Corollary 3.2. *The Grassmannian* $G_r(\mathbb{C}^n) = \pi(V_r^0(\mathbb{C}^n))$ *is a compact, connected topological space.*

In fact, $G_r(\mathbb{C}^n)$ can even be given the structure of a *complex manifold* of (complex) dimension $N = r(n-r)$. We now follow the treatment provided in [7, p.193-194] to give an outline of a proof of this very important fact.

Let $M(r,n)$ denote the set of complex $r \times n$ matrices of rank r and let

$$\text{row space} : M(r,n) \to G_r(\mathbb{C}^n)$$

be the map sending a matrix $A \in M(r,n)$ to the subspace spanned by its row vectors. Given $I = (i_1 < \cdots < i_r) \in \mathcal{P}_r^n$ and $A \in M(r,n)$, let A_I denote the $r \times r$ matrix

$$A_I = \begin{bmatrix} a_{1,i_1} & \cdots & a_{1,i_r} \\ \vdots & & \vdots \\ a_{r,i_1} & \cdots & a_{r,i_r} \end{bmatrix}$$

whose jth column is the i_jth column of A. Now,

- $U_I = \{\text{row space}(A) \ : \ A \in M(r,n),\ \det(A_I) \neq 0\}$ is an open set in $G_r(\mathbb{C}^n)$,
- the sets $\{U_I \ : \ I \in \mathcal{P}_r^n\}$ cover $G_r(\mathbb{C}^n)$, and
- for each $\Lambda \in U_I$ there is a unique matrix $\Lambda^I \in M(r,n)$ with row space$(\Lambda^I) = \Lambda$ and $(\Lambda^I)_I = I_r$, the $r \times r$ identity matrix.

Moreover letting $I^c \in \mathcal{P}_{n-r}^n$ be defined as $I^c = \{1, \ldots, n\} \backslash I$,

- the mapping $\varphi_I : U_I \to \mathbb{C}^{r(n-r)}, \quad \Lambda \mapsto (\Lambda^I)_{I^c}$ is bijective.

Proposition 3.3. *[7, p.194] The $\binom{n}{r}$ maps φ_I $(I \in \mathcal{P}_r^n)$ are charts on $G_r(\mathbb{C}^n)$, giving the space $G_r(\mathbb{C}^n)$ the structure of a compact, oriented[1] complex manifold of dimension $N = r(n-r)$.*

3.2 Cell structure on $G_r(\mathbb{C}^n)$

Let $\mathcal{F} : V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n$ be a complete flag in $\mathbb{C}^n$.

Definition 3.4. For $L \in G_r(\mathbb{C}^n)$, the *jump indices* $\mathcal{I}(L, \mathcal{F}) = (i_1 < \cdots < i_r) \in \mathcal{P}_r^n$ are defined by

$$i_j = \min\{k \ : \ \dim(L \cap V_k) = j\}$$

for $j = 1, \ldots, r$. That is, i_j is the smallest index for which $L \cap V_{i_j}$ has dimension j.

Definition 3.5. Given a complete flag $\mathcal{F}$ in $\mathbb{C}^n$ and indices $I \in \mathcal{P}_r^n$, the set

$$C(I, \mathcal{F}) := \{L \in G_r(\mathbb{C}^n) \ : \ \mathcal{I}(L, \mathcal{F}) = I\}$$

is called a *Schubert cell* of $G_r(\mathbb{C}^n)$.

It is clear from the definition that $G_r(\mathbb{C}^n)$ is the disjoint union

$$G_r(\mathbb{C}^n) = \coprod_{I \in \mathcal{P}_r^n} C(I, \mathcal{F})$$

of the Schubert cells $\{C(I, \mathcal{F}) \ : \ I \in \mathcal{P}_r^n\}$.

Lemma 3.6. *[7, p.196] For $I = (i_1 < \cdots < i_r) \in \mathcal{P}_r^n$, the Schubert cell $C(I, \mathcal{F})$ is homeomorphic to $\mathbb{C}^{d(I)}$ where the* degree *$d(I)$ is defined as*

$$d(I) = \sum_{\ell=1}^{r} (i_\ell - \ell)$$

[1]Every complex manifold carries a canonical orientation.

Proof outline. It suffices to take for $\mathcal{F}$ the standard flag $\{0\} \subset \mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^n$. One can show that each subspace $L \in C(I, \mathcal{F})$ is the row space of a unique matrix $A \in M(r, n)$ satisfying the following conditions for $j = 1, \ldots, r$:

(a) $A_{j,i_j} = 1$,

(b) $A_{k,i_j} = 0$ for $k \neq j$ and

(c) $A_{j,k} = 0$ for $i_j < k \leq n$.

This is a (reverse) reduced row echelon form. Such matrices have $d(I)$ "free" entries and the remaining entries are all fixed according to (a), (b), and (c) above, hence this representation of $C(I, \mathcal{F})$ is naturally homeomorphic to $\mathbb{C}^{d(I)}$. □

Example 3.7. Say $n = 5$, $r = 3$, $I = (1 < 3 < 5)$ and $\mathcal{F}$ is the standard flag in $\mathbb{C}^5$. Each $L \in C(I, \mathcal{F})$ has the form $L = \text{row space}(A)$ for exactly one matrix with "reverse reduced row echelon" form

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \star & 1 & 0 & 0 \\ 0 & \star & 0 & \star & 1 \end{bmatrix}.$$

This contains $d(I) = (1-1) + (3-2) + (5-3) = 3$ free entries marked "$\star$". So in this case $C(I, \mathcal{F})$ is homeomorphic to $\mathbb{C}^3$. □

There exist attaching maps that turn this cell-decomposition of the Grassmannian into a finite-cellular CW-complex. That is, letting $\mathbb{B}^\ell$ and $\mathbb{D}^\ell$ denote the open and closed unit balls of $\mathbb{R}^\ell$, respectively, there exists a family of continuous maps $\{\varphi_{C(I,\mathcal{F})} : I \in \mathcal{P}_r^n\}$ such that $\varphi_{C(I,\mathcal{F})} : \mathbb{D}^{2d(I)} \to G_r(\mathbb{C}^n)$ restricts to a homeomorphism of $\mathbb{B}^{2d(I)}$ onto $C(I, \mathcal{F})$. A proof of this can be found in [9, p.33-34].

The $2m$-cells of this CW-complex are of course the Schubert cells $C(I, \mathcal{F})$ with

$I \in \mathcal{P}_r^n$ and $d(I) = m$, and the $2m$-skeleton is therefore the union

$$\bigcup_{\substack{I \in \mathcal{P}_r^n \\ d(I) \leq m}} C(I, \mathcal{F}) = \bigcup_{\substack{I \in \mathcal{P}_r^n \\ d(I) = m}} S(I, \mathcal{F}).$$

Let X^ℓ denote the ℓ-skeleton of $G_r(\mathbb{C}^n)$, and let ξ_ℓ denote the number of ℓ-cells in X^ℓ. By definition, the ℓth cellular homology group of $G_r(\mathbb{C}^n)$ is the ℓth homology group of the cellular chain complex

$$\cdots \to H_k(X^k, X^{k-1}) \xrightarrow{d_k} H_{k-1}(X^{k-1}, X^{k-2}) \to \cdots \to H_1(X^1, X^0) \xrightarrow{d_1} H_0(X^0, \emptyset) \to 0$$

In our notation, $H_\ell(X^\ell, X^{\ell-1})$ is a relative singular homology group, and the boundary homomorphisms d_ℓ are as described in [18, p.213].

Proposition 3.8. $H_k(X^k, X^{k-1})$ *is a free abelian group whose basis has cardinality equal to the number of* k*-cells in* X^k*.*

Proof. See [8, Lemma 2.34]. □

Therefore, the ℓth cellular homology group of $G_r(\mathbb{C}^n)$ is the ℓth homology group of the chain complex

$$\cdots \to \mathbb{Z}^{\xi_\ell} \xrightarrow{d_\ell} \mathbb{Z}^{\xi_{\ell-1}} \to \cdots \to \mathbb{Z}^{\xi_1} \xrightarrow{d_1} \mathbb{Z}^{\xi_0} \to 0.$$

However, since each Schubert cell $C(I, \mathcal{F})$ has real dimension $2d(I)$, we see that $\xi_\ell = 0$ whenever ℓ is odd. Consequently, every other term in this chain complex is zero, hence all boundary maps are zero, and we find that the ℓth cellular homology group $H_\ell^{CW}(G_r(\mathbb{C}^n))$ of $G_r(\mathbb{C}^n)$ is

$$H_\ell^{CW}(G_r(\mathbb{C}^n)) = \left\{ \begin{matrix} \mathbb{Z}^{\xi_\ell} & 0 \leq \ell \leq 2r(n-r) \text{ is even} \\ 0 & \text{otherwise} \end{matrix} \right\} \tag{3.1}$$

The singular homology of $G_r(\mathbb{C}^n)$ now follows from the isomorphism between the singular and cellular homology of CW complexes. A proof of this isomorphism can be obtained from [18, Theorem 8.36].

Now, recall that for an r-tuple $I \in \mathcal{P}_r^n$ and complete flag $\mathcal{F}$, the Schubert variety $S(I, \mathcal{F}) \subset G_r(\mathbb{C}^n)$ is the set

$$S(I, \mathcal{F}) = \left\{ L \in G_k(\mathbb{C}^n) : \dim(L \cap V_{i_j}) \geq j \text{ for } 1 \leq j \leq n \right\}.$$

When a complete flag $\mathcal{F}$ is fixed, each Schubert variety $S(I, \mathcal{F})$ generates a distinct homology class $[S(I, \mathcal{F})] \in H_{2d(I)}(G_r(\mathbb{C}^n))$, and these homology classes freely generate $H_\bullet(G_r(\mathbb{C}^n))$ as a $\mathbb{Z}$-module. More information on how these Schubert varieties each generate a homology class is provided in Appendix A.

Proposition 3.9. *The homology class of any Schubert variety $S(I, \mathcal{F})$ is independent of the chosen flag $\mathcal{F}$.*

Proof. For any matrix $M \in GL_n(\mathbb{C})$, let $\widetilde{M} : G_r(\mathbb{C}^n) \to G_r(\mathbb{C}^n)$ denote the map $L \mapsto M(L)$.

Choose complete flags $\mathcal{F}_1 : V_0 \subset V_1 \subset \cdots \subset V_n$ and $\mathcal{F}_2 : W_0 \subset W_1 \subset \cdots \subset W_n$ in $\mathbb{C}^n$, and suppose that $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$ are orthonormal bases for V_k and W_k, respectively. Let A be the matrix such that $Av_j = w_j$ for each j.

First note that $\widetilde{A}(S(I, \mathcal{F}_1)) = S(I, \mathcal{F}_2)$. Indeed, A is an isomorphism since it takes a basis to another basis, hence for any $\Lambda \in G_r(\mathbb{C}^n)$ we have

$$\begin{aligned} \dim(\Lambda \cap V_{i_j}) &= \dim(\widetilde{A}(\Lambda \cap V_{i_j})) \\ &= \dim(\widetilde{A}(\Lambda) \cap \widetilde{A}(V_{i_j})) \\ &= \dim(\widetilde{A}(\Lambda) \cap W_{i_j}). \end{aligned}$$

It now follows from the definition of a Schubert variety that $\widetilde{A}(S(I, \mathcal{F}_1)) \subset S(I, \mathcal{F}_2)$. To prove the reverse inclusion, the same general argument can be used to show that $\widetilde{A^{-1}}(S(I, \mathcal{F}_2)) \subset S(I, \mathcal{F}_1)$. Now apply $\widetilde{A}$ to both sides of this inclusion. Since $\widetilde{A} \circ \widetilde{A^{-1}}$ is obviously the identity map of $G_r(\mathbb{C}^n)$, it follows that $S(I, \mathcal{F}_2) \subset \widetilde{A}(S(I, \mathcal{F}_1))$. Thus $\widetilde{A}(S(I, \mathcal{F}_1)) = S(I, \mathcal{F}_2)$ as claimed.

Now, let $\widetilde{A}_* : H(G_r(\mathbb{C}^n)) \to H(G_r(\mathbb{C}^n))$ be the map on homology induced by $\widetilde{A}$. Since

$$\widetilde{A}_*[S(I, \mathcal{F}_1)] = [\widetilde{A}(S(I, \mathcal{F}_1))] = [S(I, \mathcal{F}_2)], \tag{3.2}$$

we see that $\widetilde{A}_*$ sends the homology class of $S(I, \mathcal{F}_1)$ to the homology class of $S(I, \mathcal{F}_2)$.

We now show that $\widetilde{A}_*$ is the identity map id_* on homology. This will complete the proof, for it then follows from Equation (3.2) that $[S(I, \mathcal{F}_1)] = [S(I, \mathcal{F}_2)]$.

To show that $\widetilde{A}_* = id_*$, it suffices to show that $\widetilde{A}$ is homotopic to the identity map of $G_r(\mathbb{C}^n)$. Since $GL_n(\mathbb{C})$ is path connected and contains both A and the $n \times n$ identity matrix I_n, there is a continuous function $f : [0, 1] \to GL_n(\mathbb{C})$ such that $f(0) = I_n$ and $f(1) = A$. Define a homotopy $H : G_r(\mathbb{C}^n) \times I \to G_r(\mathbb{C}^n)$ via $H(L, t) = (f(t))(L)$. Then as $H(L, 0) = f(0)(L) = I_n(L) = L$ and $H(L, 1) = f(1)(L) = A(L)$, we see that H is indeed a homotopy between $id_{G_r(\mathbb{C}^n)}$ and $\widetilde{A}$. □

Therefore, we can unambiguously write $[I]$ for the homology class of any Schubert variety $S(I, \mathcal{F})$. Moreover, since $G_r(\mathbb{C}^n)$ is a compact, oriented manifold without boundary, Poincaré duality gives an isomorphism

$$PD : H_{2m}(G_r(\mathbb{C}^n)) \to H^{2(N-m)}(G_r(\mathbb{C}^n))$$

for each $m = 0, 1, \ldots, N$. For $I \in \mathcal{P}_r^n$, we define

$$\sigma_I = PD([I]) \in H^{2(N-d(I))}(G_r(\mathbb{C}^n))$$

to be the Poincaré dual of $[I]$. We call σ_I a *Schubert cocycle*.

From what we know of Grassmann homology, it is clear that the cohomology of $G_r(\mathbb{C}^n)$ is zero in odd dimensions, and that

$$H^{2m}(G_r(\mathbb{C}^n)) = \mathbb{Z}\text{-span}\{\sigma_I : d(I) = N - m\}$$

for $m = 0, 1, \ldots, N$.

3.3 Cup products in $H^\bullet(G_r(\mathbb{C}^n))$

The cohomology $H^\bullet(G_r(\mathbb{C}^n)) = \oplus_{m\geq 0} H^m(G_r(\mathbb{C}^n))$ is a graded ring with multiplication

$$\cup : H^\ell(G_r(\mathbb{C}^n)) \times H^m(G_r(\mathbb{C}^n)) \to H^{\ell+m}(G_r(\mathbb{C}^n))$$

given by the *cup product* [8, §3.2]. It should be noted that the cup product is bilinear, and thus that multiplication in the cohomology ring is completely determined by the products $\sigma_I \cup \sigma_J$ for $I, J \in \mathcal{P}_r^n$. Indeed, since the set $\{\sigma_I : I \in \mathcal{P}_r^n\}$ of Schubert cocycles forms an additive basis for $H^\bullet(G_r(\mathbb{C}^n))$, any two cohomology classes α, β can be written $\alpha = \sum_I a_I \sigma_I$, $\beta = \sum_J b_J \sigma_J$ $(a_I, b_J \in \mathbb{Z})$, and thus bilinearity gives us that

$$\alpha \cup \beta = \sum_{I,J} a_I b_J (\sigma_I \cup \sigma_J).$$

From now on we omit the "$\cup$" and simply express any cup product $\sigma_I \cup \sigma_J$ as $\sigma_I \sigma_J$.

3.3.1 Complementary indices and codegree

Let $I = (i_1 < \cdots < i_r) \in \mathcal{P}_r^n$ be given.

Definition 3.10. We define the *codegree* of I to be the number $d^{co}(I) = N - d(I)$. We say that I and $J \in \mathcal{P}_r^n$ have *complementary degrees* if $d(I) + d(J) = N$, or equivalently, if $d^{co}(J) = d(I)$.

Thus, the Schubert cocycle σ_I lies in $H^{2d^{co}(I)}(G_r(\mathbb{C}^n))$.

Recall (see Definition 2.17) that the complementary indices of I are defined as the r-tuple

$$I' = (n + 1 - i_r < \cdots < n + 1 - i_1).$$

Lemma 3.11. *If $I \in \mathcal{P}_r^n$ then $d(I) + d(I') = r(n - r) = N$.*

Proof.

$$\begin{aligned} d(I) + d(I') &= \sum_{j=1}^{r} (i_j - j) + \sum_{j=1}^{r} ((n + 1 - i_j) - j) \\ &= \sum_{j=1}^{r} (n + 1 - 2j) \\ &= r(n + 1) - r(r + 1) \\ &= r(n - r) = N. \end{aligned}$$

□

Corollary 3.12. *The r-tuples I and I' have complementary degrees. Equivalently, $d(I') = d^{co}(I)$.*

3.3.2 Evaluating cup products

We now lay out some general facts that will help us evaluate cup products.

- For any topological space X, the following identity holds for the cup product on $H^\bullet(X)$:
$$\alpha\beta = (-1)^{\deg\alpha+\deg\beta}\beta\alpha$$
where $\alpha \in H^{\deg\alpha}(X)$ and $\beta \in H^{\deg\beta}(X)$. Thus since Grassmann cohomology is nontrivial only in even dimensions, one deduces from the identity
$$\sigma_I\sigma_J = (-1)^{2d^{co}(I)+2d^{co}(J)}\sigma_J\sigma_I$$
that the cup product on $H^\bullet(G_r(\mathbb{C}^n))$ is commutative.
- If $d^{co}(I)+d^{co}(J) > N$ then $\sigma_I\sigma_J = 0$. This is because the cohomology of $G_r(\mathbb{C}^n)$ is trivial in degrees greater than $2N$, and $\sigma_I\sigma_J \in H^{2(d^{co}(I)+d^{co}(J))}(G_r(\mathbb{C}^n))$.
- If $d^{co}(I) + d^{co}(J) = N$ then $\sigma_I\sigma_J$ is nonzero if and only if $J = I'$. A proof of this can be found in [7, p.198].

Since the set $\{\sigma_I : I \in \mathcal{P}^n_r\}$ of Schubert cocycles forms an additive basis for the cohomology ring $H^\bullet(G_r(\mathbb{C}^n))$, one has for any $I, J \in \mathcal{P}^n_r$ that

$$\sigma_I\sigma_J = \sum_K c^K_{I,J}\sigma_K \qquad (c^K_{I,J} \in \mathbb{Z})$$

where the summation is over all $K \in \mathcal{P}^n_r$ such that $d^{co}(K) = d^{co}(I) + d^{co}(J)$. The numbers $c^K_{I,J}$ are called *Littlewood-Richardson coefficients.*

Remark 3.13. One should note that most (if not all) authors index their Littlewood-Richardson coefficients with partitions (weakly decreasing, finite sequences of nonnegative integers). We will stray from this convention and instead index our Littlewood-Richardson coefficients with strictly increasing sequences that belong to a common set $\mathcal{P}_r^n$. As our primary use for LR coefficients lies in their connections to (IJK) inequalities, this just seems more natural for our purposes.

