ABSTRACT

MATHEMATICAL TECHNIQUES FOR THE ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS

by

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Major Department: Mathematics

This thesis explores various solution methods for partial differential equations. The heat equation, wave equation and Laplace equation are analyzed using techniques from functional analysis, Fourier series, and Fourier transforms. The analysis techniques are studied in depth and applied to the respective partial differential equations to obtain a solution for each problem.

MATHEMATICAL TECHNIQUES FOR THE ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS

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CHAPTER 1: Introduction

Partial differential equations are used to model a wide range of physical problems. In order to solve partial differential equations, many techniques of analysis are employed. This thesis addresses both issues. The first half discusses various mathematical techniques essential to deriving the solutions to partial differential equations. The second half discusses three particular partial differential equations - the heat equation, wave equation, and Laplace's equation - and the problems associated with them. Our main focus is to address these problems using the mathematical techniques discussed.

Chapter two introduces concepts from functional analysis that are applied in various chapters, including the delta function and convolution. The delta function is used in chapter seven to derive solutions to partial differential equations via convolution with the fundamental solution. Convolution is also used to derive solutions or validate theorems in second chapter five using the heat kernel. The chapter ends with a discussion of two convergence theorems essential to integration theory.

Chapter three introduces and examines Fourier series. The context for analysis of Fourier series is Hilbert spaces, and basic results concerning Hilbert spaces are introduced. We verify the function space for Fourier series is indeed a Hilbert space by confirming completion. Afterwards, we discuss the two approaches to Fourier series, complex and trigonometric, and the advantages of each.

Chapter four introduces another mathematical technique, the Fourier transform, and explores properties such as inversion and convolution. To show that the Fourier transform is an isometry, we verify Plancherel's Theorem. At the end of the chapter, we examine the Gaussian function and its properties, which are used in the proofs of Plancherel's Theorem and inversion of the Fourier transform.

In the last three chapters, we show how to apply the mathematical techniques introduced thus far. Each chapter focuses on a particular partial differential equation and identifies the mathematical techniques that can be used to solve the initial value problems associated with them.

Chapter five introduces the heat equation. For its general initial value problem, we specify boundary and initial values of temperature, and what technique can be used to meet these conditions. The solution to the general initial value problem of the heat equation uses separation of variables and Fourier series. Separation of variables is used to meet the boundary conditions, while Fourier series are used to satisfy the initial temperature distribution. When the domain of an initial value problem is that of the real line, we introduce another mathematical technique specific to the heat equation, the heat kernel.

Chapter six introduces the wave equation. Similar to chapter five, we use separation of variables and Fourier series to meet the boundary values and initial conditions. For an initial value problem with the real line as a domain, we derive d'Alembert's formula for a general solution by applying the Fourier transform from chapter four.

Chapter seven introduces the Laplace equation, both homogeneous and inhomogeneous. The solutions to the homogeneous Laplace equations are harmonic functions. We demonstrate the fundamental solution to the Laplace equation in $\mathbb{R}^2$. We represent the Laplace equation in polar coordinates to explore rotation symmetry. Another problem explored is the boundary value problem in the disc using the Poisson Kernel.

CHAPTER 2: Functional Analysis

The general definitions and theorems in this chapter can be found in [6], [12], [17], and [20].

2.1 Distributions

Definition 2.1. A test function is a compactly supported smooth function. The topology on the set of test functions, denoted $\mathcal{D} = C_c^\infty(\mathbb{R})$, is defined so that a sequence $f_n \to f$ iff:

1. $\text{supp}(f_n)$ is contained in a compact set K for all n
2. $\lim\limits_{n\to\infty} ||f_n - f||_\infty = 0$ for all $k \geq 0$

Definition 2.2. A distribution is a linear functional, i.e., a continuous linear map from $\mathcal{D}$ to $\mathbb{C}$. The dual space $\mathcal{D}'$ is the space of continuous linear functionals $\varphi : \mathcal{D} \to \mathbb{C}$. We write $\varphi(f) = \langle \varphi, f \rangle$ where $\varphi \in \mathcal{D}'$ and $f \in \mathcal{D}$. The weak topology on $\mathcal{D}'$ is defined so that

$$\varphi_n \to \varphi \quad \text{iff} \quad \langle \varphi_n, f \rangle \to \langle \varphi, f \rangle$$

for all $f \in \mathcal{D}$.

Example 2.3. Let φ be bounded and continuous on $\mathbb{R}$. For $f \in \mathcal{D}$, define

$$\langle \varphi, f \rangle = \int_{\mathbb{R}} \varphi(x) f(x) dx.$$

Then φ is a continuous linear functional on $\mathcal{D}$.

Definition 2.4. The delta "function" is actually a distribution that assigns $f(0)$ to the function f. We denote the delta function

$$\langle \delta, f \rangle = \int_{\mathbb{R}} \delta(x) f(x) dx = f(0).$$

Definition 2.5. We define, for an arbitrary $\varphi \in \mathcal{D}'$,

$$\langle \varphi', f \rangle = -\langle \varphi, f' \rangle.$$

This defines φ'. This definition is motivated by the following: If $\varphi \in C^1$, then for $f \in \mathcal{D}$

$$\langle \varphi', f \rangle = \int \varphi'(x) f(x) dx = -\int \varphi(x) f'(x) dx = -\langle \varphi, f' \rangle.$$

Thus, for another example,

$$\langle \delta', f \rangle = -\langle \delta, f' \rangle = -f'(0).$$

Definition 2.6. A fundamental solution to a differential operator L is a distribution E such that

$$L(E) = \delta,$$

where the derivatives are applied to E in the distribution sense.

2.2 Convolution

Definition 2.7. If f and g are two integrable functions on $\mathbb{R}$, their convolution is

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y) g(y) dy = \int_{\mathbb{R}} f(y) g(x-y) dy.$$

The following properties of convolution hold:

1. Symmetry: $f * g = g * f$
2. Bilinearity: $f * (ag + bh) = a(f * g) + b(f * h)$
3. Associativity: $f * (g * h) = (f * g) * h$
4. Delta Function: $\delta * f = f$
5. Support: If $f, g \in \mathcal{D}$, then $f * g \in \mathcal{D}$ and $\text{supp}(f * g) \subseteq \text{supp}(f) + \text{supp}(g)$

Given a function φ and $f \in \mathcal{D}$, define $f_x(y) = f(x - y)$. Then,

$$(\varphi * f)(x) = \int \varphi(y) f(x - y) dy = \langle \varphi, f_x \rangle.$$

Thus we can extend convolution to distributions by defining, for $\varphi \in \mathcal{D}'$ and $f \in \mathcal{D}$,

$$(\varphi * f)(x) = \langle \varphi, f_x \rangle.$$

For example,

$$(\delta * f)(x) = \langle \delta, f_x \rangle = f_x(0) = f(x)$$

Thus,

$$\delta * f = f.$$

2.3 Integration Results

Theorem 2.8. *(Monotone Convergence Theorem) Let $\{f_n\}$ be a sequence of measurable functions on $E \subset \mathbb{R}$. If $\{f_n\}$ is non-negative and monotone increasing, then* $\lim f_n$ *exists and*

$$\lim \int_E f_n = \int_E \lim f_n.$$

Proof. Let $f(x) = \sup f_n(x)$. Then $f_n \to f$ pointwise. By Fatou's Lemma,

$$\int f \leq \liminf \int f_n.$$

Also, $f_n \leq f$, so $\int f_n \leq \int f$. Thus

$$\limsup \int f_n \leq \int f \leq \liminf \int f_n.$$

Hence $\lim \int f_n = \int f$. □

Theorem 2.9. *(Dominated Convergence Theorem) Let $\{f_n\}$ be a sequence of measurable functions on E with $f_n \to f$ almost everywhere on E. Suppose there is an integrable function g on E with $|f_n| \leq g$ on E for all n. Then $\{f_n\}$ and f are integrable and*

$$\int f = \lim \int f_n.$$

Proof. Suppose $f_n \to f$ everywhere on E. As $|f_n| \leq g$ for all n, $|f| \leq g$. Now as g is integrable, so are $\{f_n\}$ and f. We have $|f_n| \leq g$ which implies $-g \leq f_n \leq g$. Thus

$$g + f_n \geq 0 \quad \text{and} \quad g - f_n \geq 0.$$

Applying Fatou's Lemma to $g + f_n$ yields

$$\int_E g + \int_E f = \int_E g + f \leq \liminf \int_E g + f_n = \int_E g + \liminf \int_E f_n.$$

Thus

$$\int_E f \leq \liminf \int_E f_n.$$

Applying Fatou's Lemma to $g - f_n$ yields

$$\int_E g - \int_E f = \int_E g - f \leq \liminf \int_E g - f_n = \int_E g - \limsup \int_E f_n.$$

Thus

$$\int_E f \geq \limsup \int_E f_n.$$

Hence

$$\limsup \int_E f_n \leq \int_E f \leq \liminf \int_E f_n.$$

By definition,

$$\liminf \int_E f_n \leq \limsup \int_E f_n,$$

so

$$\liminf \int_E f_n = \limsup \int_E f_n = \int_E f.$$

□

CHAPTER 3: Fourier Series

A solution method to the initial value problem for partial differential equations is the use of Fourier series. The general definitions and theorems in this chapter are adapted from [2], [13], [15], [17], and [22].

3.1 Hilbert Spaces

The proper setting for Fourier series is in the context of Hilbert spaces.

Definition 3.1. An inner product on a complex vector space V is a map

$$V \times V \to \mathbb{C}, \quad (v, w) \mapsto \langle v, w \rangle$$

satisfying the following axioms for vectors $u, v, w \in V$ and scalars $c \in \mathbb{C}$:

1. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
2. $\langle cv, w \rangle = c\langle v, w \rangle$
3. $\langle w, v \rangle = \overline{\langle v, w \rangle}$
4. $\langle v, v \rangle > 0$ for $v \neq 0$

Definition 3.2. A Hilbert space is a complex vector space with an inner product which is complete with respect to the norm

$$||v|| = \sqrt{\langle v, v \rangle}.$$

Definition 3.3. Suppose V is an inner product space. Then $\{v_1, v_2, v_3, \ldots\}$ is an orthonormal system iff

$$\langle v_n, v_m \rangle = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} .$$

Definition 3.4. Let V be a normed vector space. A sequence $\{f_k\} \in V$ converges in norm to a vector $f \in V$ if

$$||f_k - f|| \to 0 \quad \text{as } k \to \infty.$$

Given a vector $f \in V$, its **generalized Fourier series** with respect to an orthonormal system $\{v_n\}$ is

$$f \sim \sum_{n=1}^{\infty} c_n v_n$$

with coefficients

$$c_n = \langle f, v_n \rangle.$$

The generalized Fourier series converges in norm to f if the partial Fourier sums

$$f_k = \sum_{n=1}^{k} c_n v_n \tag{3.1}$$

satisfy

$$||f_k - f|| \to 0 \quad \text{as } k \to \infty.$$

For a finite orthonormal system, the generalized Fourier series is the best approximation to f in the space:

Theorem 3.5. *Let $\{v_1, v_2, \ldots\}$ be an orthonormal sequence in V. Define $S_k = span\{v_1, v_2, \ldots, v_k\} \subset V$ to be the subspace spanned by the first k elements of the orthonormal system. Then the k^{th} term of the partial Fourier sum $f_k \in S_k$ is the best approximation to f. This means that $p \mapsto ||f - p||$, for $p \in S_k$, is minimized by $p = f_k$.*

Proof. Suppose we have an element $p \in S_k$ where

$$p_= \sum_{n=1}^{k} d_n v_n.$$

Its norm is

$$\begin{aligned}
||p||^2 &= \langle p, p \rangle \\
&= \left\langle \sum_{n=1}^{k} d_n v_n, \sum_{m=1}^{k} d_m v_m \right\rangle \\
&= \sum_{n,m=1}^{k} d_n d_m \langle v_n, v_m \rangle.
\end{aligned}$$

Since $\langle v_n, v_m \rangle = 0$ for $n \neq m$, we get

$$||p||^2 = \sum_{n=1}^{k} |d_n|^2.$$

In addition,

$$\begin{aligned}
||f-p||^2 &= \langle f-p, f-p \rangle \\
&= ||f||^2 - \langle f, p \rangle - \langle p, f \rangle + ||p||^2 \\
&= ||f||^2 - \sum_{n=1}^{k} d_n \langle v_n, f \rangle - \sum_{n=1}^{k} \overline{d_n} \langle f, v_n \rangle + \sum_{n=1}^{k} |d_n|^2 \\
&= ||f||^2 - \sum_{n=1}^{k} d_n \overline{c_n} - \sum_{n=1}^{k} \overline{d_n} c_n + \sum_{n=1}^{k} |d_n|^2 \\
&= ||f||^2 + \sum_{n=1}^{k} |c_n - d_n|^2 - \sum_{n=1}^{k} |c_n|^2.
\end{aligned}$$

We substitute the squared norm of (3.1),

$$\sum_{n=1}^{k} |c_n|^2 = ||f_k||^2$$

to get

$$||f-p||^2 = ||f||^2 + \sum_{n=1}^{k} |c_n - d_n|^2 - ||f_k||^2.$$

On the right hand side, the first and last terms are independent of $p \in S_k$, and the middle term is non-negative and will only be minimized if $c_n = d_n$ for all $n = 1, 2, \ldots k$. Thus $||f-p||$ is minimized overall $p \in S_k$ iff $d_n = c_n$. Hence $p = f_k$. $\square$

A very important property of the inner product spaces is the Cauchy-Schwartz inequality.

