Abstract

ESTIMATION OF THE PROBABILITY A BROWNIAN BRIDGE CROSSES A CONCAVE BOUNDARY

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April, 2010

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This thesis studies a new method to estimate the probability that a Brownian bridge crosses a concave boundary. We show that a Brownian bridge crosses a concave boundary if and only if its least concave majorant crosses said concave boundary. As such, we can equivalently simulate the least concave majorant of a Brownian bridge in order to estimate the probability that a Brownian bridge crosses a concave boundary.

We apply these theoretical results to the problem of estimating joint confidence intervals for a true CDF at every point. We compare this method to a traditional method for estimating joint confidence intervals for the true CDF at every point which is based upon the limiting distribution of what is often called the Kolmogorov-Smirnov distance, the sup-norm distance between the empirical and true CDFs. We indicate the disadvantages of the traditional approach and demonstrate how our approach addresses these weaknesses.

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A Thesis

Presented To

The Faculty of the Department of Mathematics

East Carolina University

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts in Mathematics

by

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ACKNOWLEDGEMENTS

This thesis grew out of guidance given by Dr. Christopher A. Carolan of the Mathematics Department, East Carolina University, in Spring 2010. I would like to start by thanking him for guiding me throughout the past year with wisdom and patience and for giving me continuous courage and faith. Unlike applied projects, where progress is observed continuously, theoretical research requires a certain time until the known results are understood and the new results start to appear.

I thank Dr. Said Elmahdy Said for believing in my abilities and encouraging me to apply for the doctoral degree in Statistics. I thank Dr. Chal Benson for being a great director who supported me since my arrival at ECU. Additionally, I thank Dr. Peng Xiao who is a great teacher and a friend.

Finally, I would like to express my deepest appreciation to my parents and my girlfriend for everything they have done for me. I dedicate this thesis in honor of them.

TABLE OF CONTENTS

LIST OF TABLES AND FIGURES

CHAPTER 1: MOTIVATION

Lower Band for *F*: classical approach

Consider the statistical problem of creating a lower confidence band of a true CDF. Suppose that $X_1,\ldots,X_n$ represent a sample from an arbitrary continuous distribution. Let *F* be the true CDF. Let $F_n(t)=\frac{1}{n}\sum_{i=1}^{n} I(X_i \le t), \forall t$ be the empirical CDF, where $I(\cdot)$ is the indicator function. In classical statistical theory, it is well known (see Shorak and Wellner 1986) that the distance

$$\sqrt{n}\sup_t \{F_n(t)-F(t)\} \xrightarrow{d} \sup_{0\le t\le 1}\{B(t)\} =_d Weibull(\gamma=2,\beta=0.5)$$

where $\{B(t):0\le t\le 1\}$ is the Brownian bridge process. We refer to the following procedure as the approach to estimate this lower confidence band for *F*.

Suppose we are looking for the 80% lower confidence band of the true CDF. Given that the 80th percentile of a $Weibull(\gamma=2,\beta=0.5)$ is 0.897, we have

$$\begin{aligned} & P\left(\sqrt{n}\sup_t\left(F_n(t)-F(t)\right)<0.897\right)\approx 0.8 \\ \Leftrightarrow\ & P\left(\sqrt{n}\left(F_n(t)-F(t)\right)<0.897,\forall t\right)\approx 0.8 \\ \Leftrightarrow\ & P\left(F(t)>F_n(t)-\frac{0.897}{\sqrt{n}},\forall t\right)\approx 0.8. \end{aligned}$$

Thus, an approximate level 80% joint lower confidence band for estimating F is given by

$F_n - \frac{0.897}{\sqrt{n}}$, where n is the sample size.

Now, consider the individual confidence levels of this band at a variety of values of t.

By the Central Limit Theorem, we have that at fixed t, $\sqrt{n}\frac{F_n(t)-F(t)}{\sqrt{F(t)(1-F(t))}} \cong N(0,1)$.

Then, at $t = F^{-1}(0.5)$,

$$P\left(F(t) > F_n(t) - \frac{0.897}{\sqrt{n}}\right) = P\left(\frac{\sqrt{n}(F_n(t)-0.5)}{\sqrt{0.5(1-0.5)}} < \frac{0.897}{\sqrt{0.5(1-0.5)}}\right)$$
$$= \Phi(1.794) = 0.96359$$

where Φ is the standard normal CDF. At $t = F^{-1}(0.1)\ or\ F^{-1}(0.9)$,

$$P\left(F(t) > F_n(t) - \frac{0.897}{\sqrt{n}}\right) = P\left(\frac{\sqrt{n}(F_n(t)-0.1)}{\sqrt{0.1(1-0.1)}} < \frac{0.897}{\sqrt{0.1(1-0.1)}}\right)$$
$$= \Phi(2.99) \approx 0.99861$$

The individual confidence levels of this lower band for the different t have wide variation. The confidence levels on the two sides are higher than in the middle. We seek to explore a new lower band which can make the individual confidence level at each t more even.

Lower Band for *F*: a new approach

Throughout this thesis, let *G* represent an arbitrary nonnegative, concave, and continuous function *G* on [0,1]. Consider such a function *G* such that $P\left(B \text{ crosses } G\right) = 0.2$. Since $\sqrt{n}\left(F_n - F\right) \xrightarrow{d} B\left(F\right)$, we have $P\left(\sqrt{n}\left(F_n - F\right) \text{ crosses } G\left(F\right)\right) \approx 0.2$.

Since $\left\|F_n - F\right\| \xrightarrow{a.s.} 0$, where $\left\|\bullet\right\|$ is the sup-norm, and since *G* is uniformly continuous by virtue of it being continuous on the closed interval [0,1], then $\left\|G\left(F_n\right) - G\left(F\right)\right\| \xrightarrow{a.s.} 0$. Therefore, $G\left(F\right) = G\left(F_n\right) + o\left(1\right)$. So, now we have

$$\begin{aligned} P\left(\sqrt{n}\left(F_n - F\right) \text{ crosses } G\left(F\right)\right) &= P\left(\sqrt{n}\left(F_n - F\right) \text{ crosses } G\left(F_n\right) + o\left(1\right)\right) \\ &\approx P\left(\sqrt{n}\left(F_n - F\right) \text{ crosses } G\left(F_n\right)\right) \approx 0.2 \end{aligned}$$

Thus, an approximate level 80% joint lower confidence band for estimating *F* is given by

$$F_n - \frac{G\left(F_n\right)}{\sqrt{n}}.$$

Now, we are interested in the exact form of *G*(*t*) which can make the probabilities the Brownian bridge crosses this concave function *G*(*t*) at each *t* more uniform. Unfortunately, this approach is based upon knowing *P*(*B* crosses *G*). Unless *G* is linear (there are linear boundary crossing probabilities for a Brownian bridge), this probability will need to be approximated via simulation. In order to do this, we must first introduce the Brownian motion, Brownian bridge and their least concave majorants.

CHAPTER 2: INTRODUCTION TO BROWNIAN MOTION AND BROWNIAN BRIDGE

The aim of this chapter is to introduce Brownian motion, Brownian bridge and their least concave majorants and discuss their properties, putting particular emphasis on the construction of the their least concave majorants.

Overview of Brownian motion and Brownian bridge

Brownian motion is one of the most famous and fundamental of stochastic processes. The mathematical theory of Brownian motion was given a firm foundation by Norbert Wiener in 1923. Let us examine the definition first.

Definition: A stochastic process $\{W(t): t \geq 0\}$ is said to be a standard Brownian motion process if

(i) $W(0) = 0$;

(ii) $\{W(t): t \geq 0\}$ has stationary, independent increments;

(iii) For every $t > 0$, $W(t)$ is normally distributed with mean 0 and variance *t*.

In practice, you can simulate a Brownian motion exactly over any finite set of times $T \subset [0,\infty)$. Most often, these times form an arithmetic sequence. If the constant difference in this arithmetic sequence is small, it is common to linearly interpolate between two successive times in order to estimate the Brownian motion between these successive times. Unfortunately, there is no absolute bound on the error when using this linear interpolation.