For the sake of comparison with our sources: if $I, J, K \in \mathcal{P}_r^n$ then our LR coefficient $c_{I,J}^K$ would typically be written as

$$c^{(n-r+1-k_1,\ldots,n-r+r-k_r)}_{(n-r+1-i_1,\ldots,n-r+r-i_r),\,(n-r+1-j_1,\ldots,n-r+r-j_r)}$$

in the literature.

Proposition 3.14. *For indices $I, J, K \in \mathcal{P}_r^n$ with $d^{co}(K) = d^{co}(I) + d^{co}(J)$ one has $c_{I,J}^K \neq 0$ if and only if $\sigma_I \sigma_J \sigma_{K'} \neq 0$.*

Proof. First note that

$$\sigma_I \sigma_J = \sum_L c_{I,J}^L \sigma_L$$

where the summation is over all $L \in \mathcal{P}_r^n$ such that $d^{co}(L) = d^{co}(I) + d^{co}(J)$. It therefore follows from the cohomology's ring structure that

$$\sigma_I \sigma_J \sigma_{K'} = \sum_L c_{I,J}^L \sigma_L \sigma_{K'}. \tag{3.3}$$

Now, for each L, the r-tuples K' and L have complementary degrees (since K and L have the same codegree). Thus $\sigma_L \sigma_{K'}$ is nonzero if and only if $L = K$, and so Equation (3.3) reduces to

$$\sigma_I \sigma_J \sigma_{K'} = c_{I,J}^K \sigma_K \sigma_{K'}$$

where $\sigma_K \sigma_{K'} \in H^{2N}(G_r(\mathbb{C}^n))$ is non-zero. □

We now state two formulas (see [7, p.203-206] for their proofs) that together enable one to compute arbitrary cup products in $H^\bullet(G_r(\mathbb{C}^n))$. Equivalently, these formulas enable the computation of Littlewood-Richardson coefficients.

To present these results, we require an alternate parametrization for the cohomology ring $H^\bullet(G_r(\mathbb{C}^n))$. Given $I = (i_1 < i_2 < \cdots < i_r) \in \mathcal{P}_r^n$ we denote $\sigma_I = PD([I])$ by

$$\{n - r + \ell - i_\ell\}_{\ell=1}^r \in H^\bullet(G_r(\mathbb{C}^n)).$$

Note that $I \mapsto (n-r+\ell-i_\ell)_{\ell=1}^r$ gives a bijection between the sets $\mathcal{P}_r^n$ and $\{(a_1, \ldots, a_r) : n - r \geq a_1 \geq \cdots \geq a_r \geq 0\}$. The benefit of using this alternate parameterization is that the sum of the entries equals the codegree of the cocycle (equivalently, the codimension of the corresponding Schubert variety). Indeed,

$$\begin{aligned}\sum_{j=1}^r (n - r + j - i_j) &= r(n - r) + \sum_{j=1}^r (j - i_j) \\ &= N - d(I) = d^{co}(I).\end{aligned}$$

Thus $H^{2m}(G_r(\mathbb{C}^n))$ is the free $\mathbb{Z}$-module generated by the cohomology classes $\{a_1, \ldots, a_r\}$ where $n - r \geq a_1 \geq \cdots \geq a_r \geq 0$ and $\sum a_j = m$.

For simplicity, we write any cocycle $\{a_1, \ldots, a_\ell, 0, \ldots, 0\}$ simply as $\{a_1, \ldots, a_\ell\}$. The cocycles $\{a\}$, for $1 \leq a \leq n - r$, are called *special cocycles.* It turns out that these actually generate the cohomology ring, as is made clear by Giambelli's formula, below.

Proposition 3.15 (Pieri's formula)**.** *Suppose $n - r \geq a_1 \geq \cdots \geq a_r \geq 0$, and let $0 \leq \ell \leq n - r$. Then*

$$\{a_1, \ldots, a_r\}\{\ell\} = \sum_{\substack{a_{i-1} \geq b_i \geq a_i \\ \sum b_i = \ell + \sum a_i}} \{b_1, \ldots, b_r\}.$$

(Here we formally define $a_0 = n - r$.)

Proposition 3.16 (Giambelli's formula)**.** *Let $n - r \geq a_1 \geq \cdots \geq a_r \geq 0$. Then* *the cocycle* $\{a_1, \ldots, a_r\}$ *is the determinant of the* $r \times r$ *matrix whose* ij^{th} *entry is the cocycle* $\{a_i - i + j\}$*:*

$$\{a_1, \ldots, a_r\} = \det \begin{pmatrix} \{a_1\} & \{a_1 + 1\} & \cdots & \{a_r + r - 1\} \\ \{a_2 - 1\} & \{a_2\} & \cdots & \{a_2 + r - 2\} \\ \vdots & \vdots & \ddots & \vdots \\ \{a_r - r + 1\} & \{a_r - r + 2\} & \cdots & \{a_r\} \end{pmatrix}.$$

One should note here that $\{0\}$ is the multiplicative identity for $H^\bullet(G_r(\mathbb{C}^n))$, and that $\{j\}$ is zero in $H^\bullet(G_r(\mathbb{C}^n))$ whenever $j < 0$ or $j > n - r$.

To see how these two formulas allow us to take arbitrary cup products, suppose we want to evaluate $\{a_1, \ldots, a_r\}\{b_1, \ldots, b_r\}$. First apply Giambelli's formula to rewrite $\{b_1, \ldots, b_n\}$ as a polynomial in special cocycles, say $\{b_1, \ldots, b_r\} = \sum_p \prod_q \{c_{p,q}\}$. Then

$$\{a_1, \ldots, a_r\}\{b_1, \ldots, b_r\} = \sum_p \prod_q \{a_1, \ldots, a_r\}\{c_{p,q}\}$$

and we can now apply Pieri's formula to each term in the summation to obtain the cup product $\{a_1, \ldots, a_r\}\{b_1, \ldots, b_r\}$ as a sum of cocycles.

Example 3.17. Let us compute the cup product of the cocycles $\{4, 2\}$ and $\{3, 1\}$ in $H^\bullet(G_3(\mathbb{C}^7))$. By Giambelli's formula,

$$\begin{aligned} \{3, 1\} &= \det \begin{bmatrix} \{3\} & \{4\} \\ \{0\} & \{1\} \end{bmatrix} \\ &= \{3\}\{1\} - \{4\}\{0\} \\ &= \{3\}\{1\} - \{4\}. \end{aligned}$$

Thus,

$$\{4, 2\}\{3, 1\} = \{4, 2\}\{3\}\{1\} - \{4, 2\}\{4\}. \tag{3.4}$$

By Pieri's formula, the product $\{4,2\}\{3\}$ is the sum of all cocycles $\{b_1, b_2, b_3\}$ with

$$4 \geq b_1 \geq 4 \geq b_2 \geq 2 \geq b_3 \geq 0 \qquad \text{and} \qquad b_1 + b_2 + b_3 = 4 + 2 + 3 = 9.$$

Hence $\{4,2\}\{3\} = \{4,4,1\} + \{4,3,2\}$, and Equation (3.4) becomes

$$\{4,2\}\{3,1\} = \{4,4,1\}\{1\} + \{4,3,2\}\{1\} - \{4,2\}\{4\}.$$

Routine applications of Pieri's formula to each of these summands now show that

$$\begin{aligned}\{4,2\}\{3,1\} &= \{4,4,2\} + \big(\{4,4,2\} + \{4,3,3\}\big) - \{4,4,2\} \\ &= \{4,4,2\} + \{4,3,3\}.\end{aligned}$$

3.4 Cup products and intersections of Schubert varieties

The following major result relates cup products in $H^\bullet(G_r(\mathbb{C}^n))$ to intersections of Schubert varieties. A version of this can be found in [10] which references [5] and [7] for the proof. An overview of the proof is given below in Appendix A.

Proposition 3.18. *Given $I, J \in \mathcal{P}_r^n$ with $d^{co}(I) + d^{co}(J) \leq N$, we have $\sigma_I \sigma_J \neq 0$ in $H^\bullet(G_r(\mathbb{C}^n))$ if and only if $S(I, \mathcal{F}_1) \cap S(J, \mathcal{F}_2) \neq \emptyset$ for all complete flags $\mathcal{F}_j$, $j = 1, 2$. Likewise, given $I, J, K \in \mathcal{P}_r^n$ with $d^{co}(I) + d^{co}(J) + d^{co}(K) \leq N$, we have $\sigma_I \sigma_J \sigma_K \neq 0$ in $H^\bullet(G_r(\mathbb{C}^n))$ if and only if $S(I, \mathcal{F}_1) \cap S(J, \mathcal{F}_2) \cap S(K, \mathcal{F}_3) \neq \emptyset$ for all complete flags $\mathcal{F}_j$, $j = 1, 2, 3$.*

The conditions on $d^{co}(I)$, $d^{co}(J)$, $d^{co}(K)$ are imposed to ensure that the cup products $\sigma_I \sigma_J$ and $\sigma_I \sigma_J \sigma_K$ lie in a nonzero cohomology group.

Corollary 2.19 asserts that Inequality (IJK) holds for all $n \times n$ Hermitian matrices $A, B, C = A + B$ whenever $S(I', \mathcal{F}_1) \cap S(J', \mathcal{F}_2) \cap S(K, \mathcal{F}_3) \neq \emptyset$ for all complete flags

$\mathcal{F}_j$. Thus Proposition 3.18 now has the following as an immediate consequence.

Theorem 3.19. *Given $I, J, K \in \mathcal{P}_r^n$, if $\sigma_{I'}\sigma_{J'}\sigma_K \neq 0$ in $H^\bullet(G_r(\mathbb{C}^n))$ (equivalently, $c_{I',J'}^{K'} \neq 0$) then Inequality* (IJK) *holds for all $A, B \in \mathrm{Herm}(n)$. That is, $(I, J, K) \in H_r^n$.*

3.5 Applications of Theorem 3.19

Theorem 3.19 provides us with yet another way to prove the Weyl and Lidskii inequalities: show that $\sigma_{I'}\sigma_{J'}\sigma_K \neq 0$, where I, J, and K are the indices from the statement of the relevant inequality (see Section 1.2.1).

Example 3.20 (Weyl inequalities)**.** In this case we have $I = (i)$, $J = (j)$, and $K = (i + j - 1)$ in $\mathcal{P}_1^n$, and we need to show that $\sigma_{I'}\sigma_{J'}\sigma_K \neq 0$.

First, we see that $I' = (n + 1 - i)$ and $J' = (n + 1 - j)$. Hence,

$$\sigma_{I'} = \{i - 1\}, \qquad \sigma_{J'} = \{j - 1\}, \qquad \sigma_K = \{n + 1 - i - j\},$$

and by Pieri's formula,

$$\begin{aligned}\{i - 1\}\{j - 1\}\{n + 1 - i - j\} &= \{i + j - 2\}\{n + 1 - i - j\} \\ &= \{n - 1\}.\end{aligned}$$

So $\sigma_{I'}\sigma_{J'}\sigma_K = \{n - 1\}$ generates the top-degree cohomology group $H^{2(n-1)}(G_1(\mathbb{C}^n))$, and thus is obviously nonzero.

Example 3.21 (Lidskii inequalties)**.** Here we have $I = (i_1 < \cdots < i_r)$ and $J = (1, 2, \ldots, r) \in \mathcal{P}_r^n$, and we need to show that $\sigma_{I'}\sigma_{J'}\sigma_I \neq 0$.

A quick calculation shows that $\sigma_{J'} = \{0, \ldots, 0\} \in H^0(G_r(\mathbb{C}^n))$ is the multiplicative identity of the cohomology ring. Hence $\sigma_{I'}\sigma_{J'}\sigma_I = \sigma_{I'}\sigma_I$, and this product is nonzero since I and I' are complementary indices.

CHAPTER 4: The Littlewood-Richardson Rules

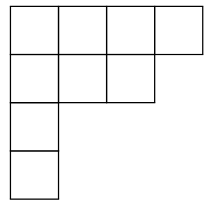
We saw in Theorem 3.19 that nonzero Littlewood-Richardson coefficients yield (IJK) eigenvalue inequalities. At the moment, our only tools for determining whether a Littlewood-Richardson coefficient is nonzero are the formulas of Pieri and Giambelli. Unfortunately, these formulas are not especially convenient for our purposes: using them to determine whether $c_{I,J}^{K}$ is nonzero usually requires us to actually compute each coefficient $c_{I,J}^{L}$ for all $L \in \mathcal{P}_r^n$ with $d^{co}(L) = d^{co}(K)$.

The Littlewood-Richardson (LR) rules provide another, more convenient algorithm for computing Littlewood-Richardson coefficients. However, explicit coefficient computations still quickly become very time consuming as r and n get bigger - even with a computer. Fortunately, the LR rules generally allow us to determine whether $c_{I,J}^{K}$ is nonzero without having to actually compute any Littlewood-Richardson coefficients.

Since the LR rules are stated in terms of Young diagrams, we begin this chapter with a very brief introduction to these combinatorial objects. The LR rules are then stated in Section 4.2, and we conclude this chapter by using the LR rules to provide extremely simple proofs of the Weyl, Lidskii, and Thompson-Freede inequalities.

4.1 Young diagrams

A *Young diagram* is a finite collection of boxes whose rows are left-justified and whose row lengths are weakly decreasing from top to bottom. If a Young diagram has r nonempty rows with lengths (listed top to bottom) $\ell_1, \ldots, \ell_r$, then the r-tuple $(\ell_1, \ldots, \ell_r)$ is called the *shape* of the diagram, and we denote this diagram by $D(\ell_1, \ldots, \ell_r)$. For example, $D(4,3,1,1)$ is the diagram



of shape $(4,3,1,1)$.

When referring to a specific Young diagram, we say "box(i,j)" to refer to the box in the ith row and jth column. Thus in the above example, box$(1,4)$ is the rightmost box, and box$(4,1)$ is the box at the very bottom of the diagram.

We can put a partial ordering $\prec$ on the collection of Young diagrams by declaring

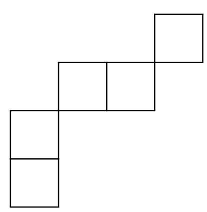
$$D(a_1, \ldots, a_\ell) \prec D(b_1, \ldots, b_r)$$

whenever $\ell \leq r$ and $a_j \leq b_j$ for each $j = 1, \ldots, \ell$. Thus $D(a_1, \ldots, a_\ell) \prec D(b_1, \ldots, b_r)$ if and only if $D(a_1, \ldots, a_\ell)$ fits inside $D(b_1, \ldots, b_r)$ when the top and left sides of the two diagrams are superimposed.

Given two diagrams $D(a_1, \ldots, a_\ell) \prec D(b_1, \ldots, b_r)$, we write

$$D(b_1, \ldots, b_r) \backslash D(a_1, \ldots, a_\ell)$$

to denote the diagram obtained by removing from $D(b_1, \ldots, b_r)$ the first a_j boxes of row j. For example, $D(4,3,1,1)\backslash D(3,1)$ looks like



4.2 The Littlewood-Richardson rules

For any $L = (\ell_1 < \cdots < \ell_r) \in P_r^n$, define

$$\text{par}(L) = (\tilde{\ell}_1 \geq \cdots \geq \tilde{\ell}_r) \qquad \text{where } \tilde{\ell}_j = n - r + j - \ell_j.$$

Note that $n - r \geq \tilde{\ell}_1$ and $\tilde{\ell}_r \geq 0$. These are the indices corresponding to our alternate parameterization of the cohomology ring from Section 3.3. That is,

$$\{\text{par}(L)\} = \sigma_L \in H^{2d^{co}(L)}(G_r(\mathbb{C}^n)),$$

and as we saw in Chapter 3, the sum of the entries of $\text{par}(L)$ is $|\text{par}(L)| = d^{co}(L)$. So $\text{par}(L)$ is a *partition* of the number $d^{co}(L)$ with at most r parts, each of size at most $n - r$.

Now, suppose that

$$\text{par}(I) = (a_1 \geq \cdots \geq a_r), \quad \text{par}(J) = (b_1 \geq \cdots \geq b_r), \quad \text{par}(K) = (c_1 \geq \cdots \geq c_r).$$

Since $|\text{par}(\cdot)| = d^{co}(\cdot)$, we already know from Chapter 4 that $c_{I,J}^K = 0$ unless $|\text{par}(K)| = |\text{par}(I)| + |\text{par}(J)|$. It can also be shown that $c_{I,J}^K = 0$ unless $a_\ell, b_\ell \leq c_\ell$ for each $\ell = 1, \ldots, r$. Assuming that these two conditions are satisfied, the following theorem describes how to compute the Littlewood-Richardson coefficient $c_{I,J}^K$.

Theorem 4.1 (LR Rules)**.** *Let $\mathcal{D} = D(par(K)) \backslash D(par(I))$. Of the $\sum b_\ell$ boxes in $\mathcal{D}$, we want to label b_1 of the boxes with a "1", b_2 boxes with a "2", …, b_r boxes with an "r". The coefficient $c_{I,J}^K$ is the number of ways this can be done subject to the following constraints:*

- *The numbers are weakly increasing along rows and strictly increasing down* *columns.*

- *Reading the numbered entries from right to left, top to bottom, we obtain a lattice word $\ell_1, \ldots, \ell_{|par(J)|}$. That is, for each $i = 1, 2, \ldots, |par(J)|$ and each $j = 1, \ldots, r-1$, the sequence $\ell_1, \ldots, \ell_i$ contains at least as many j's as $(j+1)$'s.*

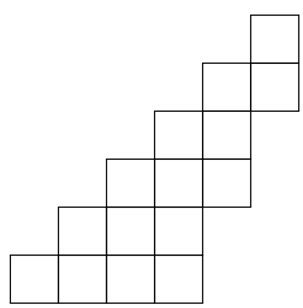
Note that, by the commutativity of the cup product, we could have instead used $\mathrm{par}(I)$ to number $D(\mathrm{par}(K))\backslash D(\mathrm{par}(J))$ in the above algorithm.

As any proof of the LR rules would land well outside of the scope of this thesis, we simply refer the reader to the proof given in [17].

Example 4.2. Let $I = (2, 4, 6, 8, 10, 12)$, and $K = (1, 2, 4, 5, 7, 8)$ be elements of P_6^{12}. We want to find $c_{I,I}^K$. First of all,

$$\mathrm{par}(I) = (5, 4, 3, 2, 1, 0) \qquad \text{and} \qquad \mathrm{par}(K) = (6, 6, 5, 5, 4, 4).$$

Now, $D(\mathrm{par}(K))\backslash D(\mathrm{par}(I))$ is the diagram



and we need to number these boxes with five 1's, four 2's, three 3's, two 4's, and one 5 in accordance with the rules from Theorem 4.1. There are in fact 9 ways to do this:

					1
				1	2
			1	3	
		1	2	4	
	2	2	3		
1	3	4	5		

					1
				1	2
			1	3	
		1	2	4	
	1	2	3		
2	3	4	5		

					1
				1	2
			1	3	
		2	2	4	
	1	3	3		
1	2	4	5		

					1
				1	2
			2	3	
		1	3	4	
	1	2	4		
1	2	3	5		

					1
				1	2
			1	2	
		1	3	3	
	1	2	4		
2	3	4	5		

					1
				1	2
			1	2	
		1	3	3	
	2	2	4		
1	3	4	5		

					1
				1	2
			1	2	
		2	3	3	
	1	3	4		
1	2	4	5		

					1
				1	2
			1	2	
		1	2	3	
	1	3	4		
2	3	4	5		

					1
				1	2
			1	2	
		1	2	3	
	2	3	4		
1	3	4	5		

.