Corollary 3.6. *(Cauchy-Schwartz Inequality) Given $f, g \in V$,*

$$|\langle f, g\rangle| \leq ||f|| \cdot ||g||.$$

Proof. Suppose we have two vectors $f, g \in V$. Without loss of generality, we assume $g \neq 0$. Let $c \in \mathbb{R}$ be arbitrary. Using sesquilinearity, we get

$$0 \leq ||f+cg||^2 = \langle f+cg, f+cg\rangle = ||f||^2 + 2c\mathrm{Re}\langle f, g\rangle + c^2||g||^2$$

with equality iff $f = -cg$. We consider the right hand side as a real-valued function of c whose minimum occurs at $c = -\frac{\langle f,g\rangle}{||g||^2}$. We substitute into the right hand side to get

$$0 \leq ||f||^2 - 2\frac{|\langle f, g\rangle|^2}{||g||^2} + \frac{|\langle f, g\rangle|^2}{||g||^2} = ||f||^2 - \frac{|\langle f, g\rangle|^2}{||g||^2}.$$

Thus,

$$|\langle f, g\rangle|^2 \leq ||f||^2||g||^2.$$

Taking the square root of both sides yields

$$|\langle f, g\rangle| \leq ||f|| \cdot ||g||$$

as desired. $\square$

Theorem 3.7. *(Bessel's Inequality) The sum of the squares of the general Fourier coefficients of $f \in V$ is bounded. In particular,*

$$\sum_{n=1}^{\infty} |c_n|^2 \leq ||f||^2.$$

Proof. Suppose we have the partial Fourier series

$$f_k = \sum_{n=1}^{k} c_n v_n.$$

By the proof of Theorem 3.5, we have

$$0 \leq ||f - f_k||^2 = ||f||^2 - ||f_k||^2 = ||f||^2 - \sum_{n=1}^{k} |c_n|^2.$$

Thus, we have the inequality

$$\sum_{n=1}^{k} |c_n|^2 \leq ||f||^2$$

for all k as desired. $\square$

Corollary 3.8. *(Riemann-Lebesgue Lemma) If $\sum_{n=1}^{\infty} c_n v_n$ is the general Fourier series for f, then*

$$\lim_{n\to\infty} c_n = 0.$$

Definition 3.9. An orthonormal system $\{v_n\} \in H$ is complete iff, for every $f \in H$, its generalized Fourier series converges in norm to f. That is,

$$||f - f_k|| \to 0 \quad \text{as } k \to \infty$$

with $f_k = \sum_{n=1}^{k} c_n v_n$ and $c_n = \langle f, v_n \rangle$.

Theorem 3.10. *The orthonormal system V is complete iff Plancherel's formula holds for every $f \in V$. That is,*

$$||f||^2 = \sum_{n=1}^{\infty} |c_n|^2.$$

Proof. Taking the limit as $k \to \infty$ in the proof of Theorem 3.7 yields

$$\begin{aligned}
\lim_{k\to\infty} ||f - f_k||^2 &= \lim_{k\to\infty} ||f||^2 - \sum_{n=1}^{k} |c_n|^2 \\
&= ||f||^2 - \lim_{n\to\infty} \sum_{n=1}^{k} |c_n|^2 \\
&= ||f||^2 - \sum_{n=1}^{\infty} |c_n|^2.
\end{aligned}$$

Thus f_k converges in norm to f iff

$$||f||^2 = \sum_{n=1}^{\infty} |c_n|^2.$$

$\square$

3.2 L^2 Spaces

Definition 3.11. A function f is square integrable on $[-\pi, \pi]$ if it has finite L^2 norm

$$||f||_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx < \infty.$$

The space $L^2(I)$, where $I = [-\pi, \pi]$ or $\mathbb{R}$, is defined by

$$L^p(I) = \left\{ f : I \to \mathbb{C} : \int_I |f|^2 < \infty \right\}.$$

The inner product of two functions $f(x)$ and $g(x)$ in the complex Hilbert space L^2 is

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)} dx,$$

with

$$\langle f, f \rangle = ||f||_2^2.$$

We have

$$|\langle f, g \rangle| \le ||f||_2 ||g||_2,$$

so $\langle f, g \rangle$ is well defined.

Theorem 3.12. *L^2 is complete.*

Proof. Suppose that $\{f_n\}$ is a Cauchy sequence in L^2. Then there exists a subsequence $\{f_{n_k}\}$ with

$$||f_{n_k} - f_{n_{k-1}}||_2 < \frac{1}{2^k}.$$

To ease notation, we go to a subsequence and assume that

$$||f_n - f_{n-1}||_2 < \frac{1}{2^n}.$$

Let $f_0 = 0$. We observe that

$$f_n = (f_n - f_{n-1}) + (f_{n-1} - f_{n-2}) + \ldots + (f_2 - f_1) + (f_1 - f_0),$$

so

$$||f_n||_2 \leq \sum_{k=2}^{n} ||f_k - f_{k-1}||_2 + ||f_1||_2 < 1 + ||f_1||_2 = M.$$

We define

$$g_n(x) = \sum_{k=1}^{n} |f_k(x) - f_{k-1}(x)|.$$

Then $\{g_n\}$ is an increasing sequence of real valued functions, so

$$g(x) = \lim_{n\to\infty} g_n(x)$$

exists for all x. Also,

$$||g_n||_2 \leq \sum_{k=1}^{n} ||f_k - f_{k-1}||_2 \leq M,$$

i.e.,

$$\int |g_n|^2 \leq M^2 \quad \text{for all } n.$$

So by the Monotone Convergence Theorem,

$$\int |g|^2 \leq M^2.$$

Thus,

$$g(x) = \sum_{k=1}^{\infty} |f_k(x) - f_{k-1}(x)|$$

is finite a.e. So

$$\sum_{k=1}^{\infty} (f_k(x) - f_{k-1}(x))$$

is absolutely convergent for x a.e. Define

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists a.e. and

$$\begin{aligned} |f_n(x)| &= \left| \sum_{k=1}^{n} f_k(x) - f_{k-1}(x) \right| \\ &\le \sum_{k=1}^{n} |f_k(x) - f_{k-1}(x)| \\ &= g_n(x) \\ &\le g(x). \end{aligned}$$

So

$$|f(x)|^2 \le g(x)^2,$$

hence f^2 is integrable. Thus

$$|f_n - f|^2 \le (|f_n|^2 + |f|^2) \le 2g^2.$$

So

$$\int |f_n - f|^2 \to 0$$

by the Dominated Convergence Theorem, i.e.,

$$||f_n - f||_2 \to 0 \quad \text{so} \quad f_n \to f \text{ in } L^2.$$

$\square$

Proposition 3.13. *The complex exponentials* $\{e^{inx} : n \in \mathbb{Z}\}$ *are an orthonormal system in* $L^2[-\pi, \pi]$.

Proof. For $n, m \in \mathbb{Z}$ and $n \neq m$,

$$\begin{aligned}
\langle e^{inx}, e^{imx} \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} e^{-imx} dx \\
&= \frac{1}{2\pi} \left[\frac{e^{ix(n-m)}}{i(n-m)} \right]_{-\pi}^{\pi} \\
&= \frac{1}{2\pi} \left(\frac{e^{i\pi(n-m)} - e^{-i\pi(n-m)}}{i(n-m)} \right) \\
&= 0.
\end{aligned}$$

Thus the e^{inx} are orthogonal. For all $n \in \mathbb{Z}$,

$$\begin{aligned}
\langle e^{inx}, e^{inx} \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} e^{-inx} dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} dx \\
&= \frac{1}{2\pi} [x]_{-\pi}^{\pi} \\
&= 1.
\end{aligned}$$

Hence, e^{inx} is an orthonormal system in $L^2[-\pi, \pi]$. $\square$

We now discuss completeness of Fourier series for sufficiently smooth functions.

Proposition 3.14. *If f is C^2 and 2π-periodic, then its Fourier series converges absolutely to f,*

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Proof. Suppose f is C^2 and 2π-periodic. Then $f(-\pi) = f(\pi)$, $f'(-\pi) = f'(\pi)$, and for $n \neq 0$

$$\begin{aligned}
c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\
&= -\frac{1}{2\pi i n} \int_{-\pi}^{\pi} f(x) \partial_x (e^{-inx}) dx \\
&= -\frac{1}{2\pi i n} \left[f(x) e^{-inx} \right]_{-\pi}^{\pi} + \frac{1}{2\pi i n} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx \\
&= -\frac{1}{2\pi n^2} \int_{-\pi}^{\pi} f''(x) e^{-inx} dx
\end{aligned}$$

using integration by parts twice. Letting $M = \frac{1}{2\pi} \max |f''|$, we obtain

$$|c_n| \leq \frac{M}{n^2}.$$

Now, fix $x_\circ \in [-\pi, \pi]$, and without loss of generality assume that $f(x_\circ) = 0$. Then

$$\begin{aligned} f_N(x_\circ) &= \sum_{-N}^{N} c_n e^{inx_\circ} \\ &= \sum_{-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \right) e^{inx_\circ} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \sum_{-N}^{N} e^{-in(x-x_\circ)} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+x_\circ) \sum_{-N}^{N} e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+x_\circ) \left(e^{iNx} \sum_{0}^{2N} e^{-inx} \right) dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+x_\circ) \frac{e^{-i(N+1)x} - e^{iNx}}{e^{-ix} - 1} dx \end{aligned}$$

for $x \neq 0$ (and $\sum = 2N+1$ for $x = 0$). Since f is continuous and differentiable, $\frac{f(x+x_\circ)}{e^{-ix}-1}$ is continuous on $[-\pi, \pi]$, so by the Riemann-Lebesgue Lemma, its Fourier coefficients go to zero. Thus

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \frac{f(x+x_\circ)}{e^{-ix}-1} e^{-i(N+1)x} dx = 0$$

and

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \frac{f(x+x_\circ)}{e^{-ix}-1} e^{iNx} dx = 0.$$

Hence $f_N \to f$, and the Fourier series converges absolutely,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

$\square$

Theorem 3.15. *The complex exponential functions e^{inx} with $n \in \mathbb{Z}$ form a complete orthonormal system in $L^2[-\pi, \pi]$.*

Proof. If f is 2π-periodic and C^2 on $[-\pi, \pi]$, its Fourier series converges absolutely

$$f = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

The same applies to the conjugate $\overline{f}$. Therefore,

$$|f(x)|^2 = f(x)\overline{f(x)} = f(x) \sum_{n=-\infty}^{\infty} \overline{c_n} e^{-inx} = \sum_{n=-\infty}^{\infty} f(x)\overline{c_n} e^{-inx}$$

which converges uniformly. We integrate both sides from $[-\pi, \pi]$,

$$\begin{aligned} ||f||_2^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx \\ &= \sum_{n=-\infty}^{\infty} \frac{\overline{c_n}}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \sum_{n=\infty}^{\infty} c_n \overline{c_n} \\ &= \sum_{n=-\infty}^{\infty} |c_n|^2. \end{aligned}$$