A standard Brownian bridge $\{B(t): 0 \le t \le 1\}$ over the interval $[0,1]$ is a standard Brownian motion $\{W(t): t \ge 0\}$ conditioned to have $W(1)=0$. People say the Brownian motion is "tied down" at time 1 to have the value 0. We know that $E(W(t) \mid W(1)) = 0$ and $Cov(W(s), W(t) \mid W(1)=0) = s(1-t)$ for $0 \le s \le t \le 1$ (see Shorak and Wellner 1986).

Definition: A standard Brownian bridge $\{B(t): 0 \le t \le 1\}$ is a Gaussian process with mean 0 and covariance function $Cov(B(s), B(t)) = s(1-t)$ for $0 \le s \le t \le 1$.

An easy way to manufacture a Brownian bridge from a standard Brownian motion is to define $B(t) = W(t) - tW(1)$ for $0 \le t \le 1$. Thus, a Brownian bridge can easily be constructed from a Brownian motion by simulating the Brownian motion over a finite arithmetic sequence of times $T \subset [0,1]$, translating those points to the Brownian bridge, and then linearly interpolating in between successive times in the arithmetic sequence. Unfortunately, the error in estimation for the Brownian bridge will again have no absolute bound.

We will now look at the Least Concave Majorants of Brownian motion and Brownian bridge.

Characterization of the Least Concave Majorant

Define the process $\{K(t): t \geq 0\}$ to be the least concave majorant of a standard Brownian motion process $\{W(t): t \geq 0\}$. That is to say that $\{K(t): t \geq 0\}$ is the lowest of all concave curves which dominate $\{W(t): t \geq 0\}$. Graphically, forming a least concave majorant of a function is like pulling a string tight over the top of the function. Since $\{W(t): t \geq 0\}$ is everywhere non-differentiable almost surely, then it is not surprising that $\{K(t): t \geq 0\}$ will be a piecewise linear curve. As such, one can characterize $\{K(t): t \geq 0\}$ by its vertices. We denote the random set of all vertex times by $V \subset (0, \infty)$, a vertex time being the location where a change in slope occurs in the process $\{K(t): t \geq 0\}$. The set of vertex times will be discrete and infinite. For any constants ε and N satisfying $0 < \varepsilon < N$, there will be an infinite number of vertex times in the interval $(0,\varepsilon)$ and an infinite number of vertex times in the interval (N,∞). There will only be a finite number of vertex times in $[\varepsilon,N]$. In order to isolate a point in V, we could fix $b > 0$ and consider the unique line with slope b that is tangent to $\{K(t): t \geq 0\}$. Define τ_b to be the last time this line touches $\{K(t): t \geq 0\}$. Given τ_b is a point in V, we proceed to index the points of V relative to τ_b. We define V_0 to be τ_b, V_n to be the n^{th} time in V larger than V_0, and V_{-n} to be the n^{th} time in V smaller than V_0. Thus, $V = \{V_i : i \in Z\}$ almost surely.

For $i \in Z - \{0\}$, define

$$T_i = \begin{cases} V_i - V_{i-1} & i \geq 1 \\ V_{i+1} - V_i & i \leq -1 \end{cases}$$

and

$$\alpha_i = \begin{cases} \left[K(V_i) - K(V_{i-1})\right]/T_i & i \geq 1 \\ \left[K(V_{i+1}) - K(V_i)\right]/T_i & i \leq -1 \end{cases}$$

Thus, α_i is the slope of the $|i|^{\text{th}}$ linear segment of $\{K(t): t \geq 0\}$ after V_0 (or before V_0 if i is negative) with horizontal run T_i. Note that the entire process $\{K(t): t \geq 0\}$ is characterized by the set of random variables $\{(T_i, \alpha_i): i \in \mathbb{Z} - \{0\}\}$.

Simulating $\{K(t): t \geq 0\}$: Step #1 – Simulating a starting vertex point

Index the vertex points of $\{K(t): t \geq 0\}$ by fixing $b > 0$ and consider the unique line with slope b that is tangent to $\{K(t): t \geq 0\}$. We define τ_b to be the last time this line touches $\{K(t): t \geq 0\}$. The point $(\tau_b, K(\tau_b))$ is a vertex point of $\{K(t): t \geq 0\}$ such that the slopes of all linear segments of $\{K(t): t \geq 0\}$ before τ_b are at least b and after τ_b are strictly smaller than b. Now, we will use the point $(\tau_b, K(\tau_b))$ as a starting point for our construction of $\{K(t): t \geq 0\}$.

The bivariate density of the starting vertex point is given in Carolan and Dykstra (2003), as described in the following lemma.

LEMMA 2.1: If U is a standard uniform random variable and independently, T is a chi-square random variable with 3 degrees of freedom, then

$$\left(\tau_b, K\left(\tau_b\right)\right) = \left(\frac{1}{b^2} T\left(1-\sqrt{U}\right)^2, \frac{1}{b} T\left(1-\sqrt{U}\right)\right)$$

Thus, we can easily simulate $\left(\tau_b, K\left(\tau_b\right)\right)$ for any arbitrary b. For convenience, we define V_0 as τ_b and continue the construction conditional on knowing $\left(V_0, K\left(V_0\right)\right)$.

Simulating $\left\{K\left(t\right): t \geq 0\right\}$: Step #2 – Simulating vertex points beyond V_0

The steps to simulate vertex points to the right of V_0 for the least concave majorant of a Brownian motion, conditional on knowing $\left(V_0, K\left(V_0\right)\right)$, are also outlined in Carolan and Dykstra (2003). The steps are as follows:

2.1. Create a vector $\left(U_1,...,U_n\right)$ of n independent uniform $U\left(0,1\right)$ realizations. Form the partial products of this vector followed by multiplying the vector by the initial slope b; the result is the vector of slopes $\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right)$, namely $\alpha_i = b\prod_{j=1}^{i} U_j$, $i=1,2,\ldots,n$.

2.2. Create a vector of n independent chi-square with one degree of freedom realizations $\left\{\chi_1^2,\chi_2^2,\ldots,\chi_n^2\right\}$. Take this vector of chi-square realizations and divide it (coordinate-wise) by the vector $\left(\alpha_1^2,\alpha_2^2,\ldots,\alpha_n^2\right)$; the result is the $\left(T_1,T_2,\ldots,T_n\right)$ vector, namely $T_i = \chi_i^2/\alpha_i^2$, $i=1,2,\ldots,n$.

2.3. Form the partial sums of the $\left(T_1,T_2,\ldots,T_n\right)$ vector and add V_0; the result is the vector $\left(V_1,V_2,\ldots V_n\right)$, namely $V_i = V_0 + \sum_{j=1}^{i} T_j$, $i=1,2,\ldots,n$.

2.4. Form the partial sums of the product (coordinate-wise) of the vector $\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right)$ and the vector $\left(T_1,T_2,\ldots,T_n\right)$, followed by adding $K\left(V_0\right)$; the result is the vector $\left(K\left(V_1\right),K\left(V_2\right),\ldots,K\left(V_n\right)\right)$, namely $K\left(V_i\right)=K\left(V_0\right)+\sum_{j=1}^{i}\alpha_j T_j$, $i=1,2,\ldots,n$.

Simulating $\left\{K\left(t\right):t\geq 0\right\}$: Alternative Step #2.1 – Poisson Process Method

Step 2.1 for constructing the $\left\{K\left(t\right):t\geq 0\right\}$ beyond our simulated initial vertex point $\left(V_0,K\left(V_0\right)\right)$ consists of simulating slopes. It is often of interest to generate vertex points until a certain slope threshold is attained. For example, our b could be 1 and we want to keep generating vertex points until the slopes between neighboring vertex points gets below 0.0001.

Using a computer, this will need to be accomplished using a while loop; where we keep adding vertex points until we finally see our slopes passing the 0.0001 threshold. It turns out that this step can be accomplished in a very efficient way by relating it to a Poisson process.

In Step 2.1, the slopes are $\alpha_i = b\prod_{j=1}^{i} U_j$ where $\left(U_1,...,U_n\right)$ are independent Uniform(0,1) random variables. We can rewrite each slope as $\alpha_i = b\exp\left[-\sum_{j=1}^{i}\left(-\ln U_j\right)\right]$. Each $-\ln U_j$ has an exponential distribution with mean 1. So $\sum_{j=1}^{i}\left(-\ln U_j\right)$ are the partial sums of exponential(1) random variables and hence can be viewed as the arrival times of a Poisson process with rate 1.