Hence $c^K_{I,I} = 9$. However, we check only that the first numbering above satisfies the LR rules:

					1
				1	2
			1	3	
		1	2	4	
	2	2	3		
1	3	4	5		

First, the correct amount of boxes are labeled 1, 2, 3, 4, and 5, the rows are weakly increasing, and the columns are strictly increasing. Since the entries, read right to left and top to bottom form the lattice word

$$1, 2, 1, 3, 1, 4, 2, 1, 3, 2, 2, 5, 4, 3, 1$$

we see that this numbering satisfies all of the requirements from the theorem.

4.3 The Littlewood-Richardson rules and Inequality (IJK)

The following results relate the Littlewood-Richardson rules to (IJK) inequalities.

Theorem 4.3. *Given $I, J, K \in P_r^n$, we have $\sigma_{I'}\sigma_{J'}\sigma_K \neq 0$ if and only if there is a way to fill the boxes of $D(par(K'))\backslash D(par(I'))$ with numbers determined by $par(J')$ according to the LR rules.*

Proof. This follows from Theorem 4.1 and Proposition 3.14. □

Corollary 4.4. *Let $I, J, K \in \mathcal{P}_r^n$ and suppose that there is a way to fill the boxes of $D(par(K'))\backslash D(par(I'))$ with numbers determined by $par(J')$ according to the LR rules. Then Inequality* (IJK) *holds for all $A, B \in \text{Herm}(n)$. That is, $(I, J, K) \in H_r^n$.*

Proof. This follows immediately from Theorem 3.19 together with Theorem 4.3. □

As was noted earlier, one can interchange the roles of I and J when applying these results. That is, one can instead label the boxes of $D(\text{par}(K'))\backslash D(\text{par}(J'))$ with numbers determined by $\text{par}(I')$ according to the LR rules.

4.4 Applications of Corollary 4.4

We can now give additional proofs of the Weyl and Lidskii inequalities, and also a proof of the Thompson-Freede inequalities.

The following lemma will make our subsequent computations simpler:

Lemma 4.5. *If* $I = (i_1 < \cdots < i_r) \in \mathcal{P}_r^n$ *then* $par(I') = (i_r - r, \cdots, i_1 - 1)$.

Proof. Since the ℓth entry of I' is $n + 1 - i_{r+1-\ell}$, the ℓth entry of $\mathrm{par}(I')$ is

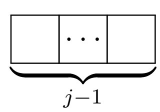
$$n - r + \ell - (n + 1 - i_{r+1-\ell}) = i_{r+1-\ell} - (r + 1 - \ell). \qquad \square$$

4.4.1 Weyl inequalities

We want to show that $c_{I',J'}^{K'} \neq 0$ whenever $I = (i)$, $J = (j)$, and $K = (i + j - 1)$ each belong to $\mathcal{P}_1^n$. First,

$$\mathrm{par}(I') = (i - 1), \qquad \mathrm{par}(J') = (j - 1), \qquad \mathrm{par}(K') = (i + j - 2),$$

and so $D(\mathrm{par}(K')) \backslash D(\mathrm{par}(I'))$ is the diagram



with $j - 1$ empty boxes that need to be labeled with $(j - 1)$ 1's. As there is exactly one way to do this, and this numbering obviously satisfies the LR rules, we conclude that $c_{I',J'}^{K'} = 1$. The Weyl inequalities now follow from Corollary 4.4.

4.4.2 Lidskii inequalities

Let $I = (i_1 < \cdots < i_r)$ and $J = (1, 2, \ldots, r) \in \mathcal{P}_r^n$. Then $\mathrm{par}(J') = (0, \ldots, 0)$, and $D(\mathrm{par}(I')) \backslash D(\mathrm{par}(I'))$ is the empty diagram. We therefore have a trivial application of the LR rules, hence $c_{I',J'}^{I'} = 1$, and so the Lidskii inequalities are valid by Corollary 4.4.

4.4.3 Thompson-Freede inequalities

Let $I = (i_1 < \cdots < i_r)$, $J = (j_1 < \cdots < j_r)$, and $K = (i_1 + j_1 - 1, \ldots, i_r + j_r - r)$ be elements of $\mathcal{P}_r^n$. Again, it suffices to show that $c_{I',J'}^{K'} \neq 0$.

Now, we have

$$\mathrm{par}(I') = (a_1, \ldots, a_r) := (i_r - r, \ldots, i_1 - 1), \quad \mathrm{par}(J') = (b_1, \ldots, b_r) := (j_r - r, \ldots, j_1 - 1)$$

and

$$\mathrm{par}(K') = (i_r + j_r - 2r, \ldots, i_1 + j_1 - 2) = (a_1 + b_1, \ldots, a_r + b_r).$$

So, $D(\mathrm{par}(K')) \backslash D(\mathrm{par}(I'))$ is the diagram whose ℓth row contains b_ℓ empty boxes, and we need to fill this diagram with b_1 1's, ..., b_r r's in a manner satisfying the LR rules. To do this, simply fill the ℓth row with all b_ℓ needed copies of the number ℓ. This numbering shows that $c_{I',J'}^{K'} \neq 0$, as was needed.

Example 4.6. Take $I = (2, 4, 5)$, $J = (3, 4, 6)$ in $\mathcal{P}_3^n$ with $n \geq 5 + 6 - 3 = 8$. We obtain

$$K = (2 + 3 - 1, 4 + 4 - 2, 5 + 6 - 3) = (4, 6, 8),$$

$$\mathrm{par}(I') = (5 - 3, 4 - 2, 2 - 1) = (2, 2, 1), \quad \mathrm{par}(J') = (6 - 3, 4 - 2, 3 - 1) = (3, 2, 2),$$

$$\mathrm{par}(K') = (8 - 3, 6 - 2, 4 - 1) = (5, 4, 3).$$

The following diagram shows that $\sigma_{I'}\sigma_{J'}\sigma_K \neq 0$ in $H^{6(n-3)}(G_3(\mathbb{C}^n), \mathbb{Z})$:

	1	1	1
	2	2	
3	3		

Thus for $n \times n$ Hermitian matrices A, B, $C = A + B$ with $n \geq 8$, we must have

$$\lambda_4^{\downarrow}(C) + \lambda_6^{\downarrow}(C) + \lambda_8^{\downarrow}(C) \leq \lambda_2^{\downarrow}(A) + \lambda_4^{\downarrow}(A) + \lambda_5^{\downarrow}(A) + \lambda_3^{\downarrow}(B) + \lambda_4^{\downarrow}(B) + \lambda_6^{\downarrow}(B).$$

CHAPTER 5: The Horn Inequalities

In 1962, Alfred Horn [11] defined sets T_r^n that he conjectured would completely characterize the possible eigenvalues of $n \times n$ Hermitian matrices $A, B, C = A + B$. He defined these sets T_r^n as follows.

Let U_r^n denote the set of triples $(I, J, K) \in (\mathcal{P}_r^n)^3$ such that

$$\sum_{i \in I} i + \sum_{j \in J} j = \sum_{k \in K} k + \frac{r(r+1)}{2}.$$

Now, define $T_1^n = U_1^n = \big\{(i, j, k) \ : \ i + j = k + 1\big\}$, and for $r > 1$ set

$$T_r^n = \{(I, J, K) \in U_r^n : \sum_{f \in F} i_f + \sum_{g \in G} j_g \leq \sum_{h \in H} k_h + p(p+1)/2$$
$$\text{for all } (F, G, H) \in T_p^r \text{ with } 1 \leq p < r\}.$$

Horn's conjecture was finally proven in the late 1990's over the course of two papers: one by Klyachko in 1998 [13], and another by Knutson and Tao in 1999 [14]. This conjecture is made precise by the following theorem.

Theorem 5.1 (Horn's Theorem)**.**

(a) One has $T_r^n \subset H_r^n$ for all $1 \leq r < n$. That is, if $(I, J, K) \in T_r^n$ then Inequality (IJK) holds for all $n \times n$ Hermitian matrices A, B, $C = A + B$.

(b) Conversely, if $(\alpha, \beta, \gamma) \in (\mathbb{R}^n)^3$ are weakly decreasing and satisfy both

- *the trace equality $\sum_{i=1}^n \gamma_i = \sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i$, and*
- *Inequality (IJK) for every $(I, J, K) \in T_r^n$ and all $1 \leq r < n$*

then there exist $A, B, C = A + B \in \mathrm{Herm}(n)$ such that

$$\lambda^{\downarrow}(A) = \alpha, \quad \lambda^{\downarrow}(B) = \beta, \quad \textit{and} \quad \lambda^{\downarrow}(C) = \gamma.$$

The goal of this chapter is to connect the sets T_r^n of "Horn triples" with the results obtained in the earlier parts of this thesis.

5.1 The relationship between T_r^n and (IJK) inequalities

Before showing how the sets T_r^n relate to (IJK) eigenvalue inequalities, it is useful to see how the sets $U_r^n \supset T_r^n$ are relevant.

Lemma 5.2. *For $I, J, K \in P_r^n$, the following are equivalent:*

(a) $(I, J, K) \in U_r^n$

(b) $\lambda = par(I')$, $\mu = par(J')$, $\nu = par(K')$ satisfy $\sum \nu_i = \sum \lambda_i + \sum \mu_i$

(c) $\sigma_{I'}\sigma_{J'}\sigma_K$ lies in the top-degree cohomology group $H^{2N}(G_r(\mathbb{C}^n))$.

Proof. For $L = (\ell_1 < \cdots < \ell_r) \in \mathcal{P}_r^n$ we have $\mathrm{par}(L') = (\ell_r - r, \ldots, \ell_1 - 1)$ by Lemma 4.5 and hence

$$\sum \mathrm{par}(L') = \sum_{j=1}^{r} \ell_j - \frac{r(r+1)}{2}.$$

Thus (b) holds if and only if

$$\sum i_\ell + \sum j_\ell - r(r+1) = \sum k_\ell - r(r+1)/2$$

which is obviously equivalent to (a).

We now show that (c) is equivalent to (a). By the gradedness of the cup product, we have $\sigma_{I'}\sigma_{J'}\sigma_K \in H^{2N}(G_r(\mathbb{C}^n), \mathbb{Z})$ if and only if $d^{co}(I') + d^{co}(J') + d^{co}(K) = N$. By Lemma 3.11 and Corollary 3.12, this happens if and only if $d^{co}(I') + d^{co}(J') = d^{co}(K')$, or equivalently, $d(I) + d(J) = d(K)$. But,

$$\begin{aligned} d(I) + d(J) = d(K) &\iff \sum (i_\ell - \ell) + \sum (j_\ell - \ell) = \sum (k_\ell - \ell) \\ &\iff \sum i_\ell + \sum j_\ell = \sum k_\ell + r(r+1)/2 \end{aligned}$$

$$\iff (I, J, K) \in U_r^n. \qquad \square$$

So U_r^n is the set of $(I, J, K) \in \mathcal{P}_r^n$ for which $\sigma_{I'}\sigma_{J'}\sigma_K \in H^{2N}(G_r(\mathbb{C}^n))$, and it turns out that $T_r^n \subset U_r^n$ is precisely the set of $(I, J, K) \in U_r^n$ for which $\sigma_{I'}\sigma_{J'}\sigma_K$ is *nonzero.* Equivalently, $T_r^n = S_r^n$ where

$$S_r^n = \{(I, J, K) \in U_r^n : c_{I',J'}^{K'} \neq 0\}.$$

This is implied by the following theorem which establishes an important connection between $H^\bullet(G_r(\mathbb{C}^n))$ and $H^\bullet(G_p(\mathbb{C}^r))$ for all $p < r$. For a proof outline, see [6, Theorem 17].

Theorem 5.3. *Let $(I, J, K) \in U_r^n$ be given, and set $\lambda = par(I'), \mu = par(J')$, and $\nu = par(K')$. The following are equivalent:*

(i) The class of σ_K occurs with nonzero coefficient in the product $\sigma_{I'}\sigma_{J'}$ in $H^\bullet(G_r(\mathbb{C}^n))$.

(ii) For all $p < r$ one has $\sum_{h\in H} \nu_h \leq \sum_{f\in F} \lambda_f + \sum_{g\in G} \mu_g$ for all $(F, G, H) \in (\mathcal{P}_p^r)^3$ such that the class of σ_H occurs in the product $\sigma_{F'}\sigma_{G'}$ in $H^\bullet(G_p(\mathbb{C}^r))$.

Corollary 5.4. *Let $(I, J, K) \in U_r^n$. With notation as in the previous theorem, one has $(I, J, K) \in S_r^n$ if and only if*

$$\sum_{h\in H} \nu_h \leq \sum_{f\in F} \lambda_f + \sum_{g\in G} \mu_g$$

for every $(F, G, H) \in S_p^r$ for all $p < r$.

We now use this corollary to show that $T_r^n = S_r^n$. It should be noted that our proof relies on the existence of a particular bijection $S_r^n \to S_{n-r}^n$ given by the map

$$(I, J, K) \mapsto (I^\perp, J^\perp, K^\perp)$$

where, for example, if $I = (i_1 < \cdots < i_r)$ then $I^\perp = (i \in \{1, \ldots, n\} : n + 1 - i \notin I)$. This bijection is established in Section 6.1.

Our proof that $S_r^n = T_r^n$ is by induction on r, expanding the proof provided in [6]. The following lemma proves our base case.

Lemma 5.5. *$S_1^n = T_1^n$ for each n.*

Proof. We have $S_1^n \subset U_1^n = T_1^n$ by definition. Conversely, if $(I, J, K) \in T_1^n = U_1^n$ then (I, J, K) are the indices of some Weyl inequality, and thus $c_{I',J'}^{K'} \neq 0$. So $T_1^n \subset S_1^n$ as well, completing the proof. □

Lemma 5.6. *Suppose $(I, J, K) \in U_r^n$ and $p \in \{1, \ldots, r - 1\}$. Then*

$$\sum_{f \in F} i_f + \sum_{g \in G} j_g \leq \sum_{h \in H} k_h + \frac{p(p+1)}{2} \tag{5.1}$$

holds for some $(F, G, H) \in T_p^r$ if and only if

$$\sum_{f \in F^\perp} \lambda_f + \sum_{g \in G^\perp} \mu_g \geq \sum_{h \in H^\perp} \nu_h. \tag{5.2}$$

where $\lambda = par(I')$, $\mu = par(J')$, and $\nu = par(K')$.

Proof. Since $\lambda_{r-\ell+1} = i_\ell - \ell$ holds for each ℓ, Equation (5.1) holds if and only if

$$\sum_{f \in F} (\lambda_{r-f+1} + f) + \sum_{g \in G} (\mu_{r-g+1} + g) \leq \sum_{h \in H} (\nu_{r-h+1} + h) + \frac{p(p+1)}{2} \tag{5.3}$$

But $(F, G, H) \in U_p^r$ implies that $\sum_{f \in F} f + \sum_{g \in G} g = \sum_{h \in H} h + p(p+1)/2$. So Equation (5.3) becomes

$$\sum_{f \in F} \lambda_{r-f+1} + \sum_{g \in G} \mu_{r-g+1} \leq \sum_{h \in H} \nu_{r-h+1}. \tag{5.4}$$

Now, since $(I, J, K) \in U_r^n$, we have $\sum \lambda_i + \sum \mu_i = \sum \nu_i$, and it follows that Equation (5.4) holds if and only if

$$\sum_{f \in F^\perp} \lambda_f + \sum_{g \in G^\perp} \mu_g \geq \sum_{h \in H^\perp} \nu_h$$

where $F^\perp = \{f \in \{1, \ldots, r\} : r - f + 1 \notin F\}$ as above, and $G^\perp$ and $H^\perp$ are defined similarly. □

Theorem 5.7. *For any* $1 \leq r < n$ *one has* $T_r^n = S_r^n$.

Proof. Since $T_1^n = S_1^n$ (Lemma 5.5), we can assume that $r > 1$ and that $T_p^m = S_p^m$ whenever $p < r \leq m$.

Let $(I, J, K) \in U_r^n$ and set $\lambda = \text{par}(I')$, $\mu = \text{par}(J')$, and $\nu = \text{par}(K')$. It follows from the previous lemma (and the definition of T_r^n) that $(I, J, K) \in T_r^n$ if and only if

$$\sum_{f \in F^\perp} \lambda_f + \sum_{g \in G^\perp} \mu_g \geq \sum_{h \in H^\perp} \nu_h \tag{5.5}$$

for every $(F, G, H) \in T_p^r = S_p^r$ for all $p < r$.

Moreover, we see in Section 6.1 that $(F, G, H) \in S_p^r$ if and only if $(F^\perp, G^\perp, H^\perp) \in S_{r-p}^r$. Thus, it is clear that Equation (5.5) holds for every $(F, G, H) \in S_p^r$ for all $p < r$ if and only if

$$\sum_{h \in H} \nu_h \leq \sum_{f \in F} \lambda_f + \sum_{g \in G} \mu_g \tag{5.6}$$

holds for every $(F, G, H) \in S_p^r$ for all $p < r$. This is equivalent to having $(I, J, K) \in S_r^n$ by Corollary 5.4. □

This next theorem summarizes the connections we have established between the sets T_r^n of "Horn triples" and the earlier results of this thesis.

Theorem 5.8. *For $(I, J, K) \in U_r^n$, conditions (a)-(e) below are equivalent.*

(a) $(I, J, K) \in T_r^n$

(b) $S(I', \mathcal{F}_1) \cap S(J', \mathcal{F}_2) \cap S(K, \mathcal{F}_3) \neq \emptyset$ *for all complete flags* $\mathcal{F}_j$ $(j = 1, 2, 3)$ *for* $\mathbb{C}^n$.

(c) $\sigma_{I'}\sigma_{J'}\sigma_K \neq 0$ *in* $H^{2N}(G_r(\mathbb{C}^n), \mathbb{Z})$

(d) $c_{I',J'}^{K'} \neq 0$

(e) It is possible to use par(J') to number the boxes in $D(par(K'))\backslash D(par(I'))$ in accordance with the LR rules.

Moreover, each of these conditions implies

(f) Inequality (IJK) *holds for all* $A, B \in \mathrm{Herm}(n)$.

Note that we can use "(a) implies (f)" above to find eigenvalue inequalities which hold for all Hermitian matrices $A, B, C = A + B$. Indeed, when n and r are small, it is easy to compute the sets T_r^n recursively from their definition, with the help of a computer. Each element of T_r^n gives rise to an eigenvalue inequality.

Example 5.9. Let $A, B, C = A + B$ be 4×4 Hermitian matrices with eigenvalues $\lambda^{\downarrow}(A) = \alpha$, $\lambda^{\downarrow}(B) = \beta$, and $\lambda^{\downarrow}(C) = \gamma$. The set T_2^4 yields the following 21 eigenvalue inequalities:

$$\left\{\begin{array}{ll}
\gamma_1+\gamma_2 \le \alpha_1+\alpha_2+\beta_1+\beta_2 & \gamma_1+\gamma_3 \le \alpha_1+\alpha_2+\beta_1+\beta_3 \\
\gamma_1+\gamma_3 \le \alpha_1+\alpha_3+\beta_1+\beta_2 & \gamma_1+\gamma_4 \le \alpha_1+\alpha_2+\beta_1+\beta_4 \\
\gamma_1+\gamma_4 \le \alpha_1+\alpha_4+\beta_1+\beta_2 & \gamma_1+\gamma_4 \le \alpha_1+\alpha_3+\beta_1+\beta_3 \\
\gamma_2+\gamma_3 \le \alpha_1+\alpha_2+\beta_2+\beta_3 & \gamma_2+\gamma_3 \le \alpha_2+\alpha_3+\beta_1+\beta_2 \\
\gamma_2+\gamma_3 \le \alpha_1+\alpha_3+\beta_1+\beta_3 & \gamma_2+\gamma_4 \le \alpha_1+\alpha_2+\beta_2+\beta_4 \\
\gamma_2+\gamma_4 \le \alpha_2+\alpha_4+\beta_1+\beta_2 & \gamma_2+\gamma_4 \le \alpha_1+\alpha_3+\beta_1+\beta_4 \\
\gamma_2+\gamma_4 \le \alpha_1+\alpha_4+\beta_1+\beta_3 & \gamma_2+\gamma_4 \le \alpha_1+\alpha_3+\beta_2+\beta_3 \\
\gamma_2+\gamma_4 \le \alpha_2+\alpha_3+\beta_1+\beta_3 & \gamma_3+\gamma_4 \le \alpha_1+\alpha_2+\beta_3+\beta_4 \\
\gamma_3+\gamma_4 \le \alpha_3+\alpha_4+\beta_1+\beta_2 & \gamma_3+\gamma_4 \le \alpha_1+\alpha_3+\beta_2+\beta_4 \\
\gamma_3+\gamma_4 \le \alpha_2+\alpha_4+\beta_1+\beta_3 & \gamma_3+\gamma_4 \le \alpha_2+\alpha_3+\beta_2+\beta_3 \\
\gamma_3+\gamma_4 \le \alpha_1+\alpha_4+\beta_1+\beta_4 &
\end{array}\right\}$$

To clarify, this last inequality must hold since $((1, 4), (1, 4), (3, 4)) \in T_2^4$.