Hence, Plancherel's formula holds for any 2π-periodic C^2 function, and hence $||f_k - f||_2 \to 0$ for all such f. Since these functions are dense in L^2, this validates Plancherel's formula for all $f \in L^2$. Hence we obtain completeness by Theorem 3.10. $\square$

Example 3.16. Let $f(x) = x^2$ on $[-\pi, \pi]$. Since f is C^2 and $f(\pi) = f(-\pi)$, it has an absolutely convergent Fourier series. For $n = 0$:

$$\begin{aligned} c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx \\ &= \frac{1}{2\pi}\left(\frac{x^3}{3}\right)\Big]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi}\left(\frac{\pi^3}{3} - \frac{-\pi^3}{3}\right) \\ &= \frac{\pi^2}{3}. \end{aligned}$$

For $n \neq 0$:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx} dx.$$

Integration by parts yields

$$\begin{aligned} c_n &= \frac{1}{2\pi}\left(\left[\frac{-x^2}{in}e^{-inx}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{-2x}{in}e^{-inx}dx\right) \\ &= \frac{1}{2\pi}\left(\frac{-\pi^2}{in}e^{-in\pi} + \frac{\pi^2}{in}e^{in\pi} - \int_{-\pi}^{\pi}\frac{-2x}{in}e^{-inx}dx\right) \\ &= \frac{1}{2\pi}\left(\frac{-\pi^2}{in}(e^{-in\pi} - e^{in\pi}) - \int_{-\pi}^{\pi}\frac{-2x}{in}e^{-inx}dx\right). \end{aligned}$$

The first term vanishes, and we integrate by parts again:

$$\begin{aligned} c_n &= \frac{1}{2in\pi}\left(\left[\frac{-2x}{in}e^{-inx}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{-2}{in}e^{-inx}dx\right) \\ &= \frac{1}{2in\pi}\left(\frac{-2\pi}{in}e^{-in\pi} - \frac{2\pi}{in}e^{in\pi} + \left[\left(\frac{2}{in}\left(\frac{-1}{in}\right)e^{-inx}\right)\right]_{-\pi}^{\pi}\right) \\ &= \frac{1}{2in\pi}\left(\frac{-2\pi}{in}(e^{-in\pi} + e^{in\pi}) + \frac{2}{n^2}(e^{-in\pi} - e^{in\pi})\right). \end{aligned}$$

Thus

$$c_n = \frac{2(-1)^n}{n^2}.$$

We combine both results to obtain the Fourier series

$$\begin{aligned}
f(x) &\sim \frac{\pi^2}{3} + \sum_{n\neq 0} \frac{2(-1)^n}{n^2} e^{inx} \\
&\sim \frac{\pi^2}{3} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} e^{inx} + \frac{(-1)^{-n}}{(-n)^2} e^{-inx} \\
&\sim \frac{\pi^2}{3} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (e^{inx} + e^{-inx}) \\
&\sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos(nx)}{n^2}.
\end{aligned}$$

The series is an absolutely convergent series, so the Fourier series is absolutely convergent and converges pointwise to f. Thus

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos(nx)}{n^2}.$$

We can use Fourier series to evaluate familiar infinite sums. For example, setting $x = 0$ yields

$$f(0) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = 0,$$

and we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

3.3 Trigonometric Fourier Series

There are two approaches to Fourier series. One involves complex exponential functions, while the other involves the trigonometric functions cosine and sine.

Definition 3.17. An infinite sum of sines and cosines that forms an orthogonal system over $[-\pi, \pi]$, the Fourier series of a function f is

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)].$$

One may switch from complex to trigonometric Fourier series using

$$a_n = c_n + c_{-n} \quad \text{and} \quad b_n = i(c_n - c_{-n}), \quad n = 0, 1, \ldots$$

and vice versa using

$$c_n = \frac{a_n - ib_n}{2} \quad \text{and} \quad c_{-n} = \frac{a_n + ib_n}{2}, \quad n = 0, 1, \ldots.$$

Example 3.18. Let $f(x) = x$ on $[-\pi, \pi]$. The advantage here is that $f(x)$ is an odd function, so only sine coefficients are non-zero. For $n = 0$:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x dx = 0,$$

and for $n \neq 0$, we have an odd function with an even function:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx) dx = 0.$$

Thus, the coefficients a_n vanish. Next, we determine b_n for $n = 1, 2, \ldots,$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\sin(nx)dx \\
&= \frac{2}{\pi}\left(\left[\frac{-x\cos(nx)}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{-\cos(nx)}{n}dx\right) \\
&= \frac{2}{\pi}\left(\frac{-\pi\cos(\pi n)}{n} + \left[\frac{\sin(nx)}{n^2}\right]_0^{\pi}\right) \\
&= \frac{2}{n}(-1)^{n+1}.
\end{aligned}$$

Hence, the Fourier series for $f(x) = x$ on $[-\pi, \pi]$ is

$$\begin{aligned}
x &\sim 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx) \\
&\sim 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \frac{\sin(4x)}{4} + \ldots\right)
\end{aligned}$$

where $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is conditionally convergent by the alternating series test.

There are cases where only half of the interval is taken into consideration.

Definition 3.19. Suppose we have a function f over $[0, \pi]$. We have a half range Fourier series, and it can be written as either a cosine series or sine series. If evaluated as a Fourier cosine series,

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx)$$

with coefficients

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx.$$

If evaluated as a Fourier sine series,

$$f = \sum_{n=1}^{\infty} b_n \sin(nx)$$

with coefficients

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx.$$

CHAPTER 4: Fourier Transform

The Fourier transform can be used to analyze solutions of partial differential equations supported on the real line or on $\mathbb{R}^n$. The Fourier transform converts differentiation into multiplication and vice-versa. The general definitions and theorems of this chapter can be found in [12], [13], [17], [18], and [20].

4.1 Fourier Transform

Definition 4.1. Given $f \in L^1(\mathbb{R})$, the space consisting of absolute integrable functions, define the Fourier transform

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x} dx = \mathcal{F}[f(x)]$$

where $\mathcal{F}$ is denoted as the Fourier transform operator.

The Fourier transform acts linearly on function spaces

$$\mathcal{F}[f(x) + g(x)] = \mathcal{F}[f(x)] + \mathcal{F}[g(x)] = \hat{f}(\xi) + \hat{g}(\xi)$$

and

$$\mathcal{F}[cf(x)] = c\mathcal{F}[f(x)] = c\hat{f}(\xi).$$

For a function $f \in L^1(\mathbb{R})$, the Fourier transform $\hat{f}$ is defined for all ξ and is bounded,

$$|\hat{f}(\xi)| \leq \frac{1}{\sqrt{2\pi}} ||f||_1.$$

Lemma 4.2. *If $f \in C_c^\infty(\mathbb{R})$, then $\hat{f}$ is in $L^1 \cap L^2$.*

Proof.

$$\int_{\mathbb{R}} |\hat{f}(\xi)| d\xi = \int_{\mathbb{R}} \left| \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x} dx \right| d\xi.$$

We take the second derivative of the exponential with respect to x

$$f(x)\partial_x^2(e^{-i\xi x}) = -f(x)\xi^2 e^{-i\xi x}$$

and have

$$f(x)e^{-i\xi x} = -\frac{1}{\xi^2} f(x)\partial_x^2(e^{-i\xi x}).$$

There is a problem as ξ approaches to zero. We define $\mathbb{R} = I \cup \tilde{I}$ where $I = [-1, 1]$ and have

$$\int_{I=[-1,1]} |\hat{f}(\xi)| d\xi \leq 2||f||_1.$$

For $\tilde{I}$, we substitute in for $f(x)e^{-i\xi x}$ to get

$$\int_{\tilde{I}} |\hat{f}(\xi)| d\xi = \int_{\tilde{I}} \frac{1}{\xi^2} \left| \int_{\mathbb{R}} f(x)\partial_x^2(e^{-i\xi x}) dx \right| d\xi.$$

On the right hand side, the integral with respect to x is bounded

$$\left| \int_{\mathbb{R}} f(x)\partial_x^2(e^{-inx}) dx \right| = \left| \int_{\mathbb{R}} f''(x)e^{-inx} dx \right| \leq ||f''||_1.$$

Thus,

$$\int_{\tilde{I}} |\hat{f}(\xi)| d\xi \leq ||f''||_1 \int_{\tilde{I}} \frac{1}{\xi^2} d\xi < \infty.$$

Likewise, we do the same for L^2. □

4.2 Plancherel's Theorem

Our goal is to prove that $\mathcal{F}$ can be extended to an isometry from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$.

Theorem 4.3. *(Plancherel's Theorem) If $f \in C_c^\infty(\mathbb{R})$ then $\hat{f} \in L^2$ and*

$$||\hat{f}||_2^2 = ||f||_2^2.$$

First, we introduce the Gaussian function. For the Gaussian function

$$\hat{g}_t(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t\xi^2}{2}},$$

its Fourier transform is

$$\hat{\hat{g}}_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

(see remark 4.13 below). As $t \to 0^+$, $e^{-\frac{t\xi^2}{2}} \to 1$ pointwise and $\hat{\hat{g}}_t \to \delta$ weakly, in the sense that

$$\lim_{t \to 0^+} \int_{\mathbb{R}} f(x)\hat{\hat{g}}_t(x)dx = f(0).$$

We prove Plancherel's theorem.

Proof. For $f \in C_c^\infty(\mathbb{R})$, the Fourier transform $\hat{f}$ is bounded and

$$\begin{aligned} ||\hat{f}||_2^2 &= \lim_{t \to 0^+} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \left(\sqrt{2\pi}\hat{g}_t(\xi)\right) d\xi \\ &= \lim_{t \to 0^+} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 e^{-\frac{t\xi^2}{2}} d\xi \end{aligned}$$

is defined, since $\hat{f}$ is bounded and $e^{-\frac{t\xi^2}{2}}$ is integrable by the Dominated Convergence Theorem. Then,

$$\begin{aligned} ||\hat{f}||_2^2 &= \lim_{t \to 0^+} \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{f}(\xi)} e^{-\frac{t\xi^2}{2}} d\xi \\ &= \lim_{t \to 0^+} \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-ix\xi}dx\right) \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \overline{f(y)}e^{iy\xi}dy\right) e^{-\frac{t\xi^2}{2}} d\xi. \end{aligned}$$

By Fubini's Theorem,

$$||\hat{f}||_2^2 = \lim_{t\to 0^+} \int_{\mathbb{R}} f(x) \int_{\mathbb{R}} \overline{f(y)} \int_{\mathbb{R}} \frac{1}{2\pi} e^{-\frac{t\xi^2}{2}} e^{-i\xi(x-y)} d\xi dy dx.$$

The integral with respect to ξ is the Fourier transform of the Gaussian function $\hat{g}_t$. Thus,

$$||\hat{f}||_2^2 = \lim_{t\to 0^+} \int_{\mathbb{R}} f(x) \int_{\mathbb{R}} \overline{f(y)} \hat{\hat{g}}_t(y-x) dy dx = \lim_{t\to 0^+} \int_{\mathbb{R}} \overline{f(y)} \int_{\mathbb{R}} f(x) \hat{\hat{g}}_t(y-x) dx dy$$

and

$$\int_{\mathbb{R}} f(x) \hat{\hat{g}}_t(y-x) dx = (f * \hat{\hat{g}}_t)(y).$$

As $t \to 0^+$, $\hat{\hat{g}}_t \to \delta$ and

$$(f * \hat{\hat{g}}_t)(y) \to f(y).$$

Hence,

$$||\hat{f}||_2^2 = \int_{\mathbb{R}} \overline{f(y)} f(y) dy = ||f||_2^2$$

as desired. □

Theorem 4.4. *The Fourier transform can be extended to an isometry from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$.*

Proof. Given $f \in L^2$, there exists a sequence $\{\varphi_n\} \in C_c^\infty(\mathbb{R})$ such that

$$\varphi_n \to f \text{ in } L^2,$$

in the sense that

$$||\varphi_n - f||_2 \to 0.$$

Then $\{\varphi_n\}$ is a Cauchy sequence in L^2. Let $\varepsilon > 0$ be given. There exists N such that, for $m, n > N$,

$$||\varphi_n - \varphi_m||_2^2 < \varepsilon.$$

Applying the Fourier transform,

$$||\hat{\varphi_n} - \hat{\varphi_m}||_2^2 = ||\varphi_n - \varphi_m||_2^2,$$

we see that $\{\hat{\varphi_n}\}$ is a Cauchy sequence. Thus $\{\hat{\varphi_n}\}$ converges in L^2, and we define

$$\hat{f} = \lim_{n\to\infty} \hat{\varphi_n}.$$

This defines the Fourier transform on $L^2(\mathbb{R})$, and it is an isometry. That is,

$$||\hat{f}||_2 = \lim_{n\to\infty} ||\hat{\varphi_n}||_2 = \lim_{n\to\infty} ||\varphi_n||_2 = ||f||_2.$$

Note: $\lim \hat{\varphi}_n$ is independent of the choice of sequence $\varphi_n \to f$ in $L^2(\mathbb{R})$. $\square$

4.3 Fourier Inversion Formula

Definition 4.5. Define

$$\tilde{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{i\xi x} d\xi$$

for $f \in L^1(\mathbb{R})$.