One can generate the arrival times for a Poisson process directly by doing partial sums of independent exponentials, but there is an alternative way to generate the arrival times of a Poisson process with rate λ. For any time t, the number of arrivals in $[0,t]$ has a Poisson distribution with mean λt. And conditional on the number of arrivals in $[0,t]$ being N, the N arrival times are distributed as the order statistics from a Uniform$(0,t)$ distribution. Thus, one can pick a time t, generate how many arrivals will occur before this time t by generating a Poisson random variable and then generate the specific times by obtaining the order statistics of the appropriate number of independent Uniform$(0,t)$ random variables.

So, if we start with b and we want to get vertex points until the slopes are less than ε ($<b$), we need to generate arrival times in $[0,\ln(b/\varepsilon)]$ since $\alpha = b\exp\left[-\ln\left(b/\varepsilon\right)\right] = \varepsilon$.

2.1*. Select a target slope value ε in $(0,b)$. Simulate a value n from a Poisson distribution with mean $\ln(b/\varepsilon)$. Then simulate and sort n independent $\text{Uniform}\left(0, \ln\left(b/\varepsilon\right)\right)$ realizations, forming a vector $\left(Y_1,\ldots,Y_n\right)$. The resulting vector of slopes $\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right)$ $= b\left(e^{-Y_1},\ldots e^{-Y_n}\right)$ or $\alpha_i = b\exp\left[-Y_i\right]$, $i=1,2,\ldots,n$.

Note that it is possible that there not be any vertex points with slopes between b and ε. This happens when our Poisson realization is 0. Also, if one wishes to get additional vertex points after doing an iteration of Step 2.1*, it must be done conditional that the proceeding slopes are less than ε.

Simulating $\left\{K\left(t\right):t\geq 0\right\}$: Step #3 – Simulating vertex points before V_0

The key to constructing $\left\{K\left(t\right):t\geq 0\right\}$ before V_0 is to translate $\left\{K\left(t\right):t\geq 0\right\}$ to its time-reversed process, which itself is distributed as the least concave majorant of a Brownian motion. It can be demonstrated that

$$\left\{K\left(t\right):t\geq 0\right\} = \left\{tK\left(\frac{1}{t}\right):t\geq 0\right\}.$$

Thus, we can construct $\left\{K\left(t\right):t\geq 0\right\}$ before V_0 by constructing its time-reversed version $\left\{K^*\left(t\right):t\geq 0\right\}$ defined by $\left\{tK\left(1/t\right):t\geq 0\right\}$ beyond some point and then mapping back. This is accomplished in the following manner.

3.1 The translated time-reversed problem has initial vertex point

$\left(V_0^*, K^*\left(V_0^*\right)\right) = \left(1/V_0, K\left(V_0\right)/V_0\right)$ with initial slope $b^* = K\left(V_0\right) - bV_0$.

3.2. Get the vector $\left(V_1^*, V_2^*, \ldots, V_m^*\right)$ and $\left(K^*\left(V_1^*\right), K^*\left(V_2^*\right), \ldots, K^*\left(V_m^*\right)\right)$ for the time-reversed process $\left\{K^*\left(t\right) : t \geq 0\right\}$ using Step 2.

3.3. We obtain the vector $\left(V_{-m}, V_{-(m-1)}, \ldots, V_{-1}\right)$ by taking the (coordinate-wise) reciprocal of the $\left(V_1^*, V_2^*, \ldots, V_m^*\right)$ vector and reversing the order of the vector.

3.4. We obtain the $\left(K\left(V_{-m}\right), K\left(V_{-(m-1)}\right), \ldots, K\left(V_{-1}\right)\right)$ vector by dividing (coordinate-wise) the vector $\left(K^*\left(V_1^*\right), K^*\left(V_2^*\right), \ldots, K^*\left(V_m^*\right)\right)$ by the vector $\left(V_1^*, V_2^*, \ldots, V_m^*\right)$ and reversing the order.

Constructing $\left\{K_0\left(t\right) : 0 \leq t \leq 1\right\}$

The procedure to construct $\left\{K\left(t\right) : t \geq 0\right\}$ can also be used to construct the least concave majorant of a Brownian bridge over any closed subinterval of $\left(0,1\right)$. We define $\left\{K_0\left(t\right) : 0 \leq t \leq 1\right\}$ to be the least concave majorant of a standard Brownian bridge process $\left\{B\left(t\right) : 0 \leq t \leq 1\right\}$. By Doob's transformation, we can define $\left\{B\left(t\right) : 0 \leq t \leq 1\right\}$ to be $\left\{\left(1-t\right)W\left(\frac{t}{1-t}\right) : 0 \leq t \leq 1\right\}$. Since Doob's transformation maps lines to lines, it follows that

$\{K_0(t): 0 \le t \le 1\}$ is given by $\left\{(1-t)K\left(\frac{t}{1-t}\right): 0 \le t \le 1\right\}$. Thus, the set of constructed vertex points $\{(V_i, K(V_i)): i = -m, \ldots, n\}$ of the process $\{K(t): t \ge 0\}$ maps to the set of vertex points $\left\{\left(\frac{V_i}{1+V_i}, \frac{K(V_i)}{1+V_i}\right): i = -m, \ldots, n\right\}$ of the process $\{K_0(t): 0 \le t \le 1\}$.

It turns out that the slope between adjacent vertex points in $\{K(t): t \ge 0\}$ has an interesting property when translated to a Brownian bridge. This property is stated in the Lemmas below.

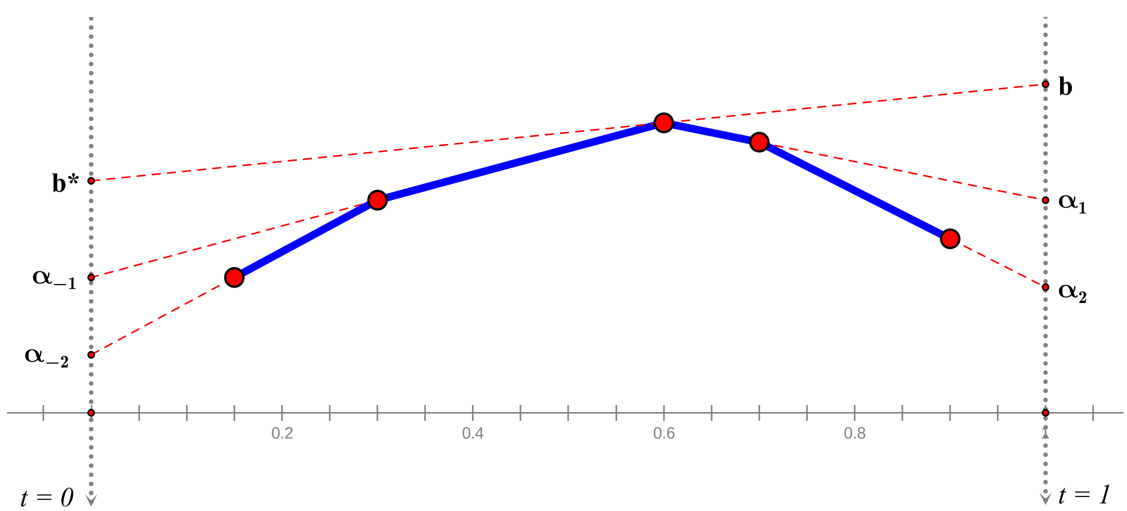
Lemma 2.2: If the slope between the two vertex points $(V_{i-1}, K(V_{i-1}))$ and $(V_i, K(V_i))$ is α_i, then the line adjoining the translated points $\left(\frac{V_{i-1}}{1+V_{i-1}}, \frac{K(V_{i-1})}{1+V_{i-1}}\right)$ and $\left(\frac{V_i}{1+V_i}, \frac{K(V_i)}{1+V_i}\right)$ intersects the line $t = 1$ at $(1, \alpha_i)$.