There are also 10 eigenvalue inequalities given by both T_1^4 and T_3^4. Therefore, the sets T_r^4 provide us with a total of $10 + 21 + 10 = 41$ eigenvalue inequalities which necessarily hold for $A, B, C = A + B \in \mathrm{Herm}(4)$.

Example 5.10. The set T_3^5 has 56 elements which yield Horn inequalities involving sums of three of the eigenvalues $\alpha = \lambda^{\downarrow}(A)$, $\beta = \lambda^{\downarrow}(B)$, $\gamma = \lambda^{\downarrow}(A+B)$ for matrices $A, B \in \mathrm{Herm}(5)$.

$$
\left\{
\begin{array}{ll}
\gamma_1+\gamma_2+\gamma_3 \le \alpha_1+\alpha_2+\alpha_3+\beta_1+\beta_2+\beta_3 & \gamma_1+\gamma_2+\gamma_4 \le \alpha_1+\alpha_2+\alpha_3+\beta_1+\beta_2+\beta_4 \\
\gamma_1+\gamma_2+\gamma_5 \le \alpha_1+\alpha_2+\alpha_3+\beta_1+\beta_2+\beta_5 & \gamma_1+\gamma_3+\gamma_4 \le \alpha_1+\alpha_2+\alpha_3+\beta_1+\beta_3+\beta_4 \\
\gamma_1+\gamma_3+\gamma_5 \le \alpha_1+\alpha_2+\alpha_3+\beta_1+\beta_3+\beta_5 & \gamma_1+\gamma_4+\gamma_5 \le \alpha_1+\alpha_2+\alpha_3+\beta_1+\beta_4+\beta_5 \\
\gamma_2+\gamma_3+\gamma_4 \le \alpha_1+\alpha_2+\alpha_3+\beta_2+\beta_3+\beta_4 & \gamma_2+\gamma_3+\gamma_5 \le \alpha_1+\alpha_2+\alpha_3+\beta_2+\beta_3+\beta_5 \\
\gamma_2+\gamma_4+\gamma_5 \le \alpha_1+\alpha_2+\alpha_3+\beta_2+\beta_4+\beta_5 & \gamma_3+\gamma_4+\gamma_5 \le \alpha_1+\alpha_2+\alpha_3+\beta_3+\beta_4+\beta_5 \\
\gamma_1+\gamma_2+\gamma_4 \le \alpha_1+\alpha_2+\alpha_4+\beta_1+\beta_2+\beta_3 & \gamma_1+\gamma_2+\gamma_5 \le \alpha_1+\alpha_2+\alpha_4+\beta_1+\beta_2+\beta_4 \\
\gamma_1+\gamma_3+\gamma_4 \le \alpha_1+\alpha_2+\alpha_4+\beta_1+\beta_2+\beta_4 & \gamma_1+\gamma_3+\gamma_5 \le \alpha_1+\alpha_2+\alpha_4+\beta_1+\beta_2+\beta_5 \\
\gamma_1+\gamma_3+\gamma_5 \le \alpha_1+\alpha_2+\alpha_4+\beta_1+\beta_3+\beta_4 & \gamma_2+\gamma_3+\gamma_4 \le \alpha_1+\alpha_2+\alpha_4+\beta_1+\beta_3+\beta_4 \\
\gamma_1+\gamma_4+\gamma_5 \le \alpha_1+\alpha_2+\alpha_4+\beta_1+\beta_3+\beta_5 & \gamma_2+\gamma_3+\gamma_5 \le \alpha_1+\alpha_2+\alpha_4+\beta_1+\beta_3+\beta_5 \\
\gamma_2+\gamma_4+\gamma_5 \le \alpha_1+\alpha_2+\alpha_4+\beta_1+\beta_4+\beta_5 & \gamma_2+\gamma_3+\gamma_5 \le \alpha_1+\alpha_2+\alpha_4+\beta_2+\beta_3+\beta_4 \\
\gamma_2+\gamma_4+\gamma_5 \le \alpha_1+\alpha_2+\alpha_4+\beta_2+\beta_3+\beta_5 & \gamma_3+\gamma_4+\gamma_5 \le \alpha_1+\alpha_2+\alpha_4+\beta_2+\beta_4+\beta_5 \\
\gamma_1+\gamma_2+\gamma_5 \le \alpha_1+\alpha_2+\alpha_5+\beta_1+\beta_2+\beta_3 & \gamma_1+\gamma_3+\gamma_5 \le \alpha_1+\alpha_2+\alpha_5+\beta_1+\beta_2+\beta_4 \\
\gamma_1+\gamma_4+\gamma_5 \le \alpha_1+\alpha_2+\alpha_5+\beta_1+\beta_2+\beta_5 & \gamma_2+\gamma_3+\gamma_5 \le \alpha_1+\alpha_2+\alpha_5+\beta_1+\beta_3+\beta_4 \\
\gamma_2+\gamma_4+\gamma_5 \le \alpha_1+\alpha_2+\alpha_5+\beta_1+\beta_3+\beta_5 & \gamma_3+\gamma_4+\gamma_5 \le \alpha_1+\alpha_2+\alpha_5+\beta_1+\beta_4+\beta_5 \\
\gamma_1+\gamma_3+\gamma_4 \le \alpha_1+\alpha_3+\alpha_4+\beta_1+\beta_2+\beta_3 & \gamma_1+\gamma_3+\gamma_5 \le \alpha_1+\alpha_3+\alpha_4+\beta_1+\beta_2+\beta_4 \\
\gamma_2+\gamma_3+\gamma_4 \le \alpha_1+\alpha_3+\alpha_4+\beta_1+\beta_2+\beta_4 & \gamma_2+\gamma_3+\gamma_5 \le \alpha_1+\alpha_3+\alpha_4+\beta_1+\beta_2+\beta_5 \\
\gamma_1+\gamma_4+\gamma_5 \le \alpha_1+\alpha_3+\alpha_4+\beta_1+\beta_3+\beta_4 & \gamma_2+\gamma_3+\gamma_5 \le \alpha_1+\alpha_3+\alpha_4+\beta_1+\beta_3+\beta_4 \\
\gamma_2+\gamma_4+\gamma_5 \le \alpha_1+\alpha_3+\alpha_4+\beta_1+\beta_3+\beta_5 & \gamma_2+\gamma_4+\gamma_5 \le \alpha_1+\alpha_3+\alpha_4+\beta_2+\beta_3+\beta_4 \\
\gamma_3+\gamma_4+\gamma_5 \le \alpha_1+\alpha_3+\alpha_4+\beta_2+\beta_3+\beta_5 & \gamma_1+\gamma_3+\gamma_5 \le \alpha_1+\alpha_3+\alpha_5+\beta_1+\beta_2+\beta_3 \\
\gamma_1+\gamma_4+\gamma_5 \le \alpha_1+\alpha_3+\alpha_5+\beta_1+\beta_2+\beta_4 & \gamma_2+\gamma_3+\gamma_5 \le \alpha_1+\alpha_3+\alpha_5+\beta_1+\beta_2+\beta_4 \\
\gamma_2+\gamma_4+\gamma_5 \le \alpha_1+\alpha_3+\alpha_5+\beta_1+\beta_2+\beta_5 & \gamma_2+\gamma_4+\gamma_5 \le \alpha_1+\alpha_3+\alpha_5+\beta_1+\beta_3+\beta_4 \\
\gamma_3+\gamma_4+\gamma_5 \le \alpha_1+\alpha_3+\alpha_5+\beta_1+\beta_3+\beta_5 & \gamma_1+\gamma_4+\gamma_5 \le \alpha_1+\alpha_4+\alpha_5+\beta_1+\beta_2+\beta_3 \\
\gamma_2+\gamma_4+\gamma_5 \le \alpha_1+\alpha_4+\alpha_5+\beta_1+\beta_2+\beta_4 & \gamma_3+\gamma_4+\gamma_5 \le \alpha_1+\alpha_4+\alpha_5+\beta_1+\beta_2+\beta_5 \\
\gamma_2+\gamma_3+\gamma_4 \le \alpha_2+\alpha_3+\alpha_4+\beta_1+\beta_2+\beta_3 & \gamma_2+\gamma_3+\gamma_5 \le \alpha_2+\alpha_3+\alpha_4+\beta_1+\beta_2+\beta_4 \\
\gamma_2+\gamma_4+\gamma_5 \le \alpha_2+\alpha_3+\alpha_4+\beta_1+\beta_3+\beta_4 & \gamma_3+\gamma_4+\gamma_5 \le \alpha_2+\alpha_3+\alpha_4+\beta_2+\beta_3+\beta_4 \\
\gamma_2+\gamma_3+\gamma_5 \le \alpha_2+\alpha_3+\alpha_5+\beta_1+\beta_2+\beta_3 & \gamma_2+\gamma_4+\gamma_5 \le \alpha_2+\alpha_3+\alpha_5+\beta_1+\beta_2+\beta_4 \\
\gamma_3+\gamma_4+\gamma_5 \le \alpha_2+\alpha_3+\alpha_5+\beta_1+\beta_3+\beta_4 & \gamma_2+\gamma_4+\gamma_5 \le \alpha_2+\alpha_4+\alpha_5+\beta_1+\beta_2+\beta_3 \\
\gamma_3+\gamma_4+\gamma_5 \le \alpha_2+\alpha_4+\alpha_5+\beta_1+\beta_2+\beta_4 & \gamma_3+\gamma_4+\gamma_5 \le \alpha_3+\alpha_4+\alpha_5+\beta_1+\beta_2+\beta_3
\end{array}
\right\}
$$

The following table shows the size of T_r^n for $1 \le r < n \le 11$. This was produced by using a Prolog program to enumerate these T_r^n-sets. In particular, since $\sum_{r=1}^{10} |T_r^{11}| = 971{,}430$, we see that the sets T_r^{11} provide us with 971,430 (IJK) eigen-

r \ n	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
1	3	6	10	15	21	28	36	45	55	66
2		6	21	56	126	252	462	792	1287	2002
3			10	56	228	751	2120	5317	12140	25678
4				15	126	751	3516	13704	46208	138519
5					21	252	2120	13704	71973	319450
6						28	462	5317	46208	319450
7							36	792	12140	138519
8								45	1287	25678
9									55	2002
10										66

Table 5.1: Cardinality of T_r^n for $1 \leq r < n \leq 11$

value inequalities which necessarily hold for all $A, B, C = A + B \in \text{Herm}(11)$! The table also illustrates the fact that $|T_r^n| = |T_{n-r}^n|$. This is a consequence of the correspondence between S_r^n and S_{n-r}^n, used in the proof for Theorem 5.7 and established below in Section 6.1.

5.2 An alternate characterization of T_r^n

The following remarkable result provides another characterization of T_r^n. It is proven in more generality in [6, Theorem 17].

Theorem 5.11. *Suppose $(I, J, K) \in U_r^n$. Then $(I, J, K) \in T_r^n$ if and only if there* *exist* $r \times r$ *Hermitian matrices* $A, B, C = A + B$ *with*

$$\lambda^{\downarrow}(A) = par(I'), \qquad \lambda^{\downarrow}(B) = par(J'), \qquad \lambda^{\downarrow}(C) = par(K'). \tag{5.7}$$

Note that this theorem together with Theorem 5.8 gives us yet another way of proving the Weyl, Lidskii, and Thompson-Freede inequalities. Indeed, we only need to show that the (I, J, K) triple corresponding to each inequality belongs to T_r^n by finding $A, B, C = A + B \in \text{Herm}(r)$ that satisfy (5.7).

Example 5.12 (Weyl Inequalities)**.** Let $I = (i)$, $J = (j)$, and $K = (i + j - 1)$. Then

$$\text{par}(I') = (i - 1), \quad \text{par}(J') = (j - 1), \quad \text{par}(K') = (i + j - 2),$$

and so the 1×1 matrices $A = [i - 1]$, $B = [j - 1]$, and $C = [i + j - 2]$ satisfy (5.7). Hence $(I, J, K) \in T_r^n$.

Example 5.13 (Lidskii inequalities)**.** Choose any $I \in \mathcal{P}_r^n$ and let $J = (1, 2, \ldots, r)$. We want to show that $(I, J, I) \in T_r^n$. Since $\text{par}(J') = (0, \ldots, 0)$, the $r \times r$ Hermitian matrices $A = C = \text{diag}(\text{par}(I'))$ and $B = O_{r \times r}$ satisfy (5.7). Hence $(I, J, I) \in T_r^n$.

Example 5.14 (Thompson-Freede inequalities)**.** Let $I = (i_1 < \cdots < i_r)$, $J = (j_1 < \cdots < j_r)$, and $K = (i_1 + j_1 - 1, \ldots, i_r + j_r - r)$ be elements of $\mathcal{P}_r^n$. Then

$$\text{par}(I') = (a_1, \ldots, a_r) := (i_r - r, \ldots, i_1 - 1), \quad \text{par}(J') = (b_1, \ldots, b_r) := (j_r - r, \ldots, j_1 - 1),$$

and

$$\text{par}(K') = (i_r + j_r - 2r, \ldots, i_1 + j_1 - 2) = (a_1 + b_1, \ldots, a_r + b_r).$$

So the $r \times r$ Hermitian matrices

$$A = \operatorname{diag}(a_1, \ldots, a_r), \quad B = \operatorname{diag}(b_1, \ldots, b_r), \quad C = \operatorname{diag}(a_1 + b_1, \ldots, a_r + b_r)$$

satisfy (5.7), showing that $(I, J, K) \in T_r^n$.

5.3 Further cohomological (IJK) inequalities

We have seen that $T_r^n \subset H_r^n$ and that T_r^n is precisely the set of indices $(I, J, K) \in U_r^n$ for which $\sigma_{I'}\sigma_{J'}\sigma_K \neq 0$. However, Theorem 3.19 shows that *every* nonzero cup product $\sigma_{I'}\sigma_{J'}\sigma_K$ produces an (IJK) eigenvalue inequality, and hence that the condition $(I, J, K) \in U_r^n$ (equivalently, $\sigma_{I'}\sigma_{J'}\sigma_K \in H^{2N}(G_r(\mathbb{C}^n))$) is unnecessary. The sets T_r^n are therefore apparently much smaller than the sets

$$\begin{aligned} C_r^n &= \big\{(I, J, K) \in (\mathcal{P}_r^n)^3 \ : \ \sigma_{I'}\sigma_{J'}\sigma_K \neq 0\big\} \\ &= \big\{(I, J, K) \in (\mathcal{P}_r^n)^3 \ : \ S(I', \mathcal{F}_1) \cap S(J', \mathcal{F}_2) \cap S(K, \mathcal{F}_3) \neq \emptyset \text{ for all flags } \mathcal{F}_j\big\}. \end{aligned}$$

Since the eigenvalue inequalities produced by elements of C_r^n are consequences of the ring structure on $H^\bullet(G_r(\mathbb{C}^n))$, we will refer to them as *cohomological (IJK) inequalities*. The purpose of this section is to prove that every cohomological (IJK) inequality is implied by at least one Horn inequality. This is made precise below.

For $I = (i_1 < \cdots < i_r)$, $J = (j_1 < \cdots < j_r)$ in $\mathcal{P}_r^n$ we will write $I \leq J$ when $i_\ell \leq j_\ell$ for each $\ell = 1, \ldots, r$. The following facts are easily verified.

- If $I \leq J$ then $\sum_{j \in J} \lambda_j^\downarrow(A) \leq \sum_{i \in I} \lambda_i^\downarrow(A)$ for all $A \in \operatorname{Herm}(n)$.
- If $I \leq J$ then $S(I, \mathcal{F}) \subset S(J, \mathcal{F})$ for all complete flags $\mathcal{F}$.
- $I \leq J \iff J' \leq I'$ (complementary indices).
- $I \leq J \iff D(\operatorname{par}(J)) \prec D(\operatorname{par}(I))$ (Young diagrams).

Definition 5.15. Given $(I, J, K), (I_1, J_1, K_1) \in (\mathcal{P}_r^n)^3$ we say that (I, J, K) *covers* (I_1, J_1, K_1) when $I_1 \leq I,\ J_1 \leq J$, and $K \leq K_1$.

Lemma 5.16. *If $(I, J, K) \in H_r^n$ covers (I_1, J_1, K_1) then $(I_1, J_1, K_1) \in H_r^n$. Likewise, if $(I, J, K) \in C_r^n$ covers (I_1, J_1, K_1) then $(I_1, J_1, K_1) \in C_r^n$.*

Proof. Suppose that (I, J, K) covers (I_1, J_1, K_1) and that $(I, J, K) \in H_r^n$. Then for any $n \times n$ Hermitian matrices A, B we have

$$\sum_{k \in K_1} \lambda_k^{\downarrow}(A+B) \leq \sum_{k \in K} \lambda_k^{\downarrow}(A+B) \leq \sum_{i \in I} \lambda_i^{\downarrow}(A) + \sum_{j \in J} \lambda_j^{\downarrow}(B) \leq \sum_{i \in I_1} \lambda_i^{\downarrow}(A) + \sum_{j \in J_1} \lambda_j^{\downarrow}(B).$$

So $(I_1, J_1, K_1) \in H_r^n$ as claimed.

Next suppose that (I, J, K) covers (I_1, J_1, K_1) and that $(I, J, K) \in C_r^n$. Let $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ be complete flags. Since $I_1 \leq I$, we have $I' \leq I_1'$, and hence $S(I', \mathcal{F}_1) \subset S(I_1', \mathcal{F}_1)$. Similarly, $S(J', \mathcal{F}_2) \subset S(J_1', \mathcal{F}_2)$ and $S(K, \mathcal{F}_3) \subset S(K_1, \mathcal{F}_3)$ since $J' \leq J_1'$ and $K \leq K_1$. Therefore,

$$\emptyset \neq S(I', \mathcal{F}_1) \cap S(J', \mathcal{F}_2) \cap S(K, \mathcal{F}_3) \subset S(I_1', \mathcal{F}_1) \cap S(J_1', \mathcal{F}_2) \cap S(K_1, \mathcal{F}_3).$$

So $S(I_1', \mathcal{F}_1) \cap S(J_1', \mathcal{F}_2) \cap S(K_1, \mathcal{F}_3) \neq \emptyset$, and thus $(I_1, J_1, K_1) \in C_r^n$ as claimed. □

If $(I, J, K) \in H_r^n$ and (I, J, K) covers (I_1, J_1, K_1) then it is easily seen that the eigenvalue inequality obtained from (I_1, J_1, K_1) is just a weakened form of the eigenvalue inequality obtained from (I, J, K). This is illustrated by the following example.