Theorem 4.6. *Suppose we have a function $f \in L^1 \cap L^2$, and its Fourier transform $\hat{f} \in L^2(\mathbb{R})$. Then*

$$\tilde{\hat{f}} = f(x).$$

An issue arises when proving this theorem. For instance,

$$\begin{aligned}\tilde{\hat{f}}(x) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{f}(\xi)e^{i\xi x}d\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(y)e^{-i\xi y}dy\right)e^{i\xi x}d\xi \\ &\neq \frac{1}{2\pi}\int_{\mathbb{R}} f(y)\int_{\mathbb{R}} e^{i\xi(x-y)}d\xi dy,\end{aligned}$$

since the integral with respect to ξ is not finite. Hence, we use the Gaussian function

$$g_t(x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$$

with the following properties:

1. $\int_{\mathbb{R}} g_t = 1$ for all t

2. $g_t \to \delta$ as $t \to 0^+$ in the sense that

$$\lim_{t\to 0^+}\int_{\mathbb{R}} f(x)g_t(x) = f(0).$$

Its Fourier transform is

$$\hat{g}_t(\xi) = \frac{1}{\sqrt{2\pi}}e^{-\frac{t\xi^2}{2}}$$

(See Section 4.5 for details.) As $t \to 0^+$,

$$\sqrt{2\pi}\hat{g}_t(\xi) = e^{-\frac{t\xi^2}{2}} \to 1 \tag{4.1}$$

which we use to prove Theorem 4.6.

Proof. Let $f \in L^1 \cap L^2$, with the Fourier transform

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(y)e^{-i\xi y}dy.$$

We take the inverse Fourier transform

$$\tilde{\hat{f}}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{i\xi x} d\xi$$

and apply (4.1)

$$\tilde{\hat{f}}(x) = \lim_{t \to 0^+} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{i\xi x} e^{-\frac{t\xi^2}{2}} d\xi.$$

We substitute for $\hat{f}$,

$$\tilde{\hat{f}}(x) = \lim_{t \to 0^+} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) e^{-i\xi y} dy \right) e^{i\xi x} e^{-\frac{t\xi^2}{2}} d\xi$$

and by Fubini,

$$\tilde{\hat{f}}(x) = \lim_{t \to 0^+} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi(x-y)} e^{-\frac{t\xi^2}{2}} d\xi \right) dy.$$

We substitute for (4.1),

$$\tilde{\hat{f}}(x) = \lim_{t \to 0^+} \int_{\mathbb{R}} f(y) \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{g}_t(\xi) e^{i\xi(x-y)} d\xi \right) dy.$$

The integral with respect to ξ is the inverse Fourier transform of $\hat{g}_t$,

$$\tilde{\hat{f}}(x) = \lim_{t \to 0^+} \int_{\mathbb{R}} f(y) g_t(x-y) dy.$$

We now have a convolution and evaluating the limit yields

$$\tilde{\hat{f}}(x) = \lim_{t \to 0^+} (f * g_t)(x) = f(x)$$

as desired. $\square$

As before, we have

$$||\tilde{f}||_2 = ||f||_2$$

for $f \in L^1 \cap L^2$, so $\mathcal{F}^{-1}$ extends to an isometry on $L^2(\mathbb{R})$.

4.4 Properties of the Fourier Transform

Proposition 4.7. *Suppose the Fourier transforms of $f, g \in L^1 \cap L^2$ are $\hat{f}, \hat{g} \in L^2(\mathbb{R})$ respectively. Then the Fourier transform of $(f * g)(x)$ is the product $\sqrt{2\pi}\hat{f}(\xi)\hat{g}(\xi)$.*

Proof. Let

$$h(x) = (f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)dy.$$

Then,

$$\begin{aligned}
\hat{h}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (f * g)(x) e^{-i\xi x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x-y)g(y)dy \right] e^{-i\xi x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(y) \int_{\mathbb{R}} f(x-y) e^{-i\xi x} dx dy.
\end{aligned}$$

Let $z = x - y$ and apply the change of variable $dx = dz$,

$$\begin{aligned}
\hat{h}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(y) \left[\int_{\mathbb{R}} f(z) e^{-i\xi(z+y)} dz \right] dy \\
&= \left[\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(z) e^{-i\xi z} dz \right] \left[\frac{\sqrt{2\pi}}{\sqrt{2\pi}} \int_{\mathbb{R}} g(y) e^{-i\xi y} dy \right] \\
&= \sqrt{2\pi}\hat{f}(\xi)\hat{g}(\xi).
\end{aligned}$$

Thus $\hat{h}(\xi) = \sqrt{2\pi}\hat{f}(\xi)\hat{g}(\xi)$ as desired. □

We also note that the Fourier transform of the product $j(x) = f(x)g(x)$ is the convolution of the Fourier transform $\hat{f}(\xi)$ and $\hat{g}(\xi)$

$$\hat{j}(\xi) = \frac{1}{\sqrt{2\pi}}(\hat{f} * \hat{g})(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi - y)\hat{g}(y)dy.$$

Lemma 4.8. *(Symmetry Principle) Suppose that $f \in L^2(\mathbb{R})$. Then the Fourier transform of $\hat{f}(\xi)$ is $f(-x)$.*

Proof. Suppose we have a function $f \in L^2(\mathbb{R})$ and its Fourier transform $\hat{f} \in L^1 \cap L^2$. The Fourier transform of $\hat{f}$ is

$$\mathcal{F}[\hat{f}(\xi)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi)e^{-i\xi x}d\xi.$$

Letting $u = -x$,

$$\mathcal{F}[\hat{f}(\xi)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi)e^{i\xi u}d\xi,$$

we have the inverse Fourier transform of $\hat{f}$. Thus,

$$\mathcal{F}[\hat{f}(\xi)] = f(u).$$

Hence

$$\mathcal{F}[\hat{f}(\xi)] = f(-x)$$

as desired. □

Lemma 4.9. *The inverse Fourier transform $f \in C_c^\infty(\mathbb{R})$ is defined as*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi)e^{i\xi x}d\xi.$$

Its derivative is

$$f'(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} i\xi \hat{f}(\xi) e^{i\xi x} d\xi.$$

This implies the Fourier transform of the derivative of f is

$$\mathcal{F}[f'(x)] = i\xi \hat{f}(\xi).$$

Proof. Let $f \in L^2(\mathbb{R})$ be differentiable. Its derivative has the Fourier transform

$$\mathcal{F}[f'(x)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f'(x) e^{-ix\xi} dx.$$

Integration by parts yields

$$\mathcal{F}[f'(x)] = \frac{1}{\sqrt{2\pi}} \left(\left[f(x) e^{-ix\xi} \right]_{-\infty}^{\infty} - \int_{\mathbb{R}} f(x) e^{-ix\xi} (-i\xi) dx \right).$$

The first two terms vanish, and we have

$$\mathcal{F}[f'(x)] = \frac{i\xi}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} dx.$$

The right hand side is the Fourier transform of $f(x)$. Thus,

$$\mathcal{F}[f'(x)] = i\xi \hat{f}(\xi)$$

as desired. □

It follows that

$$\mathcal{F}[f^n(x)] = (i\xi)^n \hat{f}(\xi).$$

Lemma 4.10. $\langle \hat{g}, f \rangle = \langle g, \tilde{f} \rangle$ *for* $f, g \in L^1 \cap L^2$.

Proof.

$$
\begin{aligned}
\langle \hat{g}, f \rangle &= \int_{\mathbb{R}} \hat{g}(y)\overline{f(y)}dy \\
&= \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(\xi)e^{-i\xi y} \right) \overline{f(y)}d\xi dy \\
&= \int_{\mathbb{R}} g(\xi) \left(\frac{1}{2\pi} \int_{\mathbb{R}} \overline{f(y)}e^{-i\xi y}dy \right) d\xi \\
&= \int_{\mathbb{R}} g(\xi)\overline{\tilde{f}(\xi)}d\xi \\
&= \langle g, \tilde{f} \rangle
\end{aligned}
$$

□

Theorem 4.11. *Suppose we have $f, g \in L^1 \cap L^2$. Then*

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$$

That is

$$\int_{\mathbb{R}} f(x)\overline{g(x)}dx = \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi.$$

Proof. For

$$\langle \hat{f}, \hat{g} \rangle = \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi,$$

we substitute for $\hat{f}$ to get

$$\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi = \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-ix\xi} \right) \overline{\hat{g}(\xi)}dxd\xi.$$

By Fubini's Theorem,

$$\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi = \int_{\mathbb{R}} f(x) \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \overline{\hat{g}(\xi)e^{ix\xi}}d\xi \right) dx.$$

The integral with respect to ξ is the inverse Fourier transform of $\overline{\hat{g}}$. Thus,

$$\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi = \int_{\mathbb{R}} f(x)\overline{g(x)}.$$

Hence,

$$\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle$$

as desired. □

4.5 Gaussian Function

Lemma 4.12. *The Gaussian function for $t > 0$*

$$g_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

has the Fourier transform

$$\hat{g}_t(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t\xi^2}{2}}.$$

Proof. Suppose we have the Gaussian function, $g_t \in L^2$. Its Fourier transform is

$$\hat{g}_t(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} e^{-i\xi x} dx.$$

We apply the change of variable

$$y = \frac{x}{\sqrt{t}} \quad \text{and} \quad \sqrt{t}dy = dx$$

to get

$$\hat{g}_t(\xi) = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\sqrt{2\pi}} \right) \int_{\mathbb{R}} e^{-\frac{y^2}{2} - i\sqrt{t}\xi y} dy.$$

Before we continue, we complete the square for the power of the exponential,

$$\begin{aligned}
-\frac{y^2}{2} - i\sqrt{t}\xi y &= -\frac{1}{2}(y^2 + 2i\sqrt{t}\xi y) \\
&= -\frac{1}{2}(y^2 + 2i\sqrt{t}\xi y - t\xi^2) - \frac{1}{2}t\xi^2 \\
&= -\frac{1}{2}(y + i\sqrt{t}\xi)^2 - \frac{1}{2}t\xi^2.
\end{aligned}$$

Now

$$\hat{g}_t(\xi) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{2\pi}}\right)e^{-\frac{t\xi^2}{2}}\int_{\mathbb{R}} e^{-\frac{1}{2}(y+i\sqrt{t}\xi)^2}dy.$$

With the use of a contour integral, the integral with respect to y,

$$\int_{\mathbb{R}} e^{-\frac{1}{2}(y+i\sqrt{t}\xi)^2}dy = \int_{\mathbb{R}} e^{-\frac{y^2}{2}}dy$$

(The proof follows in Example 4.14). The Gaussian Fourier transform $\hat{g}_t$ is now

$$\hat{g}_t(\xi) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{2\pi}}\right)e^{-\frac{t\xi^2}{2}}\int_{\mathbb{R}} e^{-\frac{y^2}{2}}dy.$$

Focusing on the integral, let

$$I = \int_{\mathbb{R}} e^{-\frac{y^2}{2}}dy.$$

We square both sides to get

$$\begin{aligned}
I^2 &= \int_{\mathbb{R}} e^{-\frac{y^2}{2}}dy\int_{\mathbb{R}} e^{-\frac{x^2}{2}}dx \\
&= \iint_{\mathbb{R}} e^{-\frac{1}{2}(x^2+y^2)}dydx.
\end{aligned}$$

Changing to polar coordinates yields

$$\begin{aligned} I^2 &= \int_0^{2\pi} \int_0^{\infty} e^{-\frac{r^2}{2}} r dr d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\infty} r e^{-\frac{r^2}{2}} dr \\ &= 2\pi \left[-e^{\frac{r^2}{2}} \right]_0^{\infty} \\ &= 2\pi. \end{aligned}$$

Thus $I = \sqrt{2\pi}$ and

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t\xi^2}{2}}$$

as desired. $\square$

Remark 4.13. The Gaussian function

$$g_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

has the Fourier transform

$$\hat{g}_t(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t\xi^2}{2}}.$$

By the symmetry principle, the Fourier transform of $\hat{g}_t$ is

$$\mathcal{F}[\hat{g}_t(\xi)] = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(-x)^2}{2t}} = g_t(-x) = g_t(x).$$

Example 4.14. Suppose we have the contour integral

$$\int_{C^R} e^{-\frac{z^2}{2}} dz$$

where C^R is the boundary of the rectangle $[-R, R] \times [0, \alpha]$ oriented counterclockwise.