Lemma 2.3: If the slope between the two vertex points $(V_{i-1}^*, K^*(V_{i-1}^*))$ and $(V_i^*, K^*(V_i^*))$ on the time reversed process (Step 3) is α_i^*, then the line adjoining the translated (two translations: translate from time-reversed Brownian motion to the original Brownian motion to the Brownian bridge) points $\left(\frac{1}{1+V_{i-1}^*}, \frac{1}{1+V_{i-1}^*}K^*(V_{i-1}^*)\right)$ and $\left(\frac{1}{1+V_i^*}, \frac{1}{1+V_i^*}K^*(V_i^*)\right)$ intersects the line $t = 0$ at $(0, \alpha_i^*)$.

The proofs can be found in the Appendix A.

The previous two lemmas imply slopes of linear segments of the least concave majorant of the Brownian motion translate to projected intercepts at $t = 0$ or $t = 1$ on the Brownian bridge, see GRAPH 2.1. The alternative Step 2 for simulating to the right of V_0 of $\{K(t): t \geq 0\}$ allows us to simulate exactly the correct number of vertex points so that neighboring slopes are arbitrarily small and this is equivalent to arbitrarily small intercept projections on the Brownian bridge.

GRAPH 2.1: The Least Concave Majorant of a Brownian Bridge with Projections



CHAPTER 3: APPROXIMATING THE PROBABILITY A BROWNIAN BRIDGE CROSSES AN ARBITRARY CONCAVE FUNCTION VIA SIMULATION.

We begin this chapter with the following useful lemma, which will allow us to simulate the least concave majorant of a Brownian bridge (instead of simulating the Brownian bridge) in order to estimate the chance that a Brownian bridge crosses *G*.

LEMMA 3.1: If $\{B(t):0\le t\le 1\}$ is a standard Brownian bridge process, $\{K_0(t):0\le t\le 1\}$ is its least concave majorant, and $\{G(t):0\le t\le 1\}$ is an arbitrary nonnegative, concave, and continuous function, then

$$\sup_{0\le t\le 1}\{B(t)-G(t)\}=\sup_{0\le t\le 1}\{K_0(t)-G(t)\}$$

The proof can be found in the Appendix A.

Noting that a function f_1 will cross (exceed) another function f_2 if and only if the supremum of (f_1-f_2) exceeds 0, then this lemma gives us the theoretical guarantee that the probability a Brownian bridge crosses *G* is equal to the probability its LCM crosses *G*.

Therefore, we shall proceed to approximate the probability that a Brownian bridge crosses G via simulation of the least concave majorant of the Brownian bridge. The following lemma will allow us to only consider the vertex points of this least concave majorant when determining the supremum distance the least concave majorant attains above G.

LEMMA 3.2: If $\{K_0(t): 0 \le t \le 1\}$ is the least concave majorant of a standard Brownian bridge process possessing a set of vertex times V and $\{G(t): 0 \le t \le 1\}$ is an arbitrary nonnegative, concave, and continuous function, then

$$\sup_{0 \le t \le 1}\{K_0(t) - G(t)\} = \sup_{t \in V}\{K_0(t) - G(t)\}$$

The proof can be found in the Appendix A.

Our approximation of the boundary crossing probability will be based upon the (partial) simulation of $\{K_0(t): 0 \le t \le 1\}$, since there will be an infinite number of vertex points and we cannot simulate them all. The following Lemma allows us to bound our error in estimation of $\sup_{0 \le t \le 1}\{K_0(t) - G(t)\}$ when only partially simulating the least concave majorant of a Brownian bridge.

LEMMA 3.3: Partition the set of all vertex times V into the set of vertex times that we have simulated V_s and the set of all unsimulated vertex times V_u.

If $\sup_{t\in V_s}\left\{K_0(t)-G(t)\right\}\geq\varepsilon-\min\left\{G(0),G(1)\right\}$, then

$$\sup_{t\in V}\left\{K_0(t)-G(t)\right\}=\sup_{t\in V_s}\left\{K_0(t)-G(t)\right\}$$

and if $\sup_{t\in V_s}\left\{K_0(t)-G(t)\right\}<\varepsilon-\min\left\{G(0),G(1)\right\}$, then

$$\sup_{t\in V}\left\{K_0(t)-G(t)\right\}\leq\varepsilon-\min\left\{G(0),G(1)\right\}$$

where ε is a ceiling for the unsimulated slopes in step 2.1 (both forward and time-reversed).

The proof can be found in the Appendix A.

Lemma 3.3 tells us that if $\sup_{t\in V_s}\left\{K_0(t)-G(t)\right\}$ is sufficiently large (greater than $\varepsilon-\min\left\{G(0),G(1)\right\}$), then we actually observe the supremum distance $\sup_{t\in V}\left\{K_0(t)-G(t)\right\}$. Otherwise, the supremum distance $\sup_{t\in V}\left\{K_0(t)-G(t)\right\}$ is undetermined, but it is known to be no greater than $\varepsilon-\min\left\{G(0),G(1)\right\}$.

CHAPTER 4: MODIFYING THE EMPIRICAL CDF TO CREATE SYMMETRY

Recall, an approximate level (1–*α*)100% joint lower confidence band for estimating *F* is given by $F_n - \frac{G(F_n)}{\sqrt{n}}$, where *G* is an arbitrary nonnegative, concave, and continuous function. There is a lack of symmetry to this problem. In order to understand this, we discuss some properties of the empirical CDF in the next paragraph.

Suppose that $Y_1, \ldots, Y_n$ are the order statistics corresponding to a random sample from a $\text{Uniform}(0,1)$ distribution with CDF represented as *U*. Then $Y_i \sim Beta(\alpha_i = i, \beta_i = n - i + 1)$ with mean $\mu_i = \alpha_i / (\alpha_i + \beta_i) = t_i$ and variance $\sigma_i^2 = \alpha_i \beta_i / \left[(\alpha_i + \beta_i)^2 (\alpha_i + \beta_i + 1)\right] = t_i(1 - t_i)/(n+2)$, where $t_i = i/(n+1)$. So, $\sqrt{n+2}(Y_i - t_i)$ has mean 0 and variance $t_i(1 - t_i)$. Note that if a random variable *W* has a Beta distribution with parameters $\alpha = \alpha_0$ and $\beta = \beta_0$, then the random variable (1–*W*) also has a Beta distribution with parameters $\alpha = \beta_0$ and $\beta = \alpha_0$. Therefore, Y_i has the same distribution as $1 - Y_{n-i+1}$. This is the symmetry. How far the i^{th} smallest observation is above zero has the same distribution as how far the i^{th} largest observation is below one. On average, this distance will be $i/(n+1)$.

So, it is reasonable to approximate or predict U evaluated at the (random) point Y_i by $i/(n+1)$, whereas the traditional empirical CDF uses i/n. Thus, we define the modified uniform empirical CDF U_n^* at each observation Y_i by $U_n^*(Y_i)=i/(n+1)$ with $U_n^*(0)=0$ and $U_n^*(1)=1$. Defining $Y_0 = 0$ and $Y_{n+1} =1$, we will linearly interpolate in order to obtain U_n^* at arbitrary values of t:

$$U_n^*(t)=\frac{i}{n+1}+\frac{t-Y_i}{(n+1)(Y_{i+1}-Y_i)} \qquad Y_i \le t < Y_{i+1}\,, i=0,\ldots,n$$

If U_n represents the standard empirical CDF, then $\left\|U_n^*-U_n\right\|<1/n$, implying that their sup-norm difference converges to zero. Thus, it follows that $\sqrt{n+2}\left(U_n^*-U\right)\rightarrow B$. Let G represent any continuous, nonnegative, concave function over [0,1] such that $P\left(B \text{ crosses } G\right)=\alpha/2$. Then, $P\left(\sqrt{n+2}\left(U_n^*-U\right) \text{exceeds } G\right)\approx\alpha/2$. Since $\left\|U_n^*-U\right\|\xrightarrow{a.s.}0$ and since G is uniformly continuous by virtue of it being continuous on the closed interval [0,1], then $\left\|G\left(U_n^*\right)-G\right\|\xrightarrow{a.s.}0$. Therefore, $G=G\left(U_n^*\right)+o(1)$. So, now we have

$$\begin{aligned}P\left(\sqrt{n+2}\left(U_n^*-U\right)\text{crosses } G\right)&=P\left(\sqrt{n+2}\left(U_n^*-U\right)\text{ crosses } G\left(U_n^*\right)+o(1)\right)\\&\approx P\left(\sqrt{n+2}\left(U_n^*-U\right)\text{ crosses } G\left(U_n^*\right)\right)\approx\alpha/2\end{aligned}$$