Example 5.17. Let $A, B \in \mathrm{Herm}(7)$, and write

$$\alpha = \lambda^{\downarrow}(A), \quad \beta = \lambda^{\downarrow}(B), \quad \gamma = \lambda^{\downarrow}(A+B)$$

Now $(I, J, K) = \big((1,4,7),(1,3,6),(4,5,7)\big)$ belongs to T_3^7, and thus

$$\gamma_4 + \gamma_5 + \gamma_7 \leq \alpha_1 + \alpha_4 + \alpha_7 + \beta_1 + \beta_3 + \beta_6. \tag{5.8}$$

The triple (I, J, K) covers both $\big((1,4,7),(1,3,6),(5,6,7)\big)$ and $\big((1,3,6),(1,2,3),(4,5,7)\big)$, as well as many others. These two triples produce cohomological inequalities

$$\gamma_5 + \gamma_6 + \gamma_7 \leq \alpha_1 + \alpha_4 + \alpha_7 + \beta_1 + \beta_3 + \beta_6 \quad \text{and}$$
$$\gamma_4 + \gamma_5 + \gamma_7 \leq \alpha_1 + \alpha_3 + \alpha_6 + \beta_1 + \beta_2 + \beta_3.$$

which are weakened forms of (5.8) obtained by taking smaller eigenvalues of $A + B$ and/or larger eigenvalues of A, B.

Proposition 5.18. *Every triple in C_r^n is covered by a triple in T_r^n. Therefore, each cohomological* (IJK) *inequality is implied by a Horn inequality.*

Proof. Let $(I, J, K) \in C_r^n$ and write

$$\sigma_{I'}\sigma_{J'}\sigma_K = \sum_L b^L_{I,J,K}\sigma_L$$

for some coefficients $b^L_{I,J,K} \in \mathbb{Z}$, where the sum is over all $L \in \mathcal{P}_r^n$ with $d^{co}(L) = d^{co}(I') + d^{co}(J') + d^{co}(K)$. As $(I, J, K) \in C_r^n$ we have $\sigma_{I'}\sigma_{J'}\sigma_K \neq 0$ in $H^\bullet(G_r(\mathbb{C}^n))$. Hence $b^{L_\circ}_{I,J,K} \neq 0$ for some $L_\circ$ and it follows that

$$\big(\sigma_{I'}\sigma_{J'}\sigma_K\sigma_{L_\circ'} \neq 0\big) \in H^{2N}(G_r(\mathbb{C}^n))$$

for this $L_\circ$. Now write

$$\sigma_K\sigma_{L_\circ'} = \sum_M c^M_{K,L_\circ'}\sigma_M$$

summing over all $M \in \mathcal{P}_r^n$ with $d^{co}(M) = d^{co}(K) + d^{co}(L'_\circ)$. We have

$$\sum_M c^M_{K,L'_\circ} \sigma_{I'}\sigma_{J'}\sigma_M = \sigma_{I'}\sigma_{J'}\sigma_K\sigma_{L'_\circ} \neq 0,$$

and so there exists some $M_\circ$ with both

$$c^{M_\circ}_{K,L'_\circ} \neq 0 \quad \text{and} \quad (\sigma_{I'}\sigma_{J'}\sigma_{M_\circ} \neq 0) \in H^{2N}(G_r(\mathbb{C}^n)). \tag{5.9}$$

As $\sigma_{I'}\sigma_{J'}\sigma_{M_\circ}$ is nonzero and lies in the top-degree cohomology group $H^{2N}(G_r(\mathbb{C}^n))$, we know from Theorem 5.8 that $(I, J, M_\circ) \in T_r^n$. Moreover, since $c^{M_\circ}_{K,L'_\circ} \neq 0$, the LR rules show that $D(\text{par}(K)) \prec D(\text{par}(M_\circ))$, and it follows that $M_\circ \leq K$. Thus $(I, J, M_\circ)$ covers (I, J, K), and so the cohomological inequality produced by (I, J, K) is implied by the cohomological inequality produced by $(I, J, M_\circ)$. □

Remark 5.19. Given a triple $(I, J, K) \in C_r^n \backslash T_r^n$, there are often many different triples in T_r^n that cover (I, J, K). The above proof shows, however, that one can always obtain at least one such covering by decreasing some k_ℓ's. The eigenvalue inequality for (I, J, K) thus involves smaller eigenvalues for $A + B$ than does the covering inequality.

Proposition 5.18 shows that the sets $T_r^n = S_r^n$ can be viewed as a smaller generating set for the cohomological (IJK) inequalities produced by C_r^n. However, T_r^n is *not* minimal in this regard. Indeed, it can be shown that the inequalities produced by $T_r^n = S_r^n$ (and thus C_r^n) are implied by the inequalities produced by the set

$$R_r^n := \{(I, J, K) \in U_r^n : c^{K'}_{I',J'} = 1\} \subset S_r^n.$$

In fact, Knutson, Tao, and Woodward proved that the inequalities produced by R_r^n

are independent (see [15]), and hence that these inequalities are a minimal generating set for the cohomological inequalities produced by C_r^n.

CHAPTER 6: Additional Results

In this chapter we prove some additional facts about the sets S_r^n and T_r^n that were defined in Chapter 5. Specifically, we show in Section 1 that $|S_r^n| = |S_{n-r}^n|$ by establishing a bijection between the two sets. We then show in Section 2 that for $r \in \{1, n-2, n-1\}$ one has $(I, J, K) \in T_r^n$ if and only if there exist *diagonal* $r \times r$ matrices $A, B, C = A + B$ with $\lambda^{\downarrow}(A) = \text{par}(I')$, $\lambda^{\downarrow}(B) = \text{par}(J')$, and $\lambda^{\downarrow}(C) = \text{par}(K')$. This strengthens Theorem 5.8 significantly for these values of r.

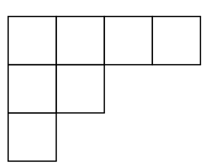
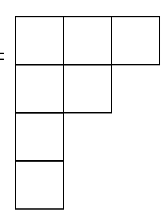
6.1 A bijection between S_r^n and S_{n-r}^n

The goal of this section is to exhibit a bijection between S_r^n and S_{n-r}^n. However, we need some preliminary results before we can do this.

6.1.1 Background material

Through the rest of this section, we regard all of our Young diagrams as having r rows and $n - r$ columns, even if this means that some rows/columns do not contain any boxes. In other words, all diagrams live inside the ambient space $D((n-r)^r)$.

Definition 6.1. The *transpose* of a Young diagram $D(a_1, \ldots, a_r)$ is the Young diagram $D^T(a_1, \ldots, a_r)$ with a_j boxes in its jth *column* for $j = 1, \ldots, n-r$. For example,

$D(4, 2, 1) =$  $\Longrightarrow$ $D^T(4, 2, 1) =$ 

Lemma 6.2. *$D^T(a_1,\ldots,a_r) = D(b_1,\ldots,b_{n-r})$ where $(b_1,\ldots,b_{n-r})$ is the sequence*

$$\mathbf{r}^{a_r} \oplus (\mathbf{r-1})^{a_{r-1}-a_r} \oplus (\mathbf{r-2})^{a_{r-2}-a_{r-1}} \oplus \cdots \oplus \mathbf{2}^{a_2-a_3} \oplus \mathbf{1}^{a_1-a_2} \oplus \mathbf{0}^{(n-r)-a_1}.$$

Here, $\mathbf{x}^m = (x,\ldots,x) \in \mathbb{N}_0^m$ and $\oplus$ is the concatenation operation:

$$(u_1,\ldots,u_\ell) \oplus (v_1,\ldots,v_m) \mapsto (u_1,\ldots,u_\ell,v_1,\ldots,v_m).$$

Proof. One easily sees that $a_j - a_{j+1}$ is the number of length j *columns* in $D(a_1,\ldots,a_r)$, each of which corresponds to a distinct length j *row* of the transpose diagram. □

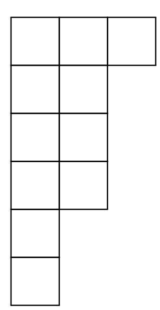
Corollary 6.3. *We have $b_{a_j} = j$ if and only if $a_j - a_{j+1} > 0$.*

Proof. By Lemma 6.2, $(b_1,\ldots,b_{n-r})$ contains exactly $a_j - a_{j+1}$ copies of the integer j for each $j = 0,\ldots,r$. (Here we define $a_0 = n-r$ and $a_{r+1} = 0$.) Hence $(b_1,\ldots,b_{n-r})$ contains exactly $\sum_{\ell=j+1}^{r}(a_\ell - a_{\ell+1}) = a_{j+1}$ integers greater than j, and entries $a_{j+1} + 1,\ldots,a_{j+1}+(a_j - a_{j+1})$ are all equal to j. The result now follows since this interval is nonempty if and only if $a_j - a_{j+1} > 0$, and its last element is $b_{a_{j+1}+(a_j-a_{j+1})} = b_{a_j}$. □

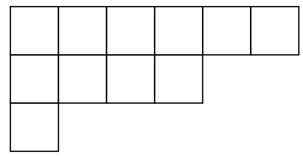
Example 6.4. We illustrate this last point with an example. Consider $I = (2,4,5,6,8,9)$ in $\mathcal{P}_6^{10}$. Note that

$$\mathrm{par}(I) = (a_1,a_2,a_3,a_4,a_5,a_6) = (3,2,2,2,1,1)$$

has Young diagram



The transposed diagram is



and thus

$$D^T(\text{par}(I)) = D(b_1, b_2, b_3) = D(6, 4, 1).$$

Therefore,

$$b_{a_1} = b_3 = 1, \quad b_{a_4} = b_2 = 4, \quad b_{a_6} = b_1 = 6$$

whereas

$$b_{a_2} = b_2 = 4 > 2, \quad b_{a_3} = b_2 = 4 > 3, \quad b_{a_5} = b_1 = 6 > 5.$$

So in this example we have

$$a_j - a_{j+1} > 0 \iff j = 1, 4, 6$$

as guaranteed by Corollary 6.3.

Let $I = (i_1 < i_2 < \cdots < i_r) \in P_r^n$ be fixed, write $\text{par}(I) = (a_1, \ldots, a_r)$, and suppose that $D^T(\text{par}(I)) = D(b_1, \ldots, b_{n-r})$ as above. Define $I^\perp \in \mathcal{P}_{n-r}^n$ to be the $(n-r)$-tuple

$$I^\perp = (r + j - b_j)_{j=1}^{n-r}.$$

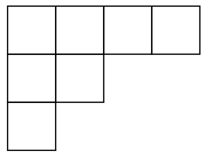
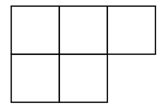
Lemma 6.5. *$par(I^\perp) = (b_1, \ldots, b_{n-r})$*

Proof. The jth entry of $\text{par}(I^\perp)$ is $n - (n-r) + j - (r + j - b_j) = b_j$. □

Corollary 6.6. *$D^T(par(I)) = D(par(I^\perp))$. Thus $|par(I)| = |par(I^\perp)|$ since a diagram and its transpose contain the same number of boxes.*

Definition 6.7. The complement of a Young diagram $D(a_1, \ldots, a_r)$ is the diagram

$D_\backslash(a_1, \ldots, a_r)$ obtained by deleting $D(a_1, \ldots, a_r)$ from $D((n-r)^r)$ and rotating the result by 180 degrees. Thus, if $r = 3$ and $n - r = 4$ then

$D(4, 2, 1) =$  $\implies$ $D_\backslash(4, 2, 1) =$ 

Proposition 6.8. *$D_\backslash(par(I)) = D(par(I'))$.*

Proof. Recall that $I' = (n + 1 - i_{r-j+1})_{j=1}^{r}$. Since the jth row of $D(\text{par}(I))$ contains $a_j = n - r + j - i_j$ boxes, and row $r - j + 1$ of $D(\text{par}(I'))$ contains

$$n - r + (r - j + 1) - (n + 1 - i_{r+1-(r-j+1)}) = i_j - j$$

boxes, these two rows contain a combined $(n-r+j-i_j)+(i_j-j) = n-r$ boxes. □

Proposition 6.9. *Given a Young diagram $D(a_1, \ldots, a_r)$, the transpose of its complement is the complement of its transpose.*

Proof. The transpose diagram $D^T(a_1, \ldots, a_r)$ has *columns* of length $a_1, \ldots, a_r$, hence its complement has columns of length $(n-r) - a_j$. On the other hand, $D_\backslash(a_1, \ldots, a_r)$ has rows of length $(n-r)-a_j$, hence its transpose has columns of these same lengths. □

Proposition 6.10. $(I')^\perp = (I^\perp)'$

Proof. Since the partition and diagram functions par and D are one-to-one for n and r fixed, it suffices to show that $D[\text{par}((I')^\perp)] = D[\text{par}((I^\perp)')]$. This holds by

Propositions 6.6, 6.8, and 6.9:

$$\begin{aligned} D[\text{par}((I')^{\perp})] &= D^T[\text{par}(I')] \\ &= D^T_{\setminus}[\text{par}(I)] \\ &= D_{\setminus}[\text{par}(I^{\perp})] \\ &= D[\text{par}((I^{\perp})')]. \end{aligned}$$ □

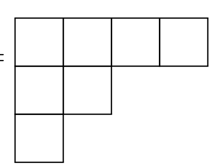
Lemma 6.11. *We have* $(I, J, K) \in U^n_r$ *if and only if* $(I^{\perp}, J^{\perp}, K^{\perp}) \in U^n_{n-r}$.

Proof. By Lemma 5.2, we have $(I, J, K) \in U^n_r$ if and only if $|\text{par}(I)| + |\text{par}(J)| = |\text{par}(K)|$. Similarly, $(I^{\perp}, J^{\perp}, K^{\perp}) \in U^n_{n-r}$ if and only if $|\text{par}(I^{\perp})| + |\text{par}(J^{\perp})| = |\text{par}(K^{\perp})|$. The result now follows from Corollary 6.6. □

Only the above material is needed to establish a bijection between S^n_r and S^n_{n-r}. The following two results are useful only in that they give us a cleaner definition of the set $I^{\perp}$. This alternate definition was used previously in the proof for Theorem 5.7.

Lemma 6.12. *Traverse* $D(\text{par}(I')) = D(i_r - r, \ldots, i_1 - 1)$ *from its lower left hand corner to its upper right hand corner along its bottom/outer edges. The numbers in* I *specify which of these steps are vertical.*

Example 6.13. Consider $I = (1, 3, 5, 8) \in \mathcal{P}^8_4$. One has $\text{par}(I') = (4, 2, 1, 0)$, and the corresponding diagram is

$D(\text{par}(I')) =$ 

We begin our traversal one "unit" below the lower left hand corner of this diagram (recall, we are assuming that our diagrams have r rows!). So the very first step in our traversal is vertical to the bottom left corner of the diagram, and our next 7 steps are, in order:

right, up, right, up, right, right, up

The vertical steps here are thus steps $(1, 3, 5, 8) = I$.

Proof of Lemma 7.12. Since the last row of $D(\text{par}(I'))$ contains $i_1 - 1$ boxes, it is clear that our first $i_1 - 1$ steps are horizontal and that step i_1 is vertical.

Now, suppose that $i_1, \ldots, i_\ell$ are vertical steps. There are exactly

$$(i_{\ell+1} - (\ell + 1)) - (i_\ell - \ell) = i_{\ell+1} - (i_\ell + 1)$$

horizontal steps before the next vertical step. Hence the $(\ell+1)$th vertical step occurs at step

$$i_\ell + i_{\ell+1} - (i_\ell + 1) + 1 = i_{\ell+1}. \qquad \square$$

Lemma 6.14. *Define* $\widetilde{I} = (i \in \{1, \ldots, n\} : n - i + 1 \notin I) \in \mathcal{P}^n_{n-r}$ *. Then* $\widetilde{I} = I^\perp$*.*

Proof. Traverse $D(\text{par}(I')) = D(i_r - r, \ldots, i_1 - 1)$ from its lower left hand corner to its upper right hand corner along its bottom edges. By the previous lemma, the numbers in I specify the vertical steps of this traversal. Hence the elements of $\widetilde{I}$ specify the horizontal steps. Consequently, $\widetilde{I}$ specifies the vertical steps one takes when performing a similar traversal of the transpose diagram $D^T(\text{par}(I')) = D(\text{par}[(I^\perp)'])$. But by the previous lemma, this diagram has its vertical steps given by the elements of $I^\perp$. Hence $\widetilde{I} = I^\perp$. $\square$

6.1.2 A bijection $S_r^n \to S_{n-r}^n$

Our first step in finding a bijection between S_r^n and S_{n-r}^n is to create a correspondence between the Schubert varieties of $G_r(\mathbb{C}^n)$ and those of $G_{n-r}(\mathbb{C}^n)$. Notice that such a map will necessarily create a correspondence between the r dimension conditions defining a Schubert variety of $G_r(\mathbb{C}^n)$ and the $n-r$ dimension conditions defining its image in $G_{n-r}(\mathbb{C}^n)$. This is clearly possible only if each Schubert variety in $G_r(\mathbb{C}^n)$ has at most $\min\{r, n-r\}$ non-redundant dimension conditions defining it. The next two lemmas prove that this is indeed the case.

Lemma 6.15. *For each $j = 1, \ldots, r-1$, if one has $i_{j+1} = i_j + 1$ then the condition $\dim(\Lambda \cap V_{i_j}) \geq j$ in the definition of $S(I, \mathcal{F})$ is redundant and can be removed.*

Proof. Suppose $\Lambda \in S(I, \mathcal{F})$ and that $i_{j+1} = i_j + 1$. Then $\dim(\Lambda \cap V_{i_{j+1}}) \geq j+1$, implying that

$$\begin{aligned}
\dim(\Lambda \cap V_{i_j}) &= \dim((\Lambda \cap V_{i_{j+1}}) \cap V_{i_j}) \\
&= \dim(\Lambda \cap V_{i_{j+1}}) + \dim(V_{i_j}) - \dim((\Lambda \cap V_{i_{j+1}}) + V_{i_j}) \\
&\geq (j+1) + i_j - \dim((\Lambda \cap V_{i_{j+1}}) + V_{i_j}) \\
&\geq (j+1) + i_j - \dim(V_{i_{j+1}}) \\
&= (j+1) + i_j - i_{j+1} \\
&= j.
\end{aligned}$$

□

Lemma 6.16. *The condition $\dim(\Lambda \cap V_{i_j}) \geq j$ in the definition for $S(I, \mathcal{F})$ is not redundant precisely when $b_{a_j} = j$.*

Proof. By Corollary 6.3

$$b_{a_j} = j \iff a_j - a_{j+1} > 0 \iff i_{j+1} > i_j + 1.$$

Now, if we know only that $i_{j+1} > i_j + 1$ and that $\dim(\Lambda \cap V_{i_{j+1}}) \geq j+1$, then we obviously cannot infer that $\dim(\Lambda \cap V_{i_j}) \geq j$. Counterexamples abound. Conversely, if $i_{j+1} = i_j + 1$ then the condition $\dim(\Lambda \cap V_{i_j}) \geq j$ is redundant by the previous lemma. $\square$

Now, let $\Lambda \mapsto \Lambda^\perp$ be the map $G_r(\mathbb{C}^n) \to G_{n-r}(\mathbb{C}^n)$ sending each r-dimensional subspace of $\mathbb{C}^n$ to its orthogonal complement. As we now show, this map yields a bijection between the Schubert varieties of $G_r(\mathbb{C}^n)$ and those of $G_{n-r}(\mathbb{C}^n)$.