The function

$$f(z) = e^{-\frac{z^2}{2}}$$

is analytic with no singularities inside the interior of the contour C^R. Thus, C^R is a simple closed contour and

$$\int_{C^R} f(z)dz = 0$$

by the Cauchy-Goursat Theorem. Since the contour is rectangular, we have $C^R = C_1^R + C_2^R - C_3^R - C_4^R$ where C_n^R, $n = 1, 2, 3, 4$, are the legs of C^R. This gives

$$\int_{C^R} f(z)dz = \int_{C_1^R} f(z)dz + \int_{C_2^R} f(z)dz - \int_{C_3^R} f(z)dz - \int_{C_4^R} f(z)dz = 0.$$

Parameterizing C_1^R as $z(w) = w$ with $-R \le w \le R$ gives

$$\int_{C_1^R} f(z)dz = \int_{-R}^{R} e^{-\frac{w^2}{2}} dw.$$

Parameterizing C_3^R as $z(w) = w + i\alpha$ with $-R \le w \le R$ gives

$$\int_{C_3^R} f(z)dz = \int_{-R}^{R} e^{-\frac{(w+i\alpha)^2}{2}} dw.$$

This gives

$$\int_{-R}^{R} e^{-\frac{w^2}{2}} dw + \int_{C_2^R} f(z)dz - \int_{-R}^{R} e^{-\frac{(w+i\alpha)}{2}} dw - \int_{C_4^R} f(z)dz = 0$$

or

$$\int_{-R}^{R} e^{-\frac{w^2}{2}} dw = \int_{-R}^{R} e^{-\frac{(w+i\alpha)^2}{2}} dw - \int_{C_2^R} f(z)dz + \int_{C_4^R} f(z)dz.$$

For points z on C_2^R, we have $z = R + iw$ with $0 \leq w \leq \alpha$ and

$$-\frac{z^2}{2} = -\frac{(R+iw)^2}{2} = -\frac{R^2}{2} - iwR + \frac{w^2}{2}.$$

So for all points $z = R + iw$ on C_2^R, we have

$$\begin{aligned} |f(z)| &= \left| e^{-\frac{z^2}{2}} \right| \\ &= \left| e^{-\frac{R^2}{2} - iwR + \frac{w^2}{2}} \right| \\ &= e^{-\frac{R^2}{2}} e^{\frac{w^2}{2}} |e^{-iwR}| \end{aligned}$$

The modulus of e^{-iwR} is 1, so

$$|f(z)| = e^{-\frac{R^2}{2}} e^{\frac{w^2}{2}} \leq e^{-\frac{R^2}{2}} e^{\frac{\alpha^2}{2}}$$

as $0 \leq w \leq \alpha$. This gives the estimate,

$$\left| \int_{C_2^R} f(z) dz \right| \leq e^{-\frac{R^2}{2}} e^{\frac{\alpha^2}{2}} \times \text{length}(C_2^R) = \alpha e^{\frac{\alpha^2}{2}} e^{-\frac{R^2}{2}}.$$

Likewise, for points z on C_4^R, we have $z = -R + iw$ with $0 \leq w \leq \alpha$ and

$$-\frac{z^2}{2} = -\frac{(-R+iw)^2}{2} = -\frac{R^2}{2} + iwR + \frac{w^2}{2}.$$

So for all points $z = -R + iw$ on C_4^R, we have

$$|f(z)| = e^{-\frac{R^2}{2}} e^{\frac{w^2}{2}} |e^{iwR}| \leq e^{-\frac{R^2}{2}} e^{\frac{\alpha^2}{2}}.$$

This gives the estimate,

$$\left|\int_{C_4^R} f(z)dz\right| \leq e^{-\frac{R^2}{2}} e^{\frac{\alpha^2}{2}} \times \text{length}(C_4^R) = \alpha e^{\frac{\alpha^2}{2}} e^{-\frac{R^2}{2}}.$$

As

$$\left|\int_{C_2^R} f(z)dz\right| \leq \alpha e^{\frac{\alpha^2}{2}} e^{-\frac{R^2}{2}}$$

and

$$\lim_{R\to\infty} e^{-\frac{R^2}{2}} = 0,$$

we have

$$\lim_{R\to\infty} \int_{C_2^R} f(z)dz = 0.$$

Similarly,

$$\lim_{R\to\infty} \int_{C_4^R} f(z)dz = 0.$$

Thus, taking the limit of both side as $R \to \infty$,

$$\begin{aligned}
\lim_{R\to\infty} \int_{-R}^{R} e^{-\frac{w^2}{2}} dw &= \lim_{R\to\infty} \int_{-R}^{R} e^{-\frac{(w+i\alpha)^2}{2}} dw - \lim_{R\to\infty} \int_{C_2^R} f(z)dz + \lim_{R\to\infty} \int_{C_4^R} f(z)dz \\
&= \lim_{R\to\infty} \int_{-R}^{R} e^{-\frac{(w+i\alpha)^2}{2}} dw - 0 + 0 \\
&= \lim_{R\to\infty} \int_{-R}^{R} e^{-\frac{(w+i\alpha)^2}{2}} dw
\end{aligned}$$

Hence,

$$\int_{\mathbb{R}} e^{-\frac{w^2}{2}} dw = \int_{\mathbb{R}} e^{-\frac{(w+i\alpha)^2}{2}} dw.$$

CHAPTER 5: Heat Equation

The first of the three partial differential equations discussed is the heat equation in one dimension. The results in this chapter have been adapted from [5], [9], [15], and [19].

5.1 Initial Value Problem for the Heat Equation

Definition 5.1. The heat equation is a parabolic partial differential equation primarily focused on the spreading of heat in an object or in space. Its formula,

$$u_t = c^2 u_{xx},$$

derives from three physical principles: conservation law, physical considerations, and Fourier heat flow. The function u is the temperature at position x at time t. The constant, c^2, is equal to $\frac{\kappa}{\rho\sigma}$ where κ is the thermal conductivity, ρ is density, and σ is the specific heat.

When solving an initial value problem for the heat equation in one spatial dimension, three conditions must be met:

1) $u(x,t)$ satisfies the heat equation $u_t = c^2 u_{xx}$ with $0 \leq x \leq L$ and $t \geq 0$

2) $u(x,t)$ satisfies the boundary condition $u(0,t) = u(L,t) = 0$

3) $u(x,t)$ satisfies the initial temperature $u(x,0) = h(x)$.

5.2 Separation of Variables

One solution method to the heat initial value problem is by separation of variables and Fourier series. Assume that a solution to the partial differential equation is of the form

$$u(x,t) = f(x)g(t). \tag{5.1}$$

We find the solution by substituting (5.1) into the heat equation, giving

$$f(x)g'(t) = c^2 f''(x)g(t)$$

or

$$\frac{g'(t)}{c^2 g(t)} = \frac{f''(x)}{f(x)}.$$

Since the left hand side does not depend on x, and the right hand side does not depend on t, both sides are equal to a constant, λ. We separate both sides into ordinary differential equations,

$$g'(t) = \lambda c^2 g(t) \quad \text{and} \quad f''(x) = \lambda f(x).$$

Applying the boundary condition to (5.1) yields

$$f(0)g(t) = f(L)g(t) = 0.$$

Assuming $g \not\equiv 0$, we have

$$f(0) = f(L) = 0.$$

(If $g \equiv 0$, the solution would be trivial, i.e., $u(x,t) \equiv 0$.) Now, we consider three cases, one of which yields a nontrivial solution.

1. Case 1: $\lambda = 0$. Here, we have

$$f''(x) = 0.$$

The general solution to this ordinary differential equation is

$$f(x) = ax + b, \quad f(0) = f(L) = 0,$$

hence $f = 0$, a trivial solution.

2. Case 2: $\lambda > 0$. Here, we have

$$f''(x) - \lambda f(x) = 0.$$

The general solution is

$$f(x) = c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x}.$$

Applying the boundary condition yields

$$f(0) = c_1 + c_2 = 0.$$

Thus $c_1 = -c_2$, and

$$\begin{aligned} f(L) &= c_1 e^{\sqrt{\lambda} L} + c_2 e^{-\sqrt{\lambda} L} \\ &= c_1 e^{\sqrt{\lambda} L} - c_1 e^{-\sqrt{\lambda} L} \\ &= c_1 \left(e^{\sqrt{\lambda} L} - e^{-\sqrt{\lambda} L} \right) \\ &= 0. \end{aligned}$$

Thus $c_1 = 0$ which means $c_2 = 0$. Hence $f = 0$ which is a trivial solution.

3. Case 3: $\lambda < 0$. Here, we have

$$f''(x) + \omega^2 f(x) = 0 \quad \text{with } \omega = \sqrt{-\lambda}.$$

The general solution corresponds to harmonic motion

$$f(x) = a\cos(\omega x) + b\sin(\omega x).$$

Applying the boundary condition yields $f(0) = 0$ which implies $a = 0$. Thus $f(x) = b\sin(\omega x)$, and $f(L) = 0$ yields $b\sin(\omega L) = 0$. Thus, either $b = 0$ or $\sin(\omega L) = 0$. If $b = 0$, we have a trivial solution. If $\sin(\omega L) = 0$, then $\omega L = n\pi$ or $\omega = \frac{n\pi}{L}$ where $n \in \mathbb{Z}$. We now have

$$f(x) = b\sin\left(\frac{n\pi x}{L}\right).$$

Thus, the only nontrivial solutions for f are

$$f(x) = \sin\left(\frac{n\pi x}{L}\right) \quad \text{with } \lambda = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$$

Now that we have the nontrivial solutions for f, we find the corresponding g satisfying

$$g'(t) = \lambda c^2 g(t) = -\left(\frac{n\pi}{L}\right)^2 c^2 g(t).$$

The general solution is

$$g(t) = be^{-(\frac{nc\pi}{L})^2 t}.$$

Hence, the nontrivial product solutions to the heat equation with the given boundary conditions are

$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right) e^{-(\frac{nc\pi}{L})^2 t}.$$

By the principal of superposition, we get

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{nc\pi}{L}\right)^2 t} \tag{5.2}$$

as a solution to the heat equation.