Also note that it is also true that

$$\begin{aligned}P\left(\sqrt{n+2}\left(U_n^*-U\right)\text{crosses } -G\right)&=P\left(\sqrt{n+2}\left(U_n^*-U\right)\text{ crosses } -G\left(U_n^*\right)+o(1)\right)\\&\approx P\left(\sqrt{n+2}\left(U_n^*-U\right)\text{ crosses } -G\left(U_n^*\right)\right)\approx\alpha/2\end{aligned}$$

Combining the two previous statements, making use of Boole's inequality, we have

$$P\left(\sqrt{n+2}\left(U_n^* - U\right) \text{ crosses } \pm G\left(U_n^*\right)\right) \approx \alpha \text{ (or less)}$$

If we restrict ourselves to the centered and rescaled empirical distribution $\sqrt{n+2}\left(U_n^* - U\right)$ crossing $\pm G$ only at the data points $Y_1,\ldots,Y_n$ yields

$$P\left(\sqrt{n+2}\left(\frac{i}{n+1} - Y_i\right) \text{exceeds} \pm G\left(\frac{i}{n+1}\right) \text{ for } i = 1,2,\ldots,\text{or } n\right) \approx \alpha \text{ (or less)}$$

Now, let F be any (unknown) continuous CDF. Then $X_1 = F^{-1}(Y_1), X_2 = F^{-1}(Y_2),\ldots, X_n = F^{-1}(Y_n)$ represent the order statistics corresponding to a random sample from F. It thereby follows that

$$P\left(\sqrt{n+2}\left(\frac{i}{n+1} - F(X_i)\right) \text{crosses} \pm G\left(\frac{i}{n+1}\right) \text{ for } i = 1,2,\ldots,\text{or } n\right) \approx \alpha \text{ (or less)}$$

Thus, we can create joint confidence intervals on the estimates of F at the observed data points $X_1, X_2,\ldots, X_n$. The joint confidence intervals for $F(X_1), F(X_2),\ldots, F(X_n)$ have approximate joint confidence level of (at least) (1–α)100% and the interval for $F(X_i)$ is given by

$$\frac{i}{n+1} \pm \frac{G\left(i/(n+1)\right)}{\sqrt{n+2}}; \quad i = 1,2,..,n$$

Therefore, we have at least (1–α)100% joint coverage probability at the observed data points. Now define, $X_0 = -\infty$ and $X_{n+1} = \infty$. Then, for free, we can bound any estimate of $F(t)$

above by $\frac{i}{n+1}+\frac{G\left(i/(n+1)\right)}{\sqrt{n+2}}$ where $X_{i-1} < t \le X_i$ and below by $\frac{i}{n+1}-\frac{G\left(i/(n+1)\right)}{\sqrt{n+2}}$ where $X_i \le t < X_{i+1}$. Thus, we borrow the upper boundary from the closest X_i that is greater than or equal to the *t* under consideration and borrow the lower boundary from the closest X_i that is less than or equal to the *t* under consideration. We hereby have created a joint confidence band for the true CDF *F*. The upper band and the lower band will both be step functions. We summarize this in the following remark.

REMARK 4.1: Let $X_1, ..., X_n$ represent the order statistics from a random sample from a continuous distribution with unknown CDF *F*. Furthermore, let *G* represent any continuous, nonnegative, and concave function over the interval [0,1] such that $P\left(B \text{ crosses } G\right)=\alpha/2$, where $\left\{B(t): 0 \le t \le 1\right\}$ is a Brownian bridge process and $\alpha \le 1$. Then, the set of all confidence intervals for *F*(*t*) of the form

$$\left[\frac{i}{n+1}-\frac{G\left(i/(n+1)\right)}{\sqrt{n+2}}, \frac{i}{n+1}+\frac{G\left(i/(n+1)\right)}{\sqrt{n+2}}\right] \quad t = X_i;\ i = 1,2,..,n$$

and

$$\left[\frac{i}{n+1}-\frac{G\left(i/(n+1)\right)}{\sqrt{n+2}}, \frac{i+1}{n+1}+\frac{G\left((i+1)/(n+1)\right)}{\sqrt{n+2}}\right] \quad X_i < t < X_{i+1};\ i = 1,2,..,n$$

has a minimum joint confidence level of approximately (1–α)100%, where $X_0 = -\infty$ and $X_{n+1} = \infty$.

CHAPTER 5: APPLICATION TO CONFIDENCE BANDS FOR CONTINUOUS CDFS

Recall that *G* represents an arbitrary nonnegative, concave, and continuous function *G* on $[0,1]$. If $P(B \text{ crosses } G) = \alpha/2$, then Remark 4.1 shows us how to get a set of confidence intervals for estimating an unknown CDF *F* at every point which has an approximate joint confidence level of (1–α)100%. In the next section, we suggest an appealing form for *G*.

Choosing a form for the concave function *G*

There are many concave functions for which we can estimate the probability the Brownian bridge crosses them. Recall, our desire is to use a concave function which possesses pointwise Brownian bridge crossing probabilities that are fairly uniform. We note that for each $0 < t < 1$, $B(t)$ has a normal distribution with mean 0 and standard deviation $\sqrt{t(1-t)}$. Thus, if our $G(t) = c\sqrt{t(1-t)}$; $c > 0$, the pointwise boundary crossing probabilities would all be $\overline{\Phi}(c)$, where $\overline{\Phi}$ is the standard normal survival function. This function *G* is half an ellipse centered at (0.5,0) with $G(0) = 0 = G(1)$. Thus, this *G* and the Brownian bridge share endpoints. This will make it difficult to determine if the Brownian bridge crosses *G* since we cannot simulate the least concave majorant of a Brownian motion in a neighborhood of the endpoints. As such, we will consider a concave function *G* of the form $G(t) = a + c\sqrt{t(1-t)}$; $c > 0, a \geq \varepsilon$, where ε is a ceiling for the unsimulated slopes in step 2.1 (both forward and time-reversed). By restricting our additive constant a to be a value greater than or equal to ε, then by Lemma 3.3, we have $\varepsilon - \min\{G(0), G(1)\} = \varepsilon - a$ is nonpositive. Thus, when our $\sup_{t \in V_s}\{K_0(t) - G(t)\} > 0$, we know for certain that the Brownian bridge has crossed our G. Most importantly however, when $\sup_{t \in V_s}\{K_0(t) - G(t)\} < 0$, then we know for certain that our Brownian bridge has not crossed G.

Finding values (*a*,*c*) with associated crossing probabilities of α/2

If $G(t)$ is of the form $a + c\sqrt{t(1-t)}$; $c > 0, a \geq \varepsilon$, then we are looking for the probability of a Brownian bridge $\{B(t) : 0 \leq t \leq 1\}$ crossing G. We seek combinations of (a,c) which make the probability α/2, α≤1. The combinations of (a,c) associated with crossing probabilities of α/2 will be estimated via simulation as described in the following paragraphs.

We will begin by arbitrarily picking any $c > 0$. We will work to estimate the value of a associated with this c such that the probability of G being crossed by a Brownian bridge is α/2. Define $G^*(t) = c\sqrt{t(1-t)}, 0 \leq t \leq 1$ and select a small value of ε > 0. We will now start simulating the least concave majorant of Brownian bridges in order to estimate $\sup_{0 \leq t \leq 1}\{K_0(t) - G^*(t)\}$. We simulate the least concave majorant of Brownian bridges because Lemma 3.1 tells us that Brownian bridge and its least concave majorant will result in the same value for the sup-distance above G^*. Furthermore, Lemma 3.2 tells us that we need only consider the vertex points of the least concave majorant of the Brownian bridge in determination of the sup-distance above G^*. When constructing the least concave majorant of the Brownian bridge, we will simulate vertex points using the alternative step 2.1 with our selected ε as a ceiling for all unsimulated slopes. In order to save time, we will choose our initial slope b to be ε. This will eliminate the need to simulate vertex points to the right of our initial vertex point.