Given a complete flag

$$\mathcal{F} : \{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n,$$

we define the complementary flag $\mathcal{F}^\perp$ via

$$\mathcal{F}^\perp : \{0\} = W_0 \subset W_1 \subset \cdots \subset W_n = \mathbb{C}^n.$$

where $W_j = V_{n-j}^\perp$ for each j. (Elsewhere we have written $\mathcal{F}'$ for this complementary flag.)

Proposition 6.17. *Given $I \in \mathcal{P}_r^n$, the image*

$$S(I, \mathcal{F})^\perp = \{\Lambda^\perp : \Lambda \in S(I, \mathcal{F})\}$$

of the Schubert variety $S(I, \mathcal{F}) \subset G_r(\mathbb{C}^n)$ under the map $\Lambda \mapsto \Lambda^\perp$ is the Schubert variety $S(I^\perp, \mathcal{F}^\perp) \subset G_{n-r}(\mathbb{C}^n)$.

Proof. Suppose that $\Lambda \in S(I, \mathcal{F})$. Then $\Lambda \in G_r(\mathbb{C}^n)$ and $\dim(\Lambda \cap V_{i_j}) \geq j$ for each

$j = 1, \dots, r$. Hence,

$$\begin{aligned} \dim(\Lambda^\perp \cap W_{n-i_j}) &= \dim(\Lambda^\perp \cap V_{i_j}^\perp) \\ &= \dim((\Lambda + V_{i_j})^\perp) \\ &= n - \dim(\Lambda + V_{i_j}) \\ &= n - \big(\dim(\Lambda) + \dim(V_{i_j}) - \dim(\Lambda \cap V_{i_j})\big) \\ &\geq n - r + j - i_j \\ &= a_j. \end{aligned}$$

Let $A = \{j : b_{a_j} = j\}$ denote the set of indices j such that the condition $\dim(\Lambda \cap V_{i_j}) \geq j$ in the definition of $S(I, \mathcal{F})$ is not redundant. For $j \in A$ we have $a_j = n - r + b_{a_j} - i_j$, and so it follows from the previous calculation that $\dim(\Lambda^\perp \cap W_{r+a_j-b_{a_j}}) \geq a_j$ for these j. Equivalently, $\dim(\Lambda^\perp \cap W_{r+\ell-b_\ell}) \geq \ell$ for all $\ell \in \{a_j : j \in A\}$. Writing $I^\perp = (i_1^\perp, \dots, i_{n-r}^\perp)$, we therefore have

$$\dim(\Lambda^\perp \cap W_{i_\ell^\perp}) \geq \ell$$

where ℓ ranges over all the non-redundant conditions in the definition of $S(I^\perp, \mathcal{F}^\perp)$. Hence $\Lambda^\perp \in S(I^\perp, \mathcal{F}^\perp)$, which shows that $S(I, \mathcal{F})^\perp \subseteq S(I^\perp, \mathcal{F}^\perp)$. A similar argument shows that $S(I^\perp, \mathcal{F}^\perp)^\perp \subseteq S(I^{\perp\perp}, \mathcal{F}^{\perp\perp}) = S(I, \mathcal{F})$, which establishes the result. □

Corollary 6.18. *For any $I, J, K \in \mathcal{P}_r^n$ and any complete flags $\mathcal{F}_j$, $j = 1, 2, 3$, one has $S(I', \mathcal{F}_1) \cap S(J', \mathcal{F}_2) \cap S(K, \mathcal{F}_3) \neq \emptyset$ if and only if*

$$S((I^\perp)', \mathcal{F}_1^\perp) \cap S((J^\perp)', \mathcal{F}_2^\perp) \cap S(K^\perp, \mathcal{F}_3^\perp) \neq \emptyset.$$

Proof. By the previous proposition, any $\Lambda \in G_r(\mathbb{C}^n)$ lies in the intersection $S(I', \mathcal{F}_1) \cap$

$S(J', \mathcal{F}_2) \cap S(K, \mathcal{F}_3)$ if and only if $\Lambda^\perp \in G_{n-r}(\mathbb{C}^n)$ belongs to the intersection

$$\begin{aligned} S(I', \mathcal{F}_1)^\perp \cap S(J', \mathcal{F}_2)^\perp \cap S(K, \mathcal{F}_3)^\perp &= S((I')^\perp, \mathcal{F}_1^\perp) \cap S((J')^\perp, \mathcal{F}_2) \cap S(K^\perp, \mathcal{F}_3^\perp) \\ &= S((I^\perp)', \mathcal{F}_1^\perp) \cap S((J^\perp)', \mathcal{F}_2^\perp) \cap S(K^\perp, \mathcal{F}_3^\perp). \end{aligned}$$

□

Corollary 6.19. *For $I, J, K \in \mathcal{P}_r^n$, one has $\sigma_{I'}\sigma_{J'}\sigma_K \neq 0 \in H^\bullet(G_r(\mathbb{C}^n))$ if and only if*

$$\sigma_{(I^\perp)'}\sigma_{(J^\perp)'}\sigma_{K^\perp}, \neq 0 \in H^\bullet(G_{n-r}(\mathbb{C}^n)).$$

Proof. This follows immediately from Proposition 3.18 and the previous corollary. □

Corollary 6.20. *For $(I, J, K) \in U_r^n$, we have $(I, J, K) \in S_r^n$ if and only if $(I^\perp, J^\perp, K^\perp) \in S_{n-r}^n$. Hence $|S_r^n| = |S_{n-r}^n|$.*

6.2 The sets T_r^n when $r \in \{1, n-2, n-1\}$

Recall that Theorem 5.11 asserts that for $(I, J, K) \in U_r^n$ one has $(I, J, K) \in T_r^n$ if and only if there exist $r \times r$ Hermitian matrices $A, B, C = A + B$ with

$$\lambda^\downarrow(A) = \mathrm{par}(I'), \qquad \lambda^\downarrow(B) = \mathrm{par}(J'), \qquad \lambda^\downarrow(C) = \mathrm{par}(K'). \tag{6.1}$$

In this section we show that for $r \in \{1, n-2, n-1\}$, this equivalence can be strengthened to having $(I, J, K) \in T_r^n$ if and only if there exist *diagonal* $r \times r$ matrices satisfying (6.1).

There is nothing to prove when $r = 1$. The situation is slightly less trivial when $r = n-1$, and we prove this case only to establish notation and provide motivation for the much less transparent (and possibly new) $r = n-2$ case.

Before we begin, given any r-tuple $L = (\ell_1, \ldots, \ell_r)$ and any permutation $\sigma \in S_r$, let $L^\sigma = (\ell_{\sigma(1)}, \ldots, \ell_{\sigma(r)})$.

Lemma 6.21. *For any $(I, J, K) \in T^n_r$ with $r = n-1$, there exists a permutation $\sigma \in S_r$ such that*

$$par(I') + par(J')^\sigma = par(K').$$

Proof. Define $\lambda = \mathrm{par}(I')$, $\mu = \mathrm{par}(J')$, and $\nu = \mathrm{par}(K')$. Each of these partitions have at most r nonzero parts of size at most $n - r = 1$. We may therefore write

$$\lambda = 1^{i_1} \oplus 0^{i_0}, \quad \mu = 1^{j_1} \oplus 0^{j_0}, \quad \nu = 1^{k_1} \oplus 0^{k_0}$$

where, for example, $1^{i_1} \oplus 0^{i_0}$ denotes the r-tuple whose first i_1 entries equal 1, and whose remaining $r - i_1 = i_0$ entries equal 0. Moreover, since $(I, J, K) \in U^n_r$, we have $|\lambda| + |\mu| = |\nu|$, and thus

$$i_1 + j_1 = k_1 \leq r. \tag{6.2}$$

Now, notice that $0^{i_1} \oplus 1^{j_1} \oplus 0^{r-i_1-j_1}$ is a permutation of μ. Indeed, the exponents i_1 and j_1 are clearly nonnegative, as is the exponent $r - i_1 - j_1 = r - k_1$ since $k_1 \leq r$. Moreover, it contains j_1 ones and $i_1 + (r - i_1 - j_1) = r - j_1 = j_0$ zeros as needed.

So, let $\sigma \in S_r$ be the permutation with $\mu^\sigma = 0^{i_1} \oplus 1^{j_1} \oplus 0^{r-i_1-j_1}$. Then

$$\begin{aligned} \lambda + \mu^\sigma &= 1^{i_1} \oplus 1^{j_1} \oplus 0^{r-i_1-j_1} \\ &= 1^{k_1} \oplus 0^{r-k_1} \\ &= \nu. \end{aligned}$$

□

Lemma 6.22. *If $(I, J, K) \in T^n_r$, $r = n-2$, then there exist permutations $\sigma_1, \sigma_2 \in S_r$*

such that

$$par(I')^{\sigma_1} + par(J')^{\sigma_2} = par(K').$$

Proof. Let $(I, J, K) \in T_r^n$ be given and write

$$\lambda = \text{par}(I'), \quad \mu = \text{par}(J'), \quad \nu = \text{par}(K').$$

Then we have, say,

$$\lambda = 2^{i_2} \oplus 1^{i_1} \oplus 0^{i_0}, \quad \mu = 2^{j_2} \oplus 1^{j_1} \oplus 0^{j_0}, \quad \nu = 2^{k_2} \oplus 1^{k_1} \oplus 0^{k_0}$$

where

$$i_2 + i_1 + i_0 = r, \quad j_2 + j_1 + j_0 = r, \quad k_2 + k_1 + k_0 = r,$$

$$|\lambda| = 2i_2 + i_1, \quad |\mu| = 2j_2 + j_1, \quad |\nu| = 2k_2 + k_1.$$

We claim that the following are permutations of λ and μ which sum to ν:

$$\lambda^{\sigma_1} := 2^{i_2} \oplus 0^{j_2} \oplus 1^{k_2 - i_2 - j_2} \oplus 1^{i_1 + i_2 + j_2 - k_2} \oplus 0^{i_2 + j_1 + j_2 - k_2} \oplus 0^{k_0},$$

$$\mu^{\sigma_2} := 0^{i_2} \oplus 2^{j_2} \oplus 1^{k_2 - i_2 - j_2} \oplus 0^{i_1 + i_2 + j_2 - k_2} \oplus 1^{i_2 + j_1 + j_2 - k_2} \oplus 0^{k_0}.$$

It therefore needs to be shown that

(i) the exponent $k_2 - i_2 - j_2$ is nonnegative,

(ii) the exponent $i_1 + i_2 + j_2 - k_2$ is nonnegative (switching the roles of λ and μ will then imply that $i_2 + j_1 + j_2 - k_2$ is also nonnegative),

(iii) each tuple contains the appropriate amount of 0's, 1's, and 2's, and

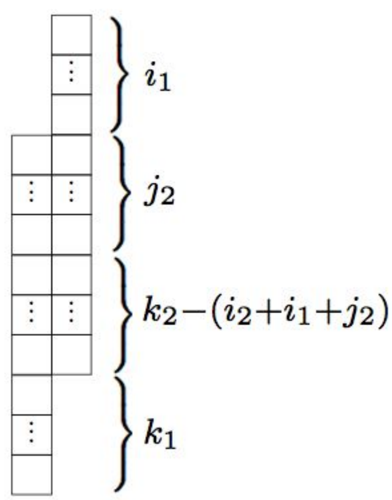
(iv) $\lambda^{\sigma_1} + \mu^{\sigma_2} = \nu$.

Proof of (i). It is easily shown that the second column of $D(\nu)\backslash D(\lambda)$ contains $k_2 - i_2$ boxes, while the first column contains $k_1 + k_2 - i_1 - i_2$ boxes. As $(I, J, K) \in T_r^n$, the LR rules tell us that it must be possible to label these boxes with

$$\text{two 1's, two 2's, } \ldots, \text{ two } j_2\text{'s;} \quad \text{one } (j_2+1), \text{ one } (j_2+2), \ldots, \text{ one } (j_2+j_1)$$

subject to the LR rules. In particular, the numbers in each column must be strictly increasing, and it is therefore immediate that the first j_2 entries in each column are $1, \ldots, j_2$. In particular, the length of the first column must be $\geq j_2$; hence $k_2 - i_2 \geq j_2$.

Proof of (ii). To see that $k_2 \leq i_1 + i_2 + j_2$, we assume that $k_2 > i_1 + i_2 + j_2$ and will derive a contradiction via the LR rules. As $k_2 > i_1 + i_2 + j_2$, the diagram $D(\nu)\backslash D(\lambda)$ is as shown in the following figure. Since $(I, J, K) \in T_r^n$, the LR rules tell us that it



must be possible to label the boxes in this diagram with

$$\text{two 1's, two 2's, } \ldots, \text{ two } j_2\text{'s;} \quad \text{one } (j_2+1), \text{ one } (j_2+2), \ldots, \text{ one } (j_2+j_1)$$

subject to the LR rules.

Now, the boxes in column 1 and rows i_1+1 through k_2-i_2 contain

$$(k_2-i_2)-(i_1+1)+1 = k_2-i_2-i_2 \ > \ j_2$$

numbers in the range $1,\ldots,j_2+j_1$ and must increase strictly. So box$(k_2-i_2,1)$ must contain a number $a>j_2$. The same reasoning shows that box$(k_2-i_2,2)$ must contain a number $b>j_2$. We must also have $a<b$ here since values increase weakly along rows and we have only one copy of each number greater than j_2. But now, listing values from right-to-left and top-to-bottom produces a sequence $\ldots,b,a,\ldots$ which fails to be a lattice word, a contradiction.

Proof of (iii). The sequence λ^{σ_1} contains i_2 twos, $(k_2-i_2-j_2)+(i_1+i_2+j_2-k_2)=i_1$ ones, and

$$\begin{aligned}
j_2+(i_2+j_1+j_2-k_2)+k_0 &= i_2+(2j_2+j_1)-k_2+k_0 \\
&= i_2+|\mu|-k_2+k_0 \\
&= (|\lambda|-i_2-i_1)+|\mu|-k_2+k_0 \\
&= (|\lambda|+|\mu|)-k_2+k_0-i_2-i_1 \\
&= (|\nu|-k_2)+k_0-i_2-i_1 \\
&= k_2+k_1+k_0-i_2-i_1 \\
&= r-i_2-i_1 \\
&= i_0
\end{aligned}$$

zeros, as needed. Similarly, μ^{σ_2} contains j_2 twos, j_1 ones, and j_0 zeros. Therefore, in combination with (i) and (ii), we see that λ^{σ_1} and μ^{σ_2} are permutations of λ and μ, respectively.

Proof of (iv). The sum $\lambda^{\sigma_1} + \mu^{\sigma_2}$ contains $i_2 + j_2 + (k_2 - i_2 - j_2) = k_2$ twos,

$$\begin{aligned}(i_1 + i_2 + j_2 - k_2) + (i_2 + j_1 + j_2 - k_2) &= (2i_2 + i_1) + (2j_2 + j_1) - 2k_2 \\ &= |\lambda| + |\mu| - 2k_2 \\ &= |\nu| - 2k_2 \\ &= k_1\end{aligned}$$

ones and k_0 zeros. Thus $\lambda^{\sigma_1} + \mu^{\sigma_2} = \nu$ as claimed. □

Proposition 6.23. *For $(I, J, K) \in U_r^n$ with $r \in \{1, n-1, n-2\}$, one has $(I, J, K) \in T_r^n$ if and only if there exist* diagonal *matrices $A, B \in \mathrm{Herm}(r)$ such that*

$$\lambda^{\downarrow}(A) = par(I'), \qquad \lambda^{\downarrow}(B) = par(J'), \qquad \lambda^{\downarrow}(A + B) = par(K').$$

Proof. Suppose $(I, J, K) \in T_r^n$ for either $r = n-1$ or $r = n-2$. If $r = n-1$ then, using the permutation σ from Lemma 6.21, set $A = \mathrm{diag}(\mathrm{par}(I'))$ and $B = \mathrm{diag}(\mathrm{par}(J')^{\sigma})$. Then $A + B = \mathrm{diag}(\mathrm{par}(K'))$ as needed. If $r = n - 2$ then, using the permutations σ_1 and σ_2 from Lemma 6.22, set $A = \mathrm{diag}(\mathrm{par}(I')^{\sigma_1})$ and $B = \mathrm{diag}(\mathrm{par}(J')^{\sigma_2})$. Then $A + B = \mathrm{diag}(\mathrm{par}(K'))$ as needed.

The converse follows immediately from Theorem 5.11. □

However, it appears as though Proposition 6.23 holds *only* for $r \in \{1, n-1, n-2\}$. A simple counterexample for both $r = 2$ and $r = n - 3$ is provided by $(I, J, K) =$

$((1,4),(1,4),(2,5)) \in T_2^5$ since here

$$\text{par}(I') = (2,0), \qquad \text{par}(J') = (2,0), \qquad \text{par}(K') = (3,1).$$

On the other hand, there do exist *non-diagonal* Hermitian matrices $A, B, C = A + B$ with $\lambda^{\downarrow}(A) = (2,0)$, $\lambda^{\downarrow}(B) = (2,0)$, and $\lambda^{\downarrow}(C) = (3,1)$, as guaranteed by Theorem 5.11. For example, it is easy to check that the Hermitian matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 3/2 \end{bmatrix}, \qquad C = A + B = \begin{bmatrix} 5/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 3/2 \end{bmatrix}$$

have eigenvalues $\lambda^{\downarrow}(A) = (2,0) = \lambda^{\downarrow}(B)$ and $\lambda^{\downarrow}(C) = (3,1)$, as required.

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APPENDIX A: Background in Algebraic Geometry

In this appendix we provide further detail on how Schubert varieties $S(I,\mathcal{F})$ in $G_r(\mathbb{C}^n)$ produce homology classes $[I]=[S(I,\mathcal{F})]\in H_{2d(I)}(G_r(\mathbb{C}^n))$. We then outline a proof for Proposition 3.18. This is a deep result that has played an important role in this thesis. Proposition 3.18 is stated in [10] with references given to [5] and [7]. Our goal here is to provide an overview; full details are beyond the scope of this thesis.

Finally, we conclude this appendix by proving that for $I,J\in\mathcal{P}_r^n$ one has $S(I,\mathcal{F}_1)\cap S(J,\mathcal{F}_2)$ nonempty for all flags $\mathcal{F}_j$ if and only if this intersection is nonempty for all flags $\mathcal{F}_1,\mathcal{F}_2$ in general position.

A.1 Projective space and projective varieties

The Grassmannian $G_1(\mathbb{C}^{n+1})$ of one-dimensional subspaces in $\mathbb{C}^{n+1}$ is the *complex projective space* $\mathbb{CP}^n$. We have seen that this is a compact, connected topological space and a complex manifold of dimension n.

Given a nonzero point $(z_0,\ldots,z_n)\in\mathbb{C}^{n+1}$, let $[z_0,\ldots,z_n]\in\mathbb{CP}^n$ denote the line

$$[z_0,\ldots,z_n]=\big\{(tz_0,\ldots,tz_n)\ :\ t\in\mathbb{C}\big\}$$

spanned by $(z_0,\ldots,z_n)$. Now, if $h_1,\ldots,h_m\in\mathbb{C}[z_0,\ldots,z_n]$ are *homogeneous* polynomials on $\mathbb{C}^{n+1}$, i.e.

$$h_j(tz_0,\ldots,tz_n)=t^{d_j}h_j(z_0,\ldots,z_n)$$

for some integers $d_j\geq 0$, then we see that the *projective variety*

$$V(h_1,\ldots,h_m)=\big\{[z_0,\ldots,z_n]\ :\ h_j(z_0,\ldots,z_n)=0 \text{ for } j=1,\ldots,m\big\}$$

is a well-defined subset of $\mathbb{CP}^n$. The collection of all projective varieties form the closed sets for a topology on $\mathbb{CP}^n$, called the *Zariski topology*. This topology is non-Hausdorff and much coarser than the manifold topology.