We now determine the coefficients b_n by applying the initial temperature to (5.2):

$$h(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

Using half range Fourier series, we identify the b_n's by evaluating $h(x)$ as a Fourier sine series. That is,

$$b_n = \frac{2}{L}\int_0^L h(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

For example, consider the initial-value problem:

1) $u_t = u_{xx}$

2) $u(0,t) = u(\pi,t) = 0$

3) $u(x,0) = h(x) = x(\pi - x)$

for $0 \leq x \leq \pi$ and $t \geq 0$. Taking $L = \pi$ and $c = 1$ in (5.2) gives

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 t}.$$

The initial temperature gives us

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx) = x(\pi - x).$$

We can identify the coefficients b_n by taking the Fourier sine series of $h(x) = x(\pi - x)$,

$$b_n = \frac{2}{\pi} \int_0^{\pi} x(\pi - x)\sin(nx)dx.$$

Integration by parts yields

$$b_n = \frac{2}{\pi}\left(\left[-\frac{(\pi x - x^2)\cos(nx)}{n}\right]_0^{\pi} - \int_0^{\pi} -\frac{(\pi - 2x)\cos(nx)}{n}dx\right).$$

The first two terms vanish, which leaves us with the integral,

$$b_n = \frac{2}{\pi} \int_0^{\pi} \frac{(\pi - 2x)\cos(nx)}{n}dx.$$

Integration by parts yields

$$b_n = \frac{2}{\pi}\left(\left[\frac{(\pi - 2x)\sin(nx)}{n^2}\right]_0^{\pi} - \int_0^{\pi} -\frac{2\sin(nx)}{n^2}dx\right).$$

The first two terms vanish which leaves us the integral

$$\begin{aligned} b_n &= \frac{4}{\pi}\int_0^{\pi} \frac{\sin(nx)}{n^2}dx \\ &= \frac{4}{\pi}\left[-\frac{\cos(nx)}{n^3}\right]_0^{\pi} \\ &= \frac{-4}{\pi n^3}(\cos(n\pi) - 1). \end{aligned}$$

Thus,

$$u(x, 0) = \sum_{n=1}^{\infty} \frac{-4((-1)^n - 1)}{\pi n^3} \sin(nx) = x(\pi - x).$$

Hence,

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} \frac{-4((-1)^n - 1)}{\pi n^3} \sin(nx) e^{-n^2 t} \\
&= \frac{8}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)x)}{(2k+1)^3} e^{-(2k+1)^2 t}.
\end{aligned}
$$

5.3 Heat Kernel

Another solution method to the one dimension heat equation is the use of the heat kernel, used in the case of an infinitely long bar, $x \in \mathbb{R}$. Thus, we have the following conditions:

1) $u(x,t)$ satisfies the heat equation $u_t = c^2 u_{xx}$ with $-\infty < x < \infty$ and $t \geq 0$
2) $u(x,t)$ satisfies the initial temperature $u(x,0) = h(x)$.

We derive the solution by examining a property of the heat equation, scale invariance. That is, if $u(x,t)$ is a solution, so is $u(\lambda x, \lambda^2 t)$ for $\lambda \in \mathbb{R}$. This scaling indicates the similarity variable $\frac{x}{\sqrt{t}}$, and the solution can be expressed as

$$
u(x,t) = v\left(\frac{x}{\sqrt{t}}\right) w(t).
$$

A property of the heat equation we also consider is the conversation of energy. Let u have the following properties:

$$
\left| \int_{\mathbb{R}} u(x,0) dx \right| < \infty
$$

and

$$
u_x(x,t) \to 0 \quad \text{as } x \to \pm\infty.
$$

Then

$$\frac{d}{dt}\int_{\mathbb{R}} u(x,t)dx = 0,$$

and thus "energy" is conserved. That is,

$$\int_{\mathbb{R}} u(x,t)dx = C \tag{5.3}$$

where $C \in \mathbb{R}$ is a constant. We substitute $u(x,t) = v\left(\frac{x}{\sqrt{t}}\right)w(t)$, to get

$$w(t)\int_{\mathbb{R}} v\left(\frac{x}{\sqrt{t}}\right) dx = C,$$

and apply the change of variable $\sqrt{t}y = x$ to obtain

$$w(t)\sqrt{t}\int_{\mathbb{R}} v(y)dy = C.$$

We take $w(t) = \frac{1}{\sqrt{t}}$ to conserve energy in terms of (5.3). Thus,

$$u(x,t) = \frac{1}{\sqrt{t}}v(y) = t^{-\frac{1}{2}}v\left(t^{-\frac{1}{2}}x\right).$$

Taking the derivative with respect to t yields

$$u_t(x,t) = -\frac{1}{2}t^{-\frac{3}{2}}v(y) - \frac{x}{2\sqrt{t}}t^{-\frac{3}{2}}v'(y).$$

Taking the derivatives with respect to x yields

$$u_x(x,t) = \frac{1}{t}v'(y) \quad \text{and} \quad u_{xx}(x,t) = t^{-\frac{3}{2}}v''(y).$$

Thus, the heat equation is of the form

$$-\frac{1}{2}t^{-\frac{3}{2}}v(y) - \frac{x}{2\sqrt{t}}t^{-\frac{3}{2}}v'(y) = c^2 t^{-\frac{3}{2}}v''(y)$$

or

$$c^2 v''(y) + \frac{1}{2}yv'(y) + \frac{1}{2}v(y) = 0.$$

We can rewrite it as

$$c^2 v''(y) + \frac{1}{2}(yv(y))' = 0$$

and take the integral of both sides to obtain

$$c^2 v'(y) + \frac{1}{2}yv(y) = C.$$

Set $C = 0$ to obtain the general solution

$$v(y) = be^{-\frac{y^2}{4c^2}}.$$

Converting back to $u(x,t)$, we get

$$u(x,t) = \frac{b}{\sqrt{t}}e^{-\frac{x^2}{4c^2 t}}.$$

We choose b so the constant in (5.3) is unity. As

$$\int_{\mathbb{R}} e^{-\frac{x^2}{4c^2 t}}dx = \sqrt{4c^2\pi t},$$

we have the fundamental solution of the heat equation

$$\Phi(x,t) = \frac{1}{\sqrt{4c^2\pi t}} e^{-\frac{x^2}{4c^2 t}} \quad \text{for } t > 0.$$

As $t \to 0$, $\Phi \to \delta$. We take the convolution of $\Phi(x,t)$ in the x variable with the function $u(x,0) = h(x)$ to get

$$u(x,t) = \frac{1}{\sqrt{4c^2\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4c^2 t}} h(y)dy.$$

So

$$\lim_{t\to 0} u(x,t) = (\delta * h)(x) = h(x),$$

and the heat kernel is

$$K(x,y,t) = \Phi(x-y,t) = \frac{1}{\sqrt{4c^2\pi t}} e^{-\frac{(x-y)^2}{4c^2 t}}.$$

Hence, for given initial temperature $h(x)$, the solution is

$$u(x,t) = \int_{\mathbb{R}} K(x,y,t)h(y)dy.$$

For example, consider the initial-value problem:

1) $u_t = c^2 u_{xx}$

2) $u(x,0) = e^{-x}$

for $-\infty < x < \infty$ and $t \geq 0$. We use the heat kernel to find $u(x,t)$. The solution is of the form

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4c^2\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4c^2t}} e^{-y} dy \\
&= \frac{1}{\sqrt{4c^2\pi t}} \int_{\mathbb{R}} e^{-\frac{(x^2-2xy+y^2+4c^2ty)}{4c^2t}} dy
\end{aligned}
$$

We compute the square with respect to y in the exponents and add/substract $4c^2tx$

$$
\begin{aligned}
y^2 + 4c^2ty + x^2 - 2xy &= (y^2 + 4c^2ty + 4c^4t^2 - 4c^2tx - 2xy + x^2) - 4c^4t^2 + 4c^2tx \\
&= (y + 2c^2t - x)^2 - 4c^4t^2 + 4c^2tx.
\end{aligned}
$$

Thus,

$$
u(x,t) = \frac{1}{\sqrt{4c^2\pi t}} \int_{\mathbb{R}} e^{-\frac{(y+2c^2t-x)^2}{4c^2t} + c^2t - x} dy.
$$

Set

$$
p = \frac{y + 2c^2t - x}{\sqrt{4c^2t}}
$$

and apply change of variable $\sqrt{4c^2t}\ dp = dy$ to the solution

$$
u(x,t) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-p^2} e^{c^2t-x} dp = e^{c^2t-x}.
$$

Applying the initial temperature to the solution yields

$$
u(x,0) = e^{-x}.
$$

Hence, the solution to the heat equation is

$$
u(x,t) = e^{c^2t-x}.
$$

CHAPTER 6: Wave Equation

The second of the three partial differential equations discussed is the wave equation in one dimension. The results are adapted from [1], [3], [5], [10], [15], [19], and [21].

6.1 Initial Value Problem for the Wave Equation

Definition 6.1. The wave equation is a hyperbolic partial differential equation primarily focused on the vibration of a finite string. Its formula,

$$u_{tt} = c^2 u_{xx}$$

derives from the application of Newton's Second Law to a medium, e.g. the vertical displacement of a string. The function u is the vertical displacement of a string at position x at time t. The constant, c^2, is equal to $\frac{T}{\rho}$ where T is tension and ρ is density.

When solving an initial value problem for the wave equation in one spatial dimension, four conditions must be met:

1) $u(x,t)$ satisfies the wave equation $u_{tt} = c^2 u_{xx}$ with $0 \leq x \leq L$ and $t \geq 0$

2) $u(x,t)$ satisfies the boundary conditions $u(0,t) = u(L,t) = 0$

3) $u(x,t)$ satisfies the initial position $u(x,0) = h(x)$

4) $u(x,t)$ satisfies the initial velocity $u_t(x,0) = j(x)$.

6.2 Separation of Variables

One solution method to the wave initial value problem is by separation of variables and Fourier series. A solution to the partial differential equation is of the form

$$u(x,t) = f(x)g(t). \tag{6.1}$$

Similar to the heat equation, we reduce to two ordinary differential equations:

$$g''(t) = \lambda c^2 g(t) \quad \text{and} \quad f''(x) = \lambda f(x).$$

Applying the boundary condition to (6.1) yields

$$f(0)g(t) = f(L)g(t) = 0$$

Assuming $g \not\equiv 0$, we have

$$f(0) = f(L) = 0.$$

As before, we are only looking for nontrivial solutions, so $g \not\equiv 0$, $\lambda \neq 0$, and $\lambda \ngtr 0$. We go to the results of Case 3: $\lambda < 0$ and find that the nontrivial solutions are

$$f(x) = \sin\left(\frac{n\pi x}{L}\right) \quad \text{with } \lambda = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$$

Now, we find the corresponding g satisfying,

$$g''(t) - \lambda c^2 g(t) = g''(t) + \left(\frac{n\pi}{L}\right)^2 c^2 g(t) = 0,$$

The general solution is

$$g(t) = a\cos\left(\frac{nc\pi t}{L}\right) + b\sin\left(\frac{nc\pi t}{L}\right).$$

Hence, the nontrivial product solutions to the wave equation with the boundary condition are

$$u_n(x,t) = \left[a_n\cos\left(\frac{nc\pi t}{L}\right) + b_n\sin\left(\frac{nc\pi t}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right).$$

By the principal of superposition, we get

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{nc\pi t}{L}\right) + b_n\sin\left(\frac{nc\pi t}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right). \tag{6.2}$$

as a solution to the wave equation.