By Lemma 3.3, if the sup-distance the least concave majorant is above G^* over the simulated vertex points exceeds ε, then we have observed the actual sup-distance over all vertex points. However, if the sup-distance the least concave majorant is above G^* over the simulated vertex points does not exceed ε, then also by Lemma 3.3, the actual sup-distance over all vertex points remains unobserved, but it is known to be less than or equal to ε.

We will construct a very large number N of least concave majorant of Brownian bridge paths. For each path, we will have a sup-distance that each least concave majorant path exceeds G^*. Provided $a > \varepsilon$, the sup-distance being greater than a implies that the least concave majorant path crosses $G = (G^* + a)$ and the sup-distance being less than or equal to a implies that the least concave majorant path does not cross $G = (G^* + a)$. Thus, the proportion of constructed least concave majorant paths that exceed G^* by more than a is also the proportion of the constructed least concave majorant paths that cross $G = (G^* + a)$. Our goal is to estimate the value for a for which $P\left(B \text{ crosses } G^* + a\right) = \alpha/2$. This value of a will be estimated as the $(1-(\alpha/2))^{\text{th}}$ percentile of the N observed sup-distances.

A criteria for selecting an optimal combination for (*a*,*c*)

There will be an infinite number of (a,c) combinations where $P\left(B \text{ crosses } G\right) = \alpha/2$. We wish to select an ($a$,$c$) combination which in general stays close to the horizontal axis. We will measure closeness to the horizontal axis as the area under the function G which is given by $a + 0.125\pi c$. Thus, our goal is to minimize $a + 0.125\pi c$, where (a,c) are certain constants such that the probability of a Brownian bridge $\left\{B(t): 0 \le t \le 1\right\}$ crossing $G(t) = a + c\sqrt{t(1-t)}$ is α/2.

Results for α/2 = 0.005, 0.025, and 0.05

The statistical package R was used to simulate the results. The function executed in R is found in Appendix B. In practice, 4 to 6 million least concave majorant of Brownian bridge paths were constructed. The associated value of a corresponding to each value of c from 1 to 5 in 0.01 increments was found for each value of α/2 = 0.005, 0.025, and 0.05. Then, the optimal combination (a,c) was identified for each value of α/2. Table 5.1 summarizes the results.

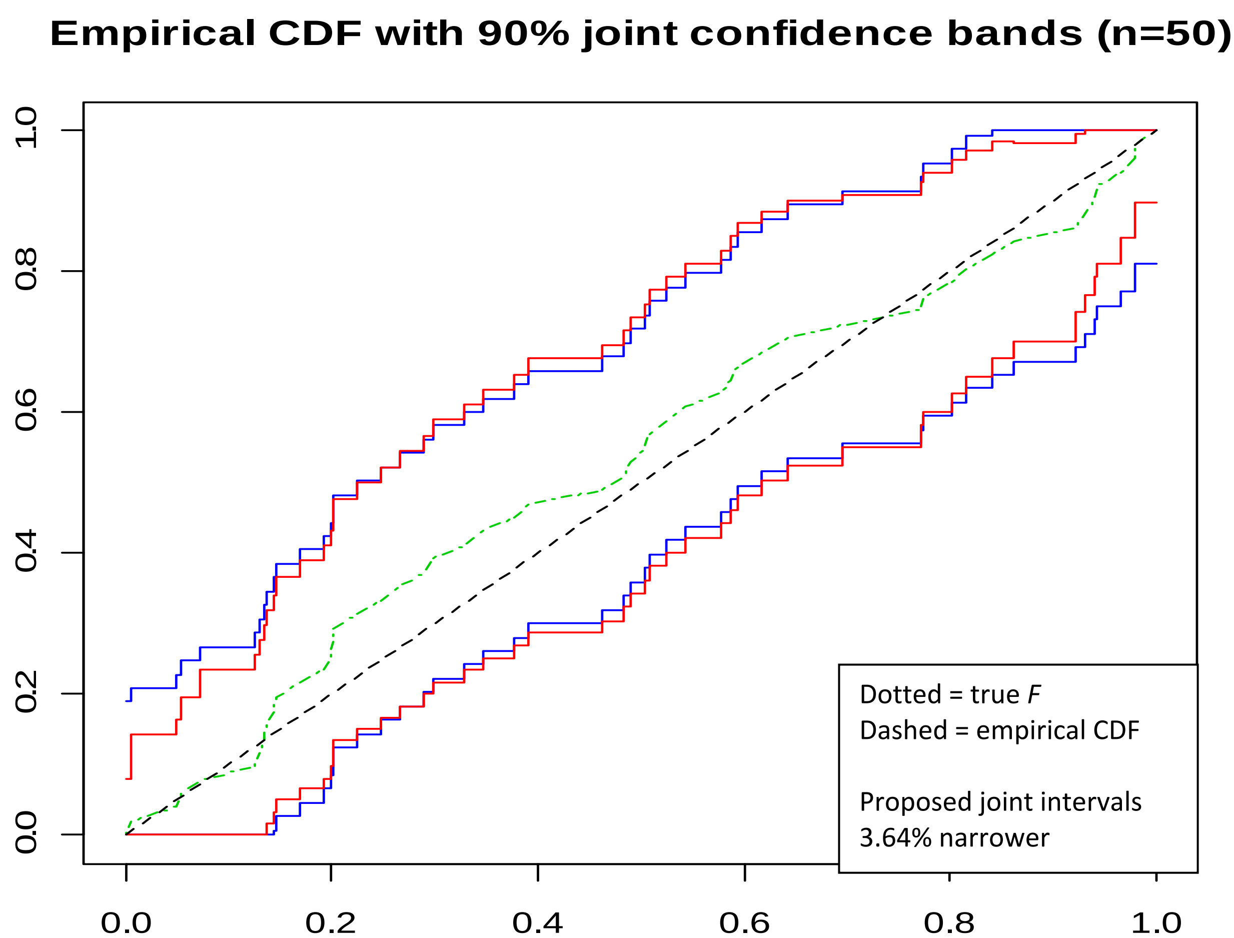
TABLE 5.1

α/2	Optimal (a,c)	Area $a + 0.125\pi c$	Kolmogorov-Smirnov Area	Percent Savings
0.05	$(0.2940, 2.10)$	1.11867	1.22387	8.6%
0.025	$(0.2655, 2.44)$	1.22369	1.35810	9.9%
0.005	$(0.2300, 3.06)$	1.43166	1.62762	12%

In Table 5.1, the Kolmogorov-Smirnov area refers to the area under *G* when *c* is 0. This is a horizontal line and its height is determined by the $(1-(\alpha/2))^{th}$ percentile of a Weibull distribution with parameters $\gamma = 2$, $\beta = 0.5$. This corresponds to the classical approach to obtaining a joint confidence band when estimating an unknown continuous CDF *F*. As measured by the area under the function *G* , we can see that our approach will tend to result in joint confident bands that are 8.6% to 12% narrower in the limit (as sample size goes to ∞). Our savings with finite samples will tend to be smaller.