Let $X \subset \mathbb{CP}^n$ be a projective variety, and suppose that X is *irreducible*, i.e., connected in the manifold and Zariski toplogies. The manifold and Zariski topologies on $\mathbb{CP}^n$ induce subspace topologies on X which we refer to as the manifold and Zariski topologies on X. We say that X is *non-singular* if X is a complex submanifold of $\mathbb{CP}^n$. In this case the *dimension* $\dim(X)$ of X is its dimension as a complex manifold. In general, X will contain a Zariski open dense subset $X \cap U$ which is a complex submanifold of $\mathbb{CP}^n$. In this case we take "$\dim(X)$" to mean the dimension of the complex manifold $X \cap U$. Thus each projective variety X is either a complex manifold or a complex manifold together with a Zariski closed singular set, which is itself a union of irreducible projective varieties of lower dimension.

A.2 The intersection pairing

Let $X \subset \mathbb{CP}^n$ be an irreducible non-singular projective variety of dimension d. We let $H_\bullet(X)$, $H^\bullet(X)$ denote the singular homology and cohomology groups for X with the manifold topology. It is known [7, p.64] that the even degree homology/cohomology groups $H_{2k}(X)$, $H^{2k}(X)$ are nonzero for $k = 0, \dots d$.

In the manifold topology, X is a compact connected complex manifold. The complex structure determines an orientation on X, viewed as a real manifold of real dimension $2d$. (See [7, p.18].) This orientation determines a fundamental class $[X] \in H_{2d}(X) \cong \mathbb{Z}$ and a Poincaré duality isomorphism

$$PD : H_k(X) \to H^{2d-k}(X)$$

for each $k = 0, \ldots, 2d$. (See [8, §3.3].) We use Poincaré duality and cup product in cohomology to define an operation in homology:

$$H_k(X) \times H_\ell(X) \to H_{k+\ell-2d}(X), \quad (c_1, c_2) \mapsto (c_1 \cdot c_2) := PD^{-1}(PD(c_1) \cup PD(c_2)),$$

called the *intersection pairing*. When $k + \ell = 2d$ (so that k and ℓ are complementary degrees), we have $(c_1 \cdot c_2) \in H_0(X) \cong \mathbb{Z}$ and can regard the intersection pairing $(c_1 \cdot c_2)$ as an integer.

A.3 The fundamental class of a subvariety

As above, let $X \subset \mathbb{CP}^n$ be an irreducible non-singular projective variety of dimension d. Let $V \subset X$ be an irreducible subvariety of dimension $k \leq d$. If V is non-singular then V is a compact oriented manifold of real dimension $2k$ with a well-defined fundamental class $[V] \in H_{2k}(V)$. Abusing notation, we also let $[V] \in H_{2k}(X)$ denote the image of $[V] \in H_{2k}(V)$ under the map $H_{2k}(V) \to H_{2k}(X)$ induced by inclusion $V \hookrightarrow X$. It is known [7, p.64] that $[V]$ is nonzero in $H_{2k}(X)$.

This construction generalizes to encompass irreducible subvarieties that contain singularities. Thus any irreducible $V \subset X$ of dimension k produces a well-defined nonzero homology class $[V] \in H_{2k}(X)$. See [4, §B.3] for details.

A.4 Geometric intersection theory

Let $X \subset \mathbb{CP}^n$ be an irreducible non-singular projective variety of dimension d and let V, W be two (possibly singular) irreducible subvarieties of dimensions $k, \ell \leq d$ satisfying

$$k + \ell \geq d.$$

The intersection $V \cap W$ is again a subvariety of X, but may be reducible. In general, $V \cap W$ decomposes as a (possibly empty) finite union of irreducible components, say

$$V \cap W = Z_1 \cup \cdots \cup Z_p$$

where each Z_j is an irreducible subvariety of X. We say that the intersection $V \cap W$ is *proper* if

$$\mathrm{codim}(Z_j) = \mathrm{codim}(V) + \mathrm{codim}(W)$$

for each component Z_j. Equivalently, each Z_j has dimension

$$\dim(Z_j) = k + \ell - d$$

and hence produces a (nonzero) fundamental class $[Z_j] \in H_{2k+2\ell-2d}(X)$.

Under the above hypotheses, the geometric intersection $V \cap W$ is related to the purely topological intersection pairing $([V] \cdot [W]) \in H_{2k+2\ell-2d}(X)$ for the fundamental classes of V and W via a formula of the form

$$([V] \cdot [W]) = m_1[Z_1] + \cdots + m_p[Z_p] \tag{A.1}$$

where the m_j's are certain geometrically defined *intersection numbers*. Note that the m_j's are *positive integers*; see [4, §B.3] and [7, p.60-65] for details.

If $V \cap W = \emptyset$ then the intersection $V \cap W$ is vacuously proper and one obtains $([V] \cdot [W]) = 0$. Otherwise, (A.1) shows that $([V] \cdot [W]) \neq 0$. As the intersection pairing $([V] \cdot [W]) \in H_\bullet(X)$ is Poincaré dual to the cup product $PD([V]) \cup PD([W]) \in H^\bullet(X)$, we can summarize this discussion as follows:

Proposition A.1. *Let X be an irreducible non-singular projective variety and let V,* *W be two (possibly singular) irreducible subvarieties with* $\dim(V)+\dim(W) \geq \dim(X)$ *and proper intersection* $V \cap W$*. Then* $PD([V]) \cup PD([W])$ *is nonzero in* $H^\bullet(X)$ *if and only if* $V \cap W \neq \emptyset$*.*

A.5 $G_r(\mathbb{C}^n)$ as a projective variety

We wish to apply the intersection theory for subvarieties of irreducible non-singular projective varieties, summarized in the preceding section, to $X = G_r(\mathbb{C}^n)$. First we must explain how the Grassmannian $G_r(\mathbb{C}^n)$ may be viewed as a projective variety.

Given an r-dimensional subspace L of $\mathbb{C}^n$, one may choose a basis $\{v_1, \dots, v_r\}$ for L and form the exterior product $v_1 \wedge \cdots \wedge v_r \in \bigwedge^r(\mathbb{C}^n)$. If $\{v_1', \dots, v_r'\}$ is another basis for L then one has $v_1' \wedge \cdots \wedge v_r' = \det(A) v_1 \wedge \cdots \wedge v_r$, where A is the $r \times r$ change of basis matrix relating $\{v_1, \dots, v_r\}$ to $\{v_1', \dots, v_r'\}$. We therefore obtain a well-defined map

$$G_r(\mathbb{C}^n) \to \mathbb{P}(\textstyle\bigwedge^r(\mathbb{C}^n)) \cong \mathbb{CP}^{(\binom{n}{r}-1)}, \quad L \mapsto \mathbb{C}v_1 \wedge \cdots \wedge v_r,$$

called the *Plücker embedding.* The Plücker embedding is injective and the image of $G_r(\mathbb{C}^n)$ in $\mathbb{P}(\bigwedge^r(\mathbb{C}^n))$ is an irreducible non-singular projective variety of dimension $N = r(n-r)$. In fact, $G_r(\mathbb{C}^n)$ is set of zeros for a system of homogeneous polynomials on $\bigwedge^r(\mathbb{C}^n)$ called the *Plücker relations.* See [7, p.209-211] or [12] for details.

Let $\mathcal{F}$ be a complete flag in $\mathbb{C}^n$. It is known that for $I \in \mathcal{P}_r^n$, the Schubert variety $S(I, \mathcal{F})$ is the closure of the Schubert cell $C(I, \mathcal{F})$ in the Zariski topology. As $S(I, \mathcal{F})$ is Zariski-closed, it is indeed a (possibly singular) projective variety.[1] Moreover, one can show that $S(I, \mathcal{F})$ is irreducible and that its dimension as a variety is $d(I)$, the topological dimension of the cell $C(I, \mathcal{F}) \cong \mathbb{C}^{d(I)}$. See [12] for details.

It follows that each $I \in \mathcal{P}_r^n$ yields a (nonzero) fundamental class $[S(I, \mathcal{F})] \in$

[1] In fact, $S(I, \mathcal{F})$ is non-singular if and only if the Young's diagram for $\mathrm{par}(I)$ is a rectangle.

$H_{2d(I)}(G_r(\mathbb{C}^n))$. We have shown (Proposition 3.9) that $[S(I,\mathcal{F})]$ does not depend on the choice of flag and write $[I] = [S(I,\mathcal{F})]$, as before.

A.6 Proof outline for Proposition 3.18

We consider only the assertion in Proposition 3.18 concerning the cup product of *two* Schubert cocycles. This is restated below for the reader's convenience.

Proposition A.2. *Given $I, J \in \mathcal{P}_r^n$ with $d(I) + d(J) \geq N$, we have $\sigma_I \sigma_J \neq 0$ in $H^\bullet(G_r(\mathbb{C}^n))$ if and only if $S(I,\mathcal{F}_1) \cap S(J,\mathcal{F}_2) \neq \emptyset$ for all complete flags $\mathcal{F}_1, \mathcal{F}_2$.*

Proof Outline. Recall that $\sigma_I = PD([I])$, $\sigma_J = PD([J])$ where $[I] = [S(I,\mathcal{F}_1)]$ and $[J] = [S(J,\mathcal{F}_2)]$ are independent of the flags $\mathcal{F}_j$. It is a fact that

- for suitably chosen flags $\mathcal{F}_j$, the intersection $S(I,\mathcal{F}_1) \cap S(J,\mathcal{F}_2)$ of irreducible subvarieties in $G_r(\mathbb{C}^n)$ is proper.

Thus if $S(I,\mathcal{F}_1) \cap S(J,\mathcal{F}_2) \neq \emptyset$ for all flags $\mathcal{F}_j$, then $S(I,\mathcal{F}_1) \cap S(J,\mathcal{F}_2) \neq \emptyset$ for some pair of flags with $S(I,\mathcal{F}_1) \cap S(J,\mathcal{F}_2)$ a proper intersection. Applying Proposition A.1, it follows that $\sigma_I \sigma_J \neq 0$ in $H^\bullet(G_r(\mathbb{C}^n))$. On the other hand, if $S(I,\mathcal{F}_1) \cap S(J,\mathcal{F}_2) = \emptyset$ for some choice of flags $\mathcal{F}_j$ then this intersection is vacuously proper and Proposition A.1 shows that $\sigma_I \sigma_J = 0$. $\square$

The condition that $S(I,\mathcal{F}_1) \cap S(J,\mathcal{F}_2)$ be a (non-empty) proper intersection means that each irreducible component Z in $S(I,\mathcal{F}_1) \cap S(J,\mathcal{F}_2)$ is a variety of dimension $\dim(Z) = d(I) + d(J) - N$. Here recall that $d(I) = \sum_{\ell=1}^{r}(i_\ell - \ell)$, $d(J) = \sum_{\ell=1}^{r}(j_\ell - \ell)$, and $N = r(n-r)$ are the dimensions for $S(I,\mathcal{F}_1)$, $S(J,\mathcal{F}_2)$, and $G_r(\mathbb{C}^n)$, respectively.

Example A.3. Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are a common flag

$$\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F} : \{0\} \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n.$$

Given $I = (i_1, \ldots, i_r)$, $J = (j_1, \ldots, j_r)$ in $\mathcal{P}_r^n$, define $K = (k_1, \ldots, k_r) \in \mathcal{P}_r^n$ via

$$k_\ell = \min(i_\ell, j_\ell).$$

We claim that

$$S(I, \mathcal{F}) \cap S(J, \mathcal{F}) = S(K, \mathcal{F}).$$

Indeed, for any $L \in G_r(\mathbb{C}^n)$ one has

$$\begin{aligned} L \in S(I, \mathcal{F}) \cap S(J, \mathcal{F}) &\iff \dim(L \cap V_{i_\ell}) \geq \ell \text{ and } \dim(L \cap V_{j_\ell}) \geq \ell \text{ for } \ell = 1, \ldots, r \\ &\iff \dim(L \cap V_{\min(i_\ell, j_\ell)}) \geq \ell \text{ for } \ell = 1, \ldots, r \\ &\iff \dim(L \cap V_{k_\ell}) \geq \ell \text{ for } \ell = 1, \ldots, r \\ &\iff L \in S(K, \mathcal{F}). \end{aligned}$$

The intersection $S(I, \mathcal{F}) \cap S(J, \mathcal{F}) = S(K, \mathcal{F})$ however fails to be proper except in the trivial situation where at least one of $S(I, \mathcal{F})$, $S(J, \mathcal{F})$ coincides with $G_r(\mathbb{C}^n)$. Recall from Proposition 2.21 that $S(I'_\circ, \mathcal{F}) = G_r(\mathbb{C}^n)$ for $I'_\circ = (n - r + 1, \ldots, n)$. As $\dim(S(K, \mathcal{F})) = d(K)$, we see that $S(I, \mathcal{F}) \cap S(J, \mathcal{F})$ is a proper intersection if and only if $d(K) = d(I) + d(J) - N$, or equivalently $d(I) + d(J) - d(K) = N$. But

$$d(I) + d(J) - d(K) = \sum_{\ell=1}^{r} \big(\max(i_\ell, j_\ell) - \ell\big) \leq \sum_{\ell=1}^{r} (n - r + \ell - \ell) = r(n - r) = N$$

with equality holding if and only if $\max(i_\ell, j_\ell) = n - r + \ell$ for $\ell = 1, \ldots, r$. If in fact $i_1 = n - r + 1$ holds then this forces $I = (n - r + 1, \ldots, n) = I'_\circ$. Thus if $S(I, \mathcal{F}) \cap S(J, \mathcal{F})$ is a proper intersection then we must have either $I = I'_\circ$ or $J = I'_\circ$.

A.7 Flags in general position

Example A.3 shows, in particular, that for any $I = (i_1, \dots, i_r)$, $J = (j_1, \dots, j_r)$ in $\mathcal{P}_r^n$ and complete flag $\mathcal{F}$, one has $S(I, \mathcal{F}) \cap S(J, \mathcal{F}) \neq \emptyset$. In connection with Proposition 3.18, this highlights the importance that the conditions on intersections of Schubert varieties must hold for *all* flags $\mathcal{F}_j$. Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are flags

$$\mathcal{F}_1 : \{0\} \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n, \qquad \mathcal{F}_2 : \{0\} \subset U_1 \subset \cdots \subset U_n = \mathbb{C}^n.$$

The assertion that $S(I, \mathcal{F}_1) \cap S(J, \mathcal{F}_2) \neq \emptyset$ means that there is an r dimensional subspace L of $\mathbb{C}^n$ for which both

$$\dim(L \cap V_{i_\ell}) \geq \ell \quad \text{and} \quad \dim(L \cap U_{j_\ell}) \geq \ell$$

hold for all $\ell = 1, \dots, r$. These conditions are most difficult to achieve when the intersections $V_i \cap U_j$ are "small." As

$$\dim(V_i \cap U_j) = \dim(V_i) + \dim(U_j) - \dim(V_i + U_j) \geq i + j - n,$$

we see that the smallest possible value for $\dim(V_i \cap U_j)$ is $\max(i + j - n, 0)$. We are therefore led to the following definition.

Definition A.4. We say that a pair of complete flags $\mathcal{F}_1$, $\mathcal{F}_2$ in $\mathbb{C}^n$ as above are *in general position* if $\dim(V_i \cap U_j) = \max(i + j - n, 0)$ for all $1 \leq i, j \leq n$.

The above discussion amounts to an informal proof for the following lemma.

Lemma A.5. *Given $I, J \in \mathcal{P}_r^n$, the following conditions are equivalent.*

- $S(I, \mathcal{F}_1) \cap S(I, \mathcal{F}_2) \neq \emptyset$ *for all complete flags $\mathcal{F}_j$.*

- *$S(I, \mathcal{F}_1) \cap S(I, \mathcal{F}_2) \neq \emptyset$ for all complete flags $\mathcal{F}_j$ with $\mathcal{F}_1$, $\mathcal{F}_2$ in general position.*

A rigorous proof for Lemma A.5 is given below. First, however, we must derive a couple of results required for the proof. The first of these provides another viewpoint on general position.

Lemma A.6. *A pair of flags $\mathcal{F}_1 : \{0\} \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$, $\mathcal{F}_2 : \{0\} \subset U_1 \subset \cdots \subset U_n = \mathbb{C}^n$ is in general position if and only if $V_i \cap U_{n-i} = \{0\}$ for $i = 1, \ldots, n-1$. (Equivalently, $\mathbb{C}^n = V_i \oplus U_{n-i}$ for $0 \leq i \leq n$ where we adopt the convention that $V_0 = \{0\} = U_0$.)*

Proof. If $\mathcal{F}_1, \mathcal{F}_2$ are in general position then $\dim(V_i \cap U_{n-i}) = \max(i+(n-i)-n, 0) = 0$, and hence $V_i \cap U_{n-i} = \{0\}$ for $i = 1, \ldots, n-1$.

Conversely, suppose that $V_i \cap U_{n-i} = \{0\}$ for $i = 1, \ldots, n-1$. Given $1 \leq i, j \leq n$, we need to show that $\dim(V_i \cap U_j) = \max(i + j - n, 0)$. First, if $i + j < n$ then $j < n - i$ and we have $V_i \cap U_j \subset V_i \cap U_{n-i} = \{0\}$. Thus $V_i \cap U_j = \{0\}$ and so $\dim(V_i \cap U_j) = 0$ as required. Alternatively, if $i + j \geq n$ then it follows from the equation $\dim(V_i \cap U_j) = \dim(V_i) + \dim(U_j) - \dim(V_i + U_j)$ that

$$\begin{aligned} \dim(V_i \cap U_j) = \max(i + j - n, 0) = i + j - n &\iff \dim(V_i + U_j) = n \\ &\iff V_i + U_j = \mathbb{C}^n. \end{aligned}$$

But as $j \geq n - i \geq 0$, one has $U_{n-i} \subset U_j$ and thus

$$V_i + U_j \supset V_i + U_{n-i} = \mathbb{C}^n.$$

Hence $V_i + U_j = \mathbb{C}^n$ as required. □

This next lemma is intuitively obvious, so we omit its proof. It is a stronger version of the well-known fact that $\{(v_1, \ldots, v_n) : v_1, \ldots, v_n \in \mathbb{C}^n$ are linearly independent$\}$ is a dense subset of the n-fold product $\mathbb{C}^n \times \cdots \times \mathbb{C}^n$.

Lemma A.7. *Suppose that $v_1, \ldots, v_{n-1} \in \mathbb{C}^n$ are linearly independent. Given any $\delta > 0$ and any $1 \leq k \leq n-1$, there exists $w \in \mathbb{C}^n$ such that $\|w - v_k\| < \delta$ and $v_1, \ldots, v_{n-1}, w$ are linearly independent.*

The following result is the key to our proof for Lemma A.5. Roughly speaking, this says that the flags in general position with respect to a given flag form a dense set.