We determine the coefficients a_n by applying the initial position to (6.2):

$$h(x) = u(x,0) = \sum_{n=1}^{\infty} a_n\sin\left(\frac{n\pi x}{L}\right).$$

Using half range Fourier series, we can identify the a_n's by evaluating $h(x)$ as a Fourier sine series. That is,

$$a_n = \frac{2}{L}\int_0^L h(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

To determine the coefficients b_n, we differentiate (6.2) with respect to t, obtaining

$$u_t(x,t) = \sum_{n=1}^{\infty}\left[-\frac{nc\pi}{L}a_n\sin\left(\frac{nc\pi t}{L}\right) + \frac{nc\pi}{L}b_n\cos\left(\frac{nc\pi t}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right) \tag{6.3}$$

and apply the initial velocity to (6.3):

$$j(x) = u_t(x,0) = \sum_{n=1}^{\infty} \frac{nc\pi}{L} b_n \sin\left(\frac{n\pi x}{L}\right).$$

We evaluate $j(x)$ as a Fourier sine series to identify the b_n's:

$$b_n = \frac{2}{nc\pi} \int_0^L j(x) \sin\left(\frac{n\pi x}{L}\right).$$

For example, consider the initial-value problem:

1) $u_{tt} = u_{xx}$

2) $u(0,t) = u(\pi,t) = 0$

3) $u(x,0) = h(x) = x(\pi - x)$

4) $u_t(x,0) = j(x) = x$

for $0 \leq x \leq \pi$ and $t \geq 0$. Taking $L = \pi$ and $c = 1$ in (6.2) yields solution

$$u(x,t) = \sum_{n=1}^{\infty} [a_n \cos(nt)\sin(nx) + b_n \sin(nt)\sin(nx)]$$

with derivative with respect to t

$$u_t(x,t) = \sum_{n=1}^{\infty} [-a_n n \sin(nt)\sin(nx) + b_n n \cos(nt)\sin(nx)].$$

The initial position gives us

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx) = x(\pi - x).$$

We can identify the coefficients a_n by using the Fourier sine series of $h(x) = x(\pi - x)$

from the heat equation's example,

$$a_n = \frac{-4}{\pi n^3}(\cos(n\pi) - 1).$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{-4((-1)^n - 1)}{\pi n^3}\cos(nt)\sin(nx) + b_n \sin(nt)\sin(nx)\right]$$

The initial velocity gives us,

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n n \sin(nx) = x.$$

We can identify the coefficients b_n by taking the Fourier sine series of $j(x) = x$

$$b_n = \frac{2}{n\pi}\int_0^{\pi} x\sin(nx)dx.$$

Integration by parts yields

$$b_n = \frac{2}{n\pi}\left(\left[\frac{-x\cos(nx)}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos(nx)}{n}dx\right)$$

The last two terms vanish which leaves

$$\begin{aligned} b_n &= \frac{2}{n\pi}\left(\frac{-\pi\cos(n\pi)}{n}\right) \\ &= \frac{-2(-1)^n}{n^2}. \end{aligned}$$

Hence,

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{-4((-1)^n - 1)}{\pi n^3} \cos(nt)\sin(nx) + \frac{-2(-1)^n}{n^2} \sin(nt)\sin(nx) \right].$$

6.3 Fourier Transform Method

When the domain is the real line, a solution to the one dimension wave equation, using the Fourier transform, is d'Alembert's formula. We have the following conditions:

1) $u(x,t)$ satisfies the wave equation $u_{tt} = c^2 u_{xx}$ with $-\infty < x < \infty$ and $t \geq 0$

2) $u(x,t)$ satisfies the initial position $u(x,0) = h(x)$

3) $u(x,t)$ satisfies the initial velocity $u_t(x,0) = j(x)$.

Fixing t, assuming a reasonable solution u, we take the Fourier transform of $u(x,t)$ with respect to x,

$$\hat{u}(\xi,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x,t) e^{-i\xi x} dx.$$

By Lemma 4.9, the Fourier transform of u_{xx} is $-\xi^2 \hat{u}(\xi,t)$. We compute the Fourier transform of u_{tt}:

$$\begin{aligned} \mathcal{F}[u_t] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u_t e^{-i\xi x} dx &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[\lim_{h\to 0} \frac{u(x,t+h) - u(x,t)}{h} \right] e^{-i\xi x} dx \\ &= \lim_{h\to 0} \frac{1}{h} \left[\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x,t+h) e^{-i\xi x} dx - \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x,t) e^{-i\xi x} dx \right] \\ &= \lim_{h\to 0} \frac{1}{h} (\hat{u}(\xi,t+h) - \hat{u}(\xi,t)) \\ &= \hat{u}_t, \end{aligned}$$

so,

$$\mathcal{F}[u_{tt}] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u_{tt} e^{-i\xi x} dx = \frac{\partial}{\partial t} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u_t e^{-i\xi x} dx = \hat{u}_{tt}.$$

The wave equation is of the form

$$\hat{u}_{tt} = -c^2\xi^2\hat{u}.$$

Fix ξ and write $\hat{u}(\xi,t) = U(t)$. The wave equation is now

$$U''(t) + c^2\xi^2 U(t) = 0$$

with the general solution

$$U(t) = a(\xi)e^{-i\xi ct} + b(\xi)e^{i\xi ct}$$

where a and b are functions of ξ. We set

$$\hat{f} = a \quad \text{and} \quad \hat{g} = b.$$

Thus,

$$\hat{u}(\xi,t) = \hat{f}e^{-i\xi ct} + \hat{g}e^{i\xi ct}.$$

We take the inverse Fourier transform to find $u(x,t)$

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{u}(\xi,t)e^{i\xi x}d\xi \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left(\hat{f}e^{-i\xi ct} + \hat{g}e^{i\xi ct}\right)e^{i\xi x}d\xi \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{f}e^{i\xi(x-ct)}d\xi + \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{g}e^{i\xi(x+ct)}d\xi \\
&= f(x-ct) + g(x+ct).
\end{aligned}$$

The solution to the wave equation is now of the form

$$u(x,t) = f(x-ct) + g(x+ct). \tag{6.4}$$

Applying the initial position to (6.4) yields

$$u(x,0) = f(x) + g(x) = h(x).$$

We differentiate $h(x)$ and multiply by the constant c to get

$$ch'(x) = cf'(x) + cg'(x).$$

Applying the initial velocity to (6.4) yields

$$u_t(x,0) = -cf'(x) + cg'(x) = j(x),$$

and we add and subtract this with $ch'(x)$ to get

$$ch'(x) + j(x) = cf'(x) + cg'(x) + [-cf'(x) + cg'(x)] = 2cg'(x)$$

and

$$ch'(x) - j(x) = cf'(x) + cg'(x) - [-cf'(x) + cg'(x)] = 2cf'(x).$$

Solving for f and g yields

$$f(x) = \frac{h(x)}{2} - \frac{1}{2c}\int_{x_\circ}^{x} j(y)dy \quad \text{and} \quad g(x) = \frac{h(x)}{2} + \frac{1}{2c}\int_{x_\circ}^{x} j(y)dy$$

for fixed $x_\circ$. Substituting both into (6.4) yields

$$u(x,t) = \frac{h(x-ct)}{2} - \frac{1}{2c}\int_{x_\circ}^{x-ct} j(y)dy + \frac{h(x+ct)}{2} + \frac{1}{2c}\int_{x_\circ}^{x+ct} j(y)dy.$$

Thus, the solution to the initial value problem is d'Alembert's formula:

$$u(x,t) = \frac{1}{2}[h(x-ct) + h(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} j(y)dy.$$

For example, consider the intial-value problem:

1) $u_{tt} = c^2 u_{xx}$

2) $u(x,0) = 0$

3) $u_t(x,0) = e^{-x}$

for $-\infty < x < \infty$ and $t \geq 0$. Taking $h(x) = 0$ and $j(x) = e^{-x}$ in d'Alembert's formula gives solution

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_{x-ct}^{x+ct} e^{-y}dy \\ &= \frac{1}{2c}\left(e^{-(x-ct)} - e^{-(x+ct)}\right) \\ &= \frac{e^{-x}\sinh(ct)}{2}. \end{aligned}$$

CHAPTER 7: Laplace Equation

The final partial differential equation we consider is the Laplace equation in $\mathbb{R}^2$. The general definitions and theorems discussed here can be found in [4], [5], [7], [8], [11], [14], [16], [17], and [20].

7.1 Laplace Equation

Definition 7.1. The Laplace equation is an elliptic partial differential equation primarily related to equilibrium equations in a variety of physical systems. Its formula, in Cartesian coordinates, is

$$\Delta u = u_{xx} + u_{yy} = 0$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator and u is a function. Its solutions are harmonic functions.

Definition 7.2. The inhomogeneous version of the Laplace equation is the Poisson equation which arises in theoretical physics. Its formula is

$$-\Delta u = f(x, y).$$

7.2 Properties of Harmonic Functions

Definition 7.3. A function is a harmonic function if $u \in C^2(\Omega)$ and satisfies the Laplace equation $\Delta u = 0$ in Ω.

Proposition 7.4. *(Mean Value Property) Let $u \in C^2$ be a harmonic function in an open domain $\Omega \subset \mathbb{R}^2$. Let $z \in \Omega$ and consider a disc $B_r(z) \subset \Omega$. Then the average value of u on the circle $S_r(z)$ is $u(z)$.*

Proof. By the Divergence Theorem,

$$\int_{B_r(z)} \Delta u = \int_{S_r(x)} \nabla u \cdot n = r \int_{S_1(x)} u'(z + r\omega)\ d\omega = r\frac{\partial}{\partial r} \int_{S_1(x)} u(z + r\omega)\ d\omega.$$

Since $\Delta u = 0$, we see that

$$0 = \frac{1}{2\pi r} \int_{B_r(z)} \Delta u = \frac{1}{2\pi} \frac{\partial}{\partial r} \int_{S_1(x)} u(z + r\omega)\ d\omega,$$

and hence

$$\frac{1}{2\pi} \int_{S_1(x)} u(z + r\omega)\ d\omega$$

is independent of r, and approaches $u(z)$ as $r \to 0$. □

Proposition 7.5. *Let $u \in C^2$ be a harmonic function in an open domain $\Omega \subset \mathbb{R}^2$. Then u is a smooth function in Ω.*

Proof. Fix $z \in \Omega$, and choose $\varepsilon > 0$ so that $B_\varepsilon(z) \subset \Omega$. Let ϕ be a smooth, radial function supported inside $|w| < \varepsilon$ with $\int_{B_\varepsilon(0)} \phi = 1$. Define ψ on $B_\varepsilon(z)$ by $\psi(z+r\omega) = \phi(r)$. Then we have the convolution

$$\begin{aligned}
(u * \psi)(z) &= \int u(z - w)\psi(w)\ dw \\
&= \int_{B_\varepsilon(z)} u(z - w)\psi(w)\ dw \\
&= \int_0^\varepsilon \int_{S_\varepsilon(0)} u(z - r\omega)\phi(r)r\ dwdr \\
&= \int_0^\varepsilon 2\pi r u(z)\phi(r)\ dr \\
&= u(z).
\end{aligned}$$

Now we have $u = (u * \psi)$ where ψ is a smooth function, hence u is smooth. □

Theorem 7.6. *Let $f(z) = u(z) + iv(z)$ and $z_\circ = x_\circ + iy_\circ$ be a point in the domain of f. If f is analytic at $z_\circ$, then the partial derivatives u_x, u_y, v_x, v_y of u and v must exist at $z_\circ$ and satisfy the Cauchy-Riemann equations*

$$u_x(z_\circ) = v_y(z_\circ) \quad \textit{and} \quad u_y(z_\circ) = -v(z_\circ).$$

It follows that the real part u and imaginary part v of the differentiable function f are solutions of the Laplace equation and are therefore harmonic functions. That is,

$$\Delta u = 0 \quad \textit{and} \quad \Delta v = 0.$$

Proof. Let $f(z) = u(z) + iv(z)$ be analytic at $z_\circ$ in the domain of f. Thus

$$f'(z) = u_x(z_\circ) + iv_x(z_\circ) = v_y(z_\circ) - iu_y(z_\circ)$$

which is analytic in the domain of f. We get

$$u_{xx}(z_\circ) = -u_{yy}(z_\circ) \quad \text{and} \quad v_{xx}(z_\circ) = -v_{yy}(z_\circ).$$

Thus,

$$u_{xx}(z_\circ) + u_{yy}(z_\circ) = 0$$

and

$$v_{xx}(z_\circ) + v_{yy}(z_\circ) = 0$$

as desired. $\square$

Example 7.7. Suppose we have the analytic function

$$f(z) = \frac{1}{z}.$$

We have the function

$$f(z) = \frac{1}{x+iy} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = u(z) + iv(z).$$

The real and imaginary parts are harmonic. Indeed, for the real part

$$u(z) = \frac{x}{x^2+y^2},$$

the partials with respect to x are

$$u_x = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

and

$$u_{xx} = \frac{-2x(x^2+y^2) - 4x(y^2-x^2)}{(x^2+y^2)^3},$$

and the partials with respect to y are

$$u_y = \frac{-2xy}{(x^2+y^2)^2}$$

and

$$u_{yy} = \frac{-2x(x^2+y^2) + 8xy^2}{(x^2+y^2)^3}.$$

Thus,

$$\Delta u = \frac{-6xy^2 + 2x^3}{(x^2+y^2)^3} + \frac{-2x^3 + 6xy^2}{(x^2+y^2)^3} = 0.$$

Likewise, for the imaginary part

$$v(z) = -\frac{y}{x^2 + y^2},$$

the second partial with respect to x is

$$v_{xx} = \frac{2y(x^2 + y^2) - 8x^2 y}{(x^2 + y^2)^3}$$

and the second partial with respect to y is

$$v_{yy} = \frac{2y(x^2 + y^2) - 4y(y^2 - x^2)}{(x^2 + y^2)^3}.$$

Thus,

$$\Delta v = \frac{2y^3 - 6x^2 y}{(x^2 + y^2)^3} + \frac{6x^2 y - 2y^3}{(x^2 + y^2)^3} = 0.$$

Hence the real and imaginary parts of $f(z)$ are harmonic.