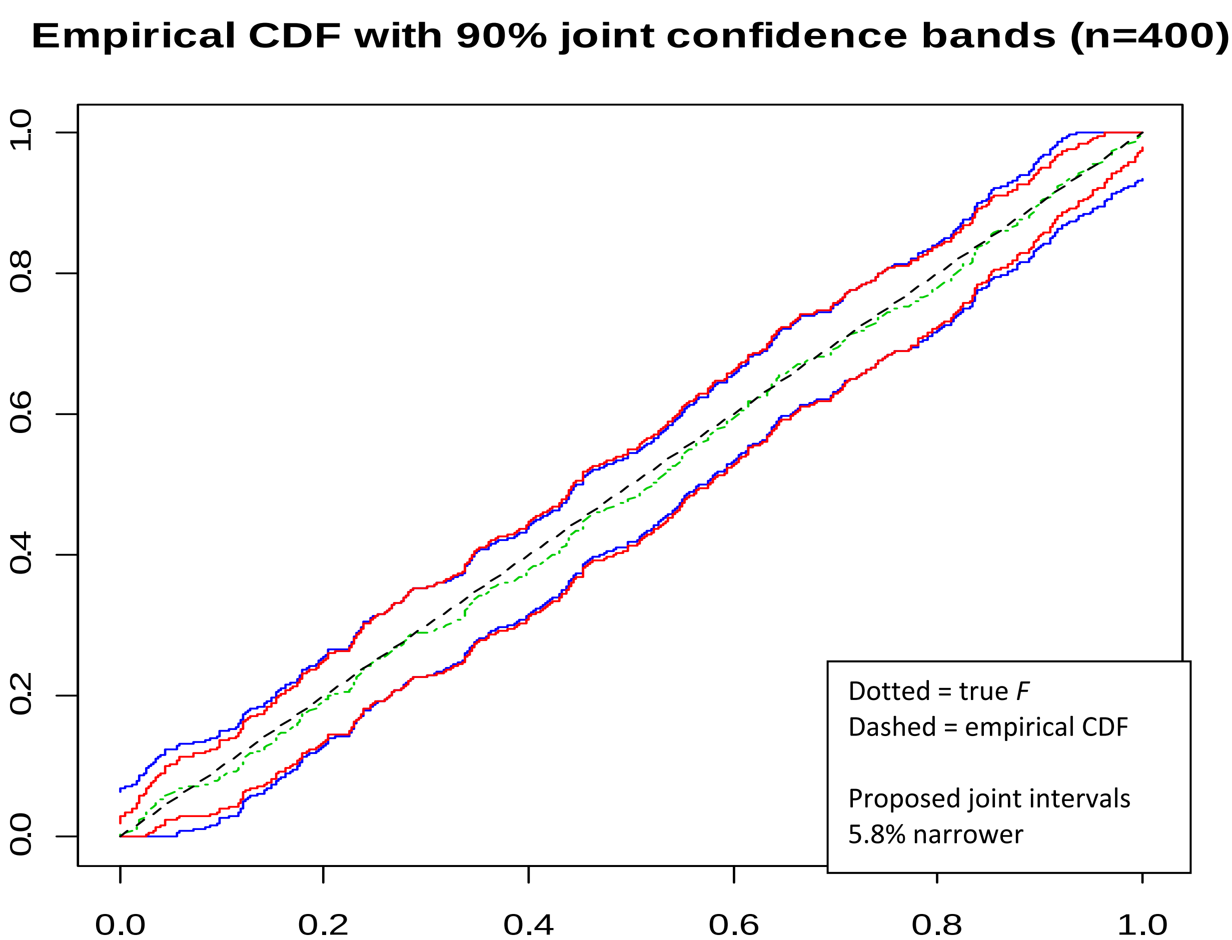
We will now compare the classical approach to obtaining a joint confidence band for and unknown continuous CDF *F*, discussed in Chapter 1, to our improved approach, summarized in Remark 4.1. In order to do this, we will take samples of size $n = 50$ and 400 from a Uniform(0,1) distribution. Using the same data, we will obtain 90% joint confidence intervals for *F* using both the classical approach and our suggested approach. The joint confidence intervals are displayed graphically. We expect that our joint confidence intervals will be slightly wider for *t* near 0.5, and moderately narrower for *t* near 0 and 1. We also check to make sure that the joint confidence intervals completely encase the known *F* they are trying to estimate.

GRAPH 5.1:



In Graph 5.1, illustrating when $n = 50$, we see that both sets of joint confidence intervals for F are correct as demonstrated by the fact that the true CDF F, represented as a dotted line, is completely between the upper and lower bounds at each point. Our proposed joint confidence intervals are wider for t approximately between 0.2 and 0.8 and dramatically narrower the closer t is to 0 or 1. Our proposed joint confidence intervals have an enclosed area that is 3.64% smaller than the area enclosed by the classical joint confidence intervals. Whenever either joint confidence interval drops below zero or goes above one, they are pulled back to the natural boundary for a probability. As illustrated by Graph 5.2, representing when $n = 400$, the region over which this pulling back occurs will tend to decrease as the sample size increases. In Graph 5.2, the difference in enclosed areas represents a 5.8% difference. As n goes to ∞, the percentage difference should be 8.6% as stated in Table 5.1.

GRAPH 5.2:



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APPENDIX A: PROOFS OF THE LEMMAS

PROOFS of Lemmas 2.2 and 2.3: We beginning by defining two neighboring vertex points $\left(V_{i-1}, K\left(V_{i-1}\right)\right)$ and $\left(V_i, K\left(V_i\right)\right)$ of $\left\{K\left(t\right): t \geq 0\right\}$. The slope between these points is α_{i} which is given by $\alpha_{\mathrm{i}} = \dfrac{K(V_i) - K(V_{i-1})}{V_i - V_{i-1}}$.

These two points on the LCM of Brownian motion map to the points $\left(\dfrac{V_{i-1}}{1+V_{i-1}}, \dfrac{1}{1+V_{i-1}} K\left(V_{i-1}\right)\right)$ and $\left(\dfrac{V_i}{1+V_i}, \dfrac{1}{1+V_i} K\left(V_i\right)\right)$ on the LCM of Brownian bridge. We first prove the line connecting these points translated points intersects $t = 1$ at $\left(1, \alpha_i\right)$.

The line $L(t)$ through this pair of translated points is given by

$$L\left(t\right) = \frac{t - \dfrac{V_i}{1+V_i}}{\dfrac{V_i}{1+V_i} - \dfrac{V_{i-1}}{1+V_{i-1}}} \left[\frac{K\left(V_i\right)}{1+V_i} - \frac{K\left(V_{i-1}\right)}{1+V_{i-1}}\right] + \frac{K\left(V_i\right)}{1+V_i}$$

Now, we seek to find the projection on the line $t = 1$

$$L\left(1\right) = \frac{1 - \dfrac{V_i}{1+V_i}}{\dfrac{V_i}{1+V_i} - \dfrac{V_{i-1}}{1+V_{i-1}}} \left[\frac{K\left(V_i\right)}{1+V_i} - \frac{K\left(V_{i-1}\right)}{1+V_{i-1}}\right] + \frac{K\left(V_i\right)}{1+V_i} = \frac{K\left(V_i\right) - K\left(V_{i-1}\right)}{V_i - V_{i-1}} = \alpha_i$$

Thus, point $\left(1, \alpha_i\right)$ is a projection onto the line $t = 1$. The lemma has been confirmed.

We use the same manner to show this line intersects $t=0$ at $(0,\alpha_i)$ when mapping the time-reversed process $\{K^*(t): t\geq 0\}$.

PROOFS of Lemma 3.1 and 3.2: In order to verify these lemmas, we will show that the distances $B(t)-G(t)$ and $K_0(t)-G(t)$ at a non-vertex time is at most the weighted average at the two neighboring vertex times. Define two sets: $V=\{t: B(t)=K_0(t)\}$ is the set of all vertex times and $V^c=\{t: B(t)<K_0(t)\}$ is the set of all non-vertex times. Consider a point $t\in V^c$. Let $t_1=\sup\{x: x\in V, x\leq t\}$ be the nearest vertex time to the left and let $t_2=\inf\{x: x\in V, x\geq t\}$ be the nearest vertex time to the right. Since the LCM is linear between vertex times, we have

$K_0(t)=\frac{t-t_1}{t_2-t_1}K_0(t_2)+\frac{t_2-t}{t_2-t_1}K_0(t_1)$ on $[t_1,t_2]$. Since G is a concave function, we agree that for $t_1\leq t\leq t_2$, $G(t)\geq\frac{t-t_1}{t_2-t_1}G(t_2)+\frac{t_2-t}{t_2-t_1}G(t_1)$ and hence

$$K_0(t)-G(t)\leq\frac{t-t_1}{t_2-t_1}\left(K_0(t_2)-G(t_2)\right)+\frac{t_2-t}{t_2-t_1}\left(K_0(t_1)-G(t_1)\right)$$

$$=\frac{t-t_1}{t_2-t_1}(K_0-G)(t_2)+\frac{t_2-t}{t_2-t_1}(K_0-G)(t_1)$$

So, we see that any distance $K_0(t)-G(t)$ at any non-vertex time at most the weighted average of the distance at the two neighboring vertex times. Thus, we have

$\sup_{t \in V}\{K_0(t)-G(t)\} \geq \sup_{t \in V^c}\{K_0(t)-G(t)\}$. This proves Lemma 3.2. Since vertex points are points where the Brownian bridge B and its LCM K_0 touch, then the sup-distance over the vertex times are equal $\sup_{t \in V}\{K_0(t)-G(t)\} = \sup_{t \in V}\{B(t)-G(t)\}$. Since K_0 dominates B, it easily follows that $\sup_{t \in V}\{B(t)-G(t)\} \geq \sup_{t \in V^c}\{K_0(t)-G(t)\} \geq \sup_{t \in V^c}\{B(t)-G(t)\}$. This proves Lemma 3.1.