Lemma A.8. *Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for $\mathbb{C}^n$ and $\mathcal{F} : \{0\} \subset U_1 \subset \cdots \subset U_n = \mathbb{C}^n$ be a flag. Then given any $\varepsilon > 0$, there exists an orthonormal basis $\{w_1, \ldots, w_n\}$ with the following properties.*

- *$\|w_i - v_i\| < \varepsilon$ for $i = 1, \ldots, n$ and*
- *setting $W_i = \operatorname{span}\{w_1, \ldots, w_i\}$, the flag $\mathcal{E} : \{0\} \subset W_1 \subset \cdots \subset W_n = \mathbb{C}^n$ is in general position with respect to $\mathcal{F}$.*

Proof. Let $\mathcal{B} \subset (\mathbb{C}^n)^n$ be the set of all n-tuples $(w_1, \ldots, w_n)$ of vectors $w_j \in \mathbb{C}^n$ with $\{w_1, \ldots, w_n\}$ linearly independent, hence a basis. The *Gram-Schmidt process* [3, §6.2] produces a *continuous* mapping

$$\mathcal{G} : \mathcal{B} \to \mathcal{B}$$

which fixes each $(v_1, \ldots, v_n) \in \mathcal{B}$ for which $\{v_1, \ldots, v_n\}$ is an orthonormal basis. Thus given an orthonormal basis $\{v_1, \ldots, v_n\}$ and any $\varepsilon > 0$ there is some $\delta > 0$ such that

- for any $(w_1, \ldots, w_n) \in \mathcal{B}$ with $\|w_i - v_i\| < \delta$ for all i, one has that $(w'_1, \ldots, w'_n) = \mathcal{G}((w_1, \ldots, w_n))$ is an orthonormal basis satisfying $\|w'_i - v_i\| < \varepsilon$ for all i.

Moreover, the Gram-Schmidt process ensures that $\operatorname{span}\{w'_1, \ldots, w'_i\} = \operatorname{span}\{w_1, \ldots, w_i\}$ for all i.

Given an orthonormal basis $\{v_1, \ldots, v_n\}$ and $\varepsilon > 0$, let $\delta > 0$ be as above. The preceding discussion shows that it suffices to produce a (not necessarily orthonormal) basis $\{w_1, \ldots, w_n\}$ for which

- $\|w_i - v_i\| < \delta$ for all i and
- setting $W_i = \text{span}\{w_1, \ldots, w_i\}$ the flag $\mathcal{E} : \{0\} \subset W_1 \subset \cdots \subset W_n = \mathbb{C}^n$ is in general position with respect to $\mathcal{F}$.

For then $\mathcal{G}((w_1, \ldots, w_n))$ is an orthonormal basis with the desired properties.

We construct this basis $\{w_1, \ldots, w_n\}$ as follows. By the previous lemma, there exists $w_1 \in \mathbb{C}^n$ such that $\|w_1 - v_1\| < \delta$ and $v_1, \ldots, v_{n-1}, w_1$ are linearly independent. Similarly, there exists $w_2 \in \mathbb{C}^n$ such that $\|w_2 - v_2\| < \delta$ and $v_1, \ldots, v_{n-2}, w_1, w_2$ are linearly independent. Continuing in this way, we obtain a basis $\{w_1, \ldots, w_n\}$ for $\mathbb{C}^n$ with $\|w_i - v_i\| < \delta$ for all i and $v_1, \ldots, v_k, w_1, \ldots, w_{n-k}$ linearly independent for each k.

Let $\mathcal{E} : \{0\} \subset W_1 \subset \cdots \subset W_n = \mathbb{C}^n$ be the flag with $W_i = \text{span}\{w_1, \ldots, w_i\}$ for each i. Then $V_k \oplus W_{n-k} = \mathbb{C}^n$ for each k, and so it follows from Lemma A.6 that $\mathcal{E}$ is in general position with respect to $\mathcal{F}$. □

We are now able to prove Lemma A.5.

Proof for Lemma A.5. Let $I, J \in \mathcal{P}_r^n$ and suppose that $S(I, \mathcal{F}_1) \cap S(J, \mathcal{F}_2) \neq \emptyset$ for every pair of flags $\mathcal{F}_1$, $\mathcal{F}_2$ in general position. Now let $\mathcal{F}_1$, $\mathcal{F}_2$ be a pair of flags which *fail* to be in general position. We need to show that $S(I, \mathcal{F}_1) \cap S(I, \mathcal{F}_2) \neq \emptyset$.

Write $\mathcal{F}_1 : \{0\} \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$ and let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for which $V_i = \text{span}\{v_1, \ldots, v_i\}$. For each $k \in \mathbb{N}$, apply Lemma A.8 to obtain an orthonormal basis $\{w_1^{(k)}, \ldots, w_n^{(k)}\}$ for $\mathbb{C}^n$ for which

- $\|w_i^{(k)} - v_i\| < 1/k$ for $i = 1, \ldots, n$ and

- setting $W_i^{(k)} = \operatorname{span}\{w_1^{(k)}, \ldots, w_i^{(k)}\}$, the flag

$$\mathcal{E}^{(k)} : \{0\} \subset W_1^{(k)} \subset \cdots \subset W_n^{(k)} = \mathbb{C}^n$$

  is in general position with respect to $\mathcal{F}_2$.

Note in particular that $\lim_k w_i^{(k)} = v_i$ for $i = 1, \ldots, n$.

By hypothesis, $S(I, \mathcal{E}^{(k)}) \cap S(J, \mathcal{F}_2) \neq \emptyset$ and we can choose an r-dimensional subspace $L^{(k)} \in S(I, \mathcal{E}^{(k)}) \cap S(J, \mathcal{F}_2)$. Since $G_r(\mathbb{C}^n)$ is compact, the sequence $\left(L^{(k)}\right)_{k=1}^{\infty}$ has a convergent subsequence. By passing to such a subsequence if necessary, we may assume that $\left(L^{(k)}\right)_{k=1}^{\infty}$ converges to say $L \in G_r(\mathbb{C}^n)$. As $L^{(k)} \in S(J, \mathcal{F}_2)$ for each k and $S(J, \mathcal{F}_2)$ is a closed subset of $G_r(\mathbb{C}^n)$, it follows that $L = \lim_k L^{(k)}$ belongs to $S(J, \mathcal{F}_2)$. We show below that also $L \in S(I, \mathcal{F}_1)$. Hence $S(I, \mathcal{F}_1) \cap S(J, \mathcal{F}_2) \neq \emptyset$, completing the proof.

To prove that $L \in S(I, \mathcal{F}_1)$, it remains only to verify that for fixed $j \in \{1, \ldots, r\}$ one has $\dim(L \cap V_{i_j}) \geq j$ (where $I = (i_1, \ldots, i_r)$). As $L^{(k)} \in S(I, \mathcal{E}^{(k)})$, we have $\dim(L^{(k)} \cap W_{i_j}^{(k)}) \geq j$ and may choose, for each $k \in \mathbb{N}$, a j-dimensional subspace $L_\circ^{(k)} \subset (L^{(k)} \cap W_{i_j}^{(k)})$. By compactness of $G_j(\mathbb{C}^n)$, we can assume, by passing to a subsequence if necessary, that $\left(L_\circ^{(k)}\right)_{k=1}^{\infty}$ converges to, say, $L_\circ \in G_j(\mathbb{C}^n)$. As $L_\circ^{(k)} \subset L^{(k)}$ for each k, it is clear that $L_\circ \subset L$. We claim that also $L_\circ \subset V_{i_j}$ and hence that $\dim(L \cap V_{i_j}) \geq j$ as desired.

Let $w \in L_\circ$ be given. We now show that $w \in V_{i_j}$. Choose vectors $w^{(k)} \in L_\circ^{(k)}$ with $\lim_k w^{(k)} = w$. As $w^{(k)} \in W_{i_j}^{(k)}$, we may write

$$w^{(k)} = c_1^{(k)} w_1^{(k)} + \cdots + c_{i_j}^{(k)} w_{i_j}^{(k)}$$

for some $c_1^{(k)}, \ldots, c_{i_j}^{(k)} \in \mathbb{C}$. Since $\{w_1^{(k)}, \ldots, w_n^{(k)}\}$ is orthonormal, we have

$$\|w^{(k)}\|^2 = |c_1^{(k)}|^2 + \cdots + |c_{i_j}^{(k)}|^2$$

for each k. As $\|w^{(k)}\|^2$ converges to $\|w\|^2$ as $k \to \infty$, it follows that the sequences $\left(c_\ell^{(k)}\right)_{k=1}^\infty$ are bounded for each $\ell = 1, \ldots, i_j$. Passing to a subsequence if necessary, we can assume that each $\left(c_\ell^{(k)}\right)_{k=1}^\infty$ converges in $\mathbb{C}$, say

$$\lim_{k\to\infty} c_\ell^{(k)} = c_\ell.$$

So finally, we see that

$$w = \lim_{k\to\infty} w^{(k)} = \lim_{k\to\infty} \left(c_1^{(k)} w_1^{(k)} + \cdots + c_{i_j}^{(k)} w_{i_j}^{(k)}\right) = c_1 v_1 + \cdots + c_{i_j} v_{i_j}$$

belongs to V_{i_j} as required. □

Working from Lemma A.5 we obtain a substantial technical refinement of Proposition 3.18. This is Theorem A.14 below. Recall that for a given flag $\mathcal{F} : \{0\} \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$, its complimentary flag $\mathcal{F}' : \{0\} \subset U_1 \subset \cdots \subset U_n = \mathbb{C}^n$ has

$$U_j = V_{n-j}^\perp.$$

Equivalently, choosing an orthonormal basis $\{v_1, \ldots, v_n\}$ for $\mathbb{C}^n$ with $V_j = \operatorname{span}\{v_1, \ldots, v_j\}$, one has

$$U_j = \operatorname{span}\{v_{n-j+1}, \ldots, v_n\}.$$

Lemma A.9. *The flags $\mathcal{F}$, $\mathcal{F}'$ are in general position for any flag $\mathcal{F}$.*

Proof. For $1 \le i \le n-1$ we have $V_i \cap U_{n-i} = V_i \cap V_i^\perp = \{0\}$. Thus $\mathcal{F}$, $\mathcal{F}'$ are in general position by Lemma A.6. □

Given a vector space isomorphism $T : \mathbb{C}^n \to \mathbb{C}^n$ and flag $\mathcal{F} : \{0\} \subset V_1 \subset \cdots \subset$

$V_n = \mathbb{C}^n$ we denote by $T(\mathcal{F})$ the flag

$$\{0\} \subset T(V_1) \subset \cdots \subset T(V_n) = \mathbb{C}^n.$$

Lemma A.10. *Given any pair of flags $\mathcal{F}_1$, $\mathcal{F}_2$ there exists a vector space isomorphism $T : \mathbb{C}^n \to \mathbb{C}^n$ with $T(\mathcal{F}_1) = \mathcal{F}_2$.*

Proof. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be $\{0\} \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$ and $\{0\} \subset U_1 \subset \cdots \subset U_n = \mathbb{C}^n$. Choose bases for $\{v_1, \ldots, v_n\}$, $\{u_1, \ldots, u_n\}$ for $\mathbb{C}^n$ with $V_j = \text{span}\{v_1, \ldots, v_j\}$ and $U_j = \text{span}\{u_1, \ldots, u_j\}$. Letting $T : \mathbb{C}^n \to \mathbb{C}^n$ be the linear transformation with $T(v_j) = u_j$ for each j, it is clear that T is an isomorphism satisfying $T(\mathcal{F}_1) = \mathcal{F}_2$. □

Lemma A.11. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be a pair of flags in general position. Then there exists a vector space isomorphism $T : \mathbb{C}^n \to \mathbb{C}^n$ with $T(\mathcal{F}_1) = \mathcal{F}_1$ and $T(\mathcal{F}_2) = \mathcal{F}_1'$, the complementary flag.*

Proof. Let $\{e_1, \ldots, e_n\}$ denote the standard basis for $\mathbb{C}^n$. Lemma A.10 ensures that there is an isomorphism $\mathbb{C}^n \to \mathbb{C}^n$ that takes $\mathcal{F}_1$ to the standard flag $0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$ with $V_j = \text{span}\{e_1, \ldots, e_j\}$. We can therefore assume without loss of generality that $\mathcal{F}_1$ is the standard flag.

We restrict our attention to the case where $n = 4$ in an effort to provide a cleaner proof. It will be obvious how this argument can be adapted to prove the general case.

Since $n = 4$, we have flags

$$\mathcal{F}_1 : \{0\} \subset V_1 \subset V_2 \subset V_3 \subset V_4 = \mathbb{C}^4, \quad \mathcal{F}_2 : \{0\} \subset U_1 \subset U_2 \subset U_3 \subset U_4 = \mathbb{C}^4$$

in general position with $V_j = \text{span}\{e_1, \ldots, e_j\} \subset \mathbb{C}^4$. As $U_1 \cap V_3 = \{0\}$, there is a nonzero vector in U_1 of the form

$$u_1 = (u_{11}, u_{12}, u_{13}, 1).$$

As $\dim(U_2 \cap V_3) = 1$ but $\dim(U_2 \cap V_2) = 0$, there is a vector $u_2 \in U_2$ of the form

$$u_2 = (u_{21}, u_{22}, 1, 0).$$

As u_1 and u_2 are linearly independent, we see that

$$U_2 = \text{span}\{u_1, u_2\}.$$

As $\dim(U_3 \cap V_2) = 1$ but $\dim(U_3 \cap V_1) = 0$, there must exist a vector $u_3 \in U_3$ of the form

$$u_3 = (u_{31}, 1, 0, 0).$$

As u_1, u_2, u_3 are linearly independent, we have

$$U_3 = \text{span}\{u_1, u_2, u_3\}.$$

Setting

$$u_4 = e_1 = (1, 0, 0, 0)$$

gives us a basis $\{u_1, u_2, u_3, u_4\}$ for $\mathbb{C}^4$ with $U_j = \text{span}\{u_1, \ldots, u_j\}$ for $j = 1, 2, 3, 4$.

Let $T : \mathbb{C}^4 \to \mathbb{C}^4$ be the vector space isomorphism satisfying

$$T(u_1) = e_4, \quad T(u_2) = e_3, \quad T(u_3) = e_2, \quad T(u_4 = e_1) = e_1.$$

By construction, $T(\mathcal{F}_2)$ is the complementary flag of the standard flag $\mathcal{F}_1$. That is,

$$T(U_1) = \mathbb{C}e_4, \quad T(U_2) = \text{span}\{e_3, e_4\}, \quad T(U_3) = \text{span}\{e_2, e_3, e_4\}, \quad T(U_4) = \mathbb{C}^4.$$

It now just remains to verify that T preserves $\mathcal{F}_1$. First, we compute

- $T(e_1) = e_1$
- $T(e_2) = T(u_3 - u_{31}e_1) = T(u_3) - u_{31}T(e_1) = e_2 - u_{31}e_1$
- $T(e_3) = T(u_2 - u_{21}e_1 - u_{22}e_2) = T(u_2) - u_{21}T(e_1) - u_{22}T(e_2) = e_3 - u_{21}e_1 - u_{22}(e_2 - u_{31}e_1) = e_3 + (u_{22}u_{31} - u_{21})e_1 - u_{22}e_2$

The following computations now show that T preserves $\mathcal{F}_1$, completing the proof:

- $T(V_1) = \text{span}\{T(e_1)\} = \text{span}\{e_1\} = V_1$
- $T(V_2) = \text{span}\{T(e_1), T(e_2)\} = \text{span}\{e_1, e_2 - u_{31}e_1\} = \text{span}\{e_1, e_2\} = V_2$
- $T(V_3) = \text{span}\{T(e_1), T(e_2), T(e_3)\} = \text{span}\{e_1, e_2, e_3 + (u_{22}u_{31} - u_{21})e_1 - u_{22}e_2\} = \text{span}\{e_1, e_2, e_3\} = V_3$ □

Lemma A.12. *Given $I, J \in \mathcal{P}^n_r$, complete flags $\mathcal{F}_1$, $\mathcal{F}_2$, and $T : \mathbb{C}^n \to \mathbb{C}^n$ a vector space isomorphism, one has*

$$S(I, \mathcal{F}_1) \cap S(J, \mathcal{F}_2) \neq \emptyset \iff S(I, T(\mathcal{F}_1)) \cap S(J, T(\mathcal{F}_2)) \neq \emptyset.$$

Proof. If $L \in S(I, \mathcal{F}_1) \cap S(J, \mathcal{F}_2)$ then $T(L) \in S(I, T(\mathcal{F}_1)) \cap S(J, T(\mathcal{F}_2))$. Conversely, if $L \in S(I, T(\mathcal{F}_1)) \cap S(J, T(\mathcal{F}_2))$ then $T^{-1}(L) \in S(I, \mathcal{F}_1) \cap S(J, \mathcal{F}_2)$. □

Lemma A.13. *For $I, J \in \mathcal{P}^n_r$, the following are equivalent:*

(1) $S(I, \mathcal{F}_1) \cap S(J, \mathcal{F}_2) \neq \emptyset$ for every pair of flags $\mathcal{F}_1$, $\mathcal{F}_2$ in general position.

(2) $S(I, \mathcal{F}) \cap S(J, \mathcal{F}') \neq \emptyset$ for every flag $\mathcal{F}$ ($\mathcal{F}'$ denotes the complementary flag).

(3) $S(I, \mathcal{F}) \cap S(J, \mathcal{F}') \neq \emptyset$ for a single flag $\mathcal{F}$.

Proof. We have (1) $\implies$ (2) by Lemma A.9, and (2) $\implies$ (3) is obvious. To see that (3) $\implies$ (1), assume that $S(I, \mathcal{F}) \cap S(J, \mathcal{F}') \neq \emptyset$ for some flag $\mathcal{F}$ and let $\mathcal{F}_1$, $\mathcal{F}_2$ be another pair of flags in general position. Lemma A.10 shows that there is an isomorphism $T : \mathbb{C}^n \to \mathbb{C}^n$ with $T(\mathcal{F}_1) = \mathcal{F}$. Moreover, it is clear that $\mathcal{F} = T(\mathcal{F}_1)$ and $T(\mathcal{F}_2)$ are in general position. Therefore, Lemma A.11 ensures that there is an isomorphism $S : \mathbb{C}^n \to \mathbb{C}^n$ satisfying $S(\mathcal{F}) = \mathcal{F}$ and $S(T(\mathcal{F}_2)) = \mathcal{F}'$. So now $R = S \circ T$ is an isomorphism $\mathbb{C}^n \to \mathbb{C}^n$ with $R(\mathcal{F}_1) = \mathcal{F}$ and $R(\mathcal{F}_2) = \mathcal{F}'$. As $S(I, \mathcal{F}) \cap S(J, \mathcal{F}') \neq \emptyset$, Lemma A.12 now implies that $S(I, \mathcal{F}_1) \cap S(J, \mathcal{F}_2) \neq \emptyset$ as desired. □

The following refinement of Proposition 3.18 now follows immediately from Proposition 3.18 in combination with Lemmas A.5 and A.13.

Theorem A.14. *Given $I, J \in \mathcal{P}_r^n$ with $d(I) + d(J) \geq N$, the following conditions are equivalent:*

(1) $\sigma_I \sigma_J \neq 0$ *in* $H^\bullet(G_r(\mathbb{C}^n))$.

(2) $S(I, \mathcal{F}_1) \cap S(J, \mathcal{F}_2) \neq \emptyset$ *for all complete flags* $\mathcal{F}_j$, $j = 1, 2$.

(3) $S(I, \mathcal{F}_1) \cap S(J, \mathcal{F}_2) \neq \emptyset$ *for every pair of flags* $\mathcal{F}_1$, $\mathcal{F}_2$ *in general position.*

(4) $S(I, \mathcal{F}) \cap S(J, \mathcal{F}') \neq \emptyset$ *for every flag* $\mathcal{F}$ *(*$\mathcal{F}'$ *denotes the complementary flag).*

(5) $S(I, \mathcal{F}) \cap S(J, \mathcal{F}') \neq \emptyset$ *for a single flag* $\mathcal{F}$.