7.3 Polar Coordinates

One can exploit the rotational symmetry of the Laplace equation by using the polar coordinate system.

Definition 7.8. Cartesian coordinates can be represented by polar coordiantes

$$x = r\cos\theta, \quad y = r\sin\theta \quad \Leftrightarrow \quad r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

Note: The transformation to θ only works for $x > 0$, in the right half plane.

Lemma 7.9. *The Laplace Equation is now*

$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 0$$

where $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$.

Proof. By the chain rule,

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

and

$$\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}.$$

Differentiating $\frac{\partial}{\partial x}$ with respect to x yields

$$\frac{\partial^2}{\partial x^2} = \cos^2\theta \frac{\partial^2}{\partial r^2} + \frac{\partial^2 r}{\partial x^2}\frac{\partial}{\partial r} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r \partial \theta} + \frac{\partial^2 \theta}{\partial x^2}\frac{\partial}{\partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Differentiating $\frac{\partial}{\partial y}$ with respect to y yields

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta \frac{\partial^2}{\partial r^2} + \frac{\partial^2 r}{\partial y^2}\frac{\partial}{\partial r} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r \partial \theta} + \frac{\partial^2 \theta}{\partial y^2}\frac{\partial}{\partial \theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Thus

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

as desired. □

Definition 7.10. In $\mathbb{R}^2$, a vector

$$z = \begin{bmatrix} x \\ y \end{bmatrix}$$

is rotated counterclockwise by the rotation matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Definition 7.11. A function u is radial, invariant under rotation, iff $u(R_\theta z) = u(z)$ for all θ and z.

Proposition 7.12. *If u is radial, then Δu is radial, and*

$$\Delta u = u_{rr} + \frac{u_r}{r} = \frac{(ru_r)_r}{r}.$$

Proof. Suppose we have a harmonic function u. We apply rotation to u

$$(u \circ R_\theta)(z) = u(\cos\theta x - \sin\theta y, -\sin\theta x + \cos\theta y) = u(R_\theta z).$$

We want to show that the Laplacian operator is radial. We take the partial derivative of u with respect to x twice:

$$\partial_x(u \circ R_\theta)(z) = \cos\theta\partial_1 u(R_\theta z) + \sin\theta\partial_2 u(R_\theta z)$$

and

$$\partial_x^2(u \circ R_\theta)(z) = \cos^2\theta\partial_1^2 u(R_\theta z) + 2\cos\theta\sin\theta\partial_1\partial_2 u(R_\theta z) + \sin^2\theta\partial_2^2 u(R_\theta z).$$

We take the partial derivative of u with respect to y twice:

$$\partial_y(u \circ R_\theta)(z) = -\sin\theta\partial_1 u(R_\theta z) + \cos\theta\partial_2 u(R_\theta z)$$

and

$$\partial_y^2(u \circ R_\theta)(z) = \sin^2\theta \partial_1^2 u(R_\theta z) - 2\cos\theta \sin\theta \partial_1 \partial_2 u(R_\theta z) + \sin^2\theta \partial_2^2 u(R_\theta z).$$

We combine $\partial_x^2(u \circ R_\theta)(z)$ and $\partial_y^2(u \circ R_\theta)(z)$

$$\begin{aligned}
\Delta(u \circ R_\theta)(z) &= \cos^2\theta \partial_1^2 u(R_\theta z) + 2\cos\theta \sin\theta \partial_1 \partial_2 u(R_\theta z) + \sin^2\theta \partial_2^2 u(R_\theta z) \\
&\quad + \sin^2\theta \partial_1^2 u(R_\theta z) - 2\cos\theta \sin\theta \partial_1 \partial_2 u(R_\theta z) + \sin^2\theta \partial_2^2 u(R_\theta z) \\
&= \partial_1^2 u(R_\theta z) + \partial_2^2 u(R_\theta z) \\
&= (\Delta u)(R_\theta z).
\end{aligned}$$

Thus

$$\Delta(u \circ R_\theta) = (\Delta u) \circ R_\theta.$$

If u is radial,

$$\Delta u \circ R_\theta = \Delta(u \circ R_\theta) = \Delta u,$$

so Δu is radial. □

7.4 Fundamental Solution

Definition 7.13. Let Γ be defined for $z \in \mathbb{R}^2 \backslash \{0\}$ by

$$\Gamma(z) = \frac{1}{2\pi} \ln |z|.$$

We show that the function Γ is the fundamental solution of the Laplace operator. In polar coordinates

$$\Gamma(r, \theta) = \frac{1}{2\pi} \ln r.$$

The function Γ is harmonic in $\mathbb{R}^2\backslash\{0\}$, i.e., $\Delta\Gamma = 0$ for $z \neq 0$. We want to show that $\Delta\Gamma = \delta$. That is,

$$\langle \Delta\Gamma, f\rangle = \langle \Gamma, \Delta f\rangle = \langle \delta, f\rangle = f(0).$$

Proposition 7.14. Γ *satisfies*

$$\Delta\Gamma = \delta.$$

That is, for all $f \in C_c^\infty$*,*

$$\int_{\mathbb{R}^2} \Gamma(z)\Delta f(z)dz = f(0).$$

Proof. First, assume g is radial and $g \in C_c^\infty(\mathbb{R}^2)$. Then

$$\begin{aligned}\langle \Delta g, \Gamma\rangle &= \int_{\mathbb{R}^2} \Delta g(z)\Gamma(z)dz\\ &= \lim_{\delta\to 0}\int_0^{2\pi}\int_\delta^\infty \Delta g(r)\left[\frac{1}{2\pi}\ln(r)\right] r dr d\theta\\ &= \lim_{\delta\to 0}\int_\delta^\infty \left(g''(r)r\ln(r) + g'(r)\ln(r)\right)dr.\end{aligned}$$

The first term

$$\int_\delta^\infty g''(r)r\ln(r)dr = [g'(r)r\ln(r)]_\delta^\infty - \int_\delta^\infty g'(r)(1+\ln(r))dr$$

by integration by parts. So

$$\begin{aligned}\langle \Delta g, \Gamma\rangle &= \lim_{\delta\to 0}\left([g'(r)r\ln(r)]_\delta^\infty - \int_\delta^\infty g'(r)(1+\ln(r))dr + \int_\delta^\infty g'(r)\ln(r)dr\right)\\ &= \lim_{\delta\to 0}[g'(r)r\ln(r)]_\delta^\infty - \int_0^\infty g'(r)dr.\end{aligned}$$

The first two terms vanish, so

$$\langle \Delta g, \Gamma \rangle = -\int_0^\infty g'(r)dr = g(0).$$

Now consider f non-radial. Define

$$f_R(z) = \frac{1}{2\pi}\int_0^{2\pi} f(R_\theta z)d\theta.$$

Since Δ commutes with rotations,

$$\Delta(f_R) = (\Delta f)_R.$$

Then f_R is radial, $f_R(0) = f(0)$ and

$$\begin{aligned}
\langle \Delta\Gamma, f \rangle &= \langle \Gamma, \Delta f \rangle \\
&= \langle \Gamma, (\Delta f)_R \rangle \\
&= \langle \Gamma, \Delta(f_R) \rangle \\
&= f_R(0) \\
&= f(0).
\end{aligned}$$

$\square$

Proposition 7.15. *For $f \in C_c^\infty(\mathbb{R}^2)$,*

$$\Delta(\Gamma * f) = f.$$

Proof. For $f \in C_c^\infty(\mathbb{R})$ and any $g \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}\langle \Delta(\Gamma * f), g\rangle &= \langle \Gamma * f, \Delta g\rangle \\ &= \int \Gamma(x-y) f(y) \Delta g(x) dx dy \\ &= \int f(y) \int \Gamma(x) \Delta g(x+y) dx dy \\ &= \int f(y) g(y) dy \\ &= \langle f, g\rangle.\end{aligned}$$

□

7.5 Poisson Kernel

A solution method to the boundary value problem in the circle is the Poisson kernel. We derive it from the following corollary and its proof.

Corollary 7.16. *Let f be at least C^2 and continuous on the circle $|z| = 1$. Then there is a harmonic function u on $|z| < 1$ that extends to a continuous function on $|z| \leq 1$ such that $u = f$ on $|z| = 1$. Namely:*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{1 - |z|^2}{|z - e^{i\theta}|^2} d\theta.$$

Proof. Suppose f is at least C^2. Then f has an absolutely convergent Fourier series. Thus

$$f(e^{it}) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int}$$

where

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} d\theta.$$

We rewrite $f(e^{it})$ as a limit of partial sums

$$\begin{aligned}
f(e^{it}) &= \lim_{N\to\infty} \sum_{|n|\le N} \hat{f}(n)e^{int} \\
&= \lim_{N\to\infty} \sum_{|n|\le N} \left(\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta\right) e^{int} \\
&= \lim_{N\to\infty} \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) \left(\sum_{|n|\le N} e^{-in\theta}e^{int}\right) d\theta \\
&= \lim_{N\to\infty} \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) \left(\sum_{n=0}^{N} e^{-in\theta}e^{int} + \sum_{n=-1}^{-N} e^{-in\theta}e^{int}\right) d\theta \\
&= \lim_{N\to\infty} \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) \left(\sum_{n=0}^{N} e^{-in\theta}e^{int} + \sum_{n=1}^{N} e^{in\theta}e^{-int}\right) d\theta.
\end{aligned}$$

For $|z| < 1$, define

$$u(z) = \lim_{N\to\infty} \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) \left(\sum_{n=0}^{N} e^{-in\theta}z^n + \sum_{n=1}^{N} e^{in\theta}\bar{z}^n\right) d\theta$$

so that $u(e^{it}) = f(e^{it})$. This is absolutely convergent, and $u = f$ on $|z| = 1$. So

$$\begin{aligned}
u(z) &= \lim_{N\to\infty} \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) \left(\frac{1-(ze^{-i\theta})^{N+1}}{1-ze^{-i\theta}} + \frac{\bar{z}e^{i\theta}-(\bar{z}e^{i\theta})^{N+1}}{1-\bar{z}e^{i\theta}}\right) d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) \left(\frac{1}{1-ze^{-i\theta}} + \frac{\bar{z}e^{i\theta}}{1-\bar{z}e^{i\theta}}\right) d\theta.
\end{aligned}$$

This is harmonic, as it is the sum of an anti-holomorphic function and a holomorphic function. We use the properties of the complex z to simplify:

$$\begin{aligned}
\frac{1}{1-ze^{-i\theta}}+\frac{\bar{z}e^{i\theta}}{1-\bar{z}e^{i\theta}} &= \frac{1-\bar{z}e^{i\theta}}{(1-ze^{-i\theta})(1-\bar{z}e^{i\theta})}+\frac{\bar{z}e^{i\theta}(1-ze^{-i\theta})}{(1-ze^{-i\theta})(1-\bar{z}e^{i\theta})}\\
&= \frac{1-\bar{z}e^{i\theta}+\bar{z}e^{i\theta}-|z|^2}{|1-ze^{-i\theta}|^2}\\
&= \frac{1-|z|^2}{|z-e^{i\theta}|^2}.
\end{aligned}$$

Thus the solution is

$$\begin{aligned}
u(z) &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\left(\frac{1-|z|^2}{|z-e^{i\theta}|^2}\right)d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})P(z,e^{i\theta})d\theta
\end{aligned}$$

where

$$P(z,e^{i\theta})=\frac{1-|z|^2}{|z-e^{i\theta}|^2}$$

is the Poisson kernel. $\square$

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