PROOF of Lemma 3.3: Let V be the set of all vertex times, V_u be the set of vertex times we have not simulated and V_s be the set of vertex times we have simulated. Define $t_{\min} = \inf\{V_s\}$ and $t_{\max} = \sup\{V_s\}$. Let $\overline{K_0}$ be an upper bound for K_0 given by

$$\overline{K_0}(t) = \begin{cases} \dfrac{K_0(t_{\min})-\varepsilon}{t_{\min}}t+\varepsilon & 0 \leq t \leq t_{\min} \\ K_0(t) & t_{\min} \leq t \leq t_{\max} \\ \dfrac{K_0(t_{\max})-\varepsilon}{1-t_{\max}}(1-t)+\varepsilon & t_{\max} \leq t \leq 1 \end{cases}$$

where ε is a ceiling for the unsimulated slopes in step 2.1 (both forward and time-reversed). Recall, by Lemmas 2.2 and 2.3, this ε will also be a ceiling for the projected heights at $t = 0$ and $t = 1$. See Figure 2, noting that $\overline{K_0}$ is essentially the simulated part of K_0, projected out to $t = 0$ and $t = 1$. Noting that $K_0 \leq \overline{K_0}$, we have

$$
\begin{aligned}
\sup_{t\in V_u}\left\{K_0(t)-G(t)\right\} &\le \sup_{t\in V_u}\left\{\overline{K_0}(t)-G(t)\right\}\\
&= \max_{t=0,t_{\min},t_{\max},1}\left\{\overline{K_0}(t)-G(t)\right\}\\
&= \max\left\{\max_{t=0,1}\left\{\overline{K_0}(t)-G(t)\right\},\ \max_{t=t_{\min},t_{\max}}\left\{\overline{K_0}(t)-G(t)\right\}\right\}\\
&= \max\left\{\varepsilon-\min_{t=0,1}\left\{G(t)\right\},\ \max_{t=t_{\min},t_{\max}}\left\{\overline{K_0}(t)-G(t)\right\}\right\}\\
&= \max\left\{\varepsilon-\min_{t=0,1}\left\{G(t)\right\},\ \max_{t=t_{\min},t_{\max}}\left\{K_0(t)-G(t)\right\}\right\}
\end{aligned}
$$

Noting that $t_{\min}$ and $t_{\max}$ belong to the set V_s , we now have

$$
\begin{aligned}
\sup_{t\in V}\left\{K_0(t)-G(t)\right\} &= \max\left\{\sup_{t\in V_s}\left\{K_0(t)-G(t)\right\},\sup_{t\in V_u}\left\{K_0(t)-G(t)\right\}\right\}\\
&\le \max\left\{\sup_{t\in V_s}\left\{K_0(t)-G(t)\right\},\varepsilon-\min_{t=0,1}\left\{G(t)\right\},\ \max_{t=t_{\min},t_{\max}}\left\{K_0(t)-G(t)\right\}\right\}\\
&= \max\left\{\sup_{t\in V_s}\left\{K_0(t)-G(t)\right\},\varepsilon-\min\left\{G(0),G(1)\right\}\right\}
\end{aligned}
$$

Clearly, if $\sup_{t\in V_s}\left\{K_0(t)-G(t)\right\}\ge\varepsilon-\min\left\{G(0),G(1)\right\}$, then

$\sup_{t\in V_s}\left\{K_0(t)-G(t)\right\}\ge\sup_{t\in V_u}\left\{K_0(t)-G(t)\right\}$ and it follows $\sup_{t\in V}\left\{K_0(t)-G(t)\right\}\le\sup_{t\in V_s}\left\{K_0(t)-G(t)\right\}$.

Also, if $\sup_{t\in V_s}\left(K_0(t)-G(t)\right)<\varepsilon-\min\left\{G(0),G(1)\right\}$ we clearly have that

$\sup_{t\in V}\left\{K_0(t)-G(t)\right\}\le\varepsilon-\min\left\{G(0),G(1)\right\}$. This proves Lemma 3.3.

APPENDIX B: R FUNCTIONS

1. Finding the Best combo of (a,c) where P(B crosses G) = $\alpha/2$. The function maxfind outputs the supremum value $B - G$ for each value of c in the coef input.

```
# our initial b will be (ε) eps
# G(t) = coef * sqrt(t(1-t))
coef <- seq(1,5,by=0.01)
eps <- 1/100000

maxfind <- function(coef,eps) {
result <- NULL
b <- eps
temp1 <- runif(1)
temp2 <- rchisq(1,3)
ihoriz <- temp2*((1-sqrt(temp1))^2)/(b^2)
ivert <-     temp2*(1-sqrt(temp1))/b
bstar <- ivert – b*ihoriz
trihoriz <- 1/ihoriz
trivert <- ivert/ihoriz
k <- rpois(1,-log(eps/bstar))
trhoriz <- NULL
trvert <- NULL
if (k>0) {
trslopes <- bstar*exp(-sort(-log(eps/bstar)*runif(k)))
temp1 <- rchisq(k,1)
```

```
trhoriz <- cumsum(temp1/(trslopes^2)) + trihoriz
trvert <- cumsum(temp1/trslopes) + trivert
}
trhoriz <- c(trihoriz,trhoriz)
trvert <- c(trivert,trvert)
bbt <- c(0,rev(1/(1+trhoriz)),1)
bby <- c(eps,rev(trvert/(1+trhoriz)),eps)
for (j in 1:length(coef)) {
result <- c(result,max(bby – coef[j]*sqrt(bbt*(1-bbt))))
}
result
}
```

2. Pick a value for a sample size *n* and pick a one-sided coverage probability (either 0.95, 0.975, 0.995) for the confidence band and Draw an empirical CDF (data from a Unif(0,1)) with confidence bands formed via Kolmogorov-Smirnov and our proposed method. The area between the bounds from the old method, the area between the bands for our new method, and the percentage difference is also outputted.

```
n <- 50
p <- 0.95
data <- sort(runif(n))
plot(c(0,data,1),c(0,(1:n)/(n+1),1), type="l", xlim=c(0,1), ylim=c(0,1), xlab="",ylab="", col = 2)
yabove <- pmin(1,((1:n)/(n+1))+(sqrt(log(1/(1-p))/2)/sqrt(n+2)))
```

```
ybelow <- pmax(0,((1:n)/(n+1))-(sqrt(log(1/(1-p))/2)/sqrt(n+2)))
points(c(0,rep(data, each = 2),1),rep(c(yabove,1),each = 2), type = "l", lty=1, col = 4)
points(c(0,rep(data, each = 2),1),rep(c(0,ybelow), each = 2), type = "l", lty=1, col = 4)
areaold <- sum((c(data,1)-c(0,data))*(c(yabove,1)-c(0,ybelow)))
if (p==0.950) {
a <- 0.294
b <- 2.10           }
if (p==0.975) {
a <- 0.2655
b <- 2.44           }
if (p==0.995) {
a <- 0.23
b <- 3.06           }
yabove <- pmin(1,((1:n)/(n+1))+(a+(b*sqrt(data*(1-data))))/sqrt(n+2))
ybelow <- pmax(0,((1:n)/(n+1))-(a+(b*sqrt(data*(1-data))))/sqrt(n+2))
points(c(0,rep(data, each = 2),1),rep(c(yabove,1),each = 2), type = "l", lty=1, col = 5)
points(c(0,rep(data, each = 2),1),rep(c(0,ybelow), each = 2), type = "l", lty=1, col = 5)
points(c(0,1),c(0,1),type="l",lty=2)
areanew <- sum((c(data,1)-c(0,data))*(c(yabove,1)-c(0,ybelow)))
savings <- (1 - (areanew/areaold))*100
title("Empirical cdf with 90% joint confidence bands (n=50)")
areaold
areanew
savings
```