ABSTRACT

THE SPECTRAL THEOREM FOR SELF-ADJOINT OPERATORS

by

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The Spectral Theorem for Self-Adjoint Operators allows one to define what it means to evaluate a function on the operator for a large class of functions defined on the spectrum of the operator. This is done by developing a functional calculus that extends the intuitive notion of evaluating a polynomial on an operator. The Spectral Theorem is fundamentally important to operator theory and has applications in many fields, especially harmonic analysis on locally compact abelian groups. This thesis represents a merging of two traditional treatments of the Spectral Theorem and includes an extended example highlighting an important application in harmonic analysis.

THE SPECTRAL THEOREM FOR SELF-ADJOINT OPERATORS

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CHAPTER 1: Introduction

Suppose that A is an element of some algebra over the complex numbers. Then it is natural to define $p(A)$ for any complex polynomial p. Are there other classes of functions defined on $\mathbb{C}$ for which $f(A)$ makes sense? If so, can we define $f(A)$ in a way that is consistent with operations such as addition and multiplication? Even for the case where A is a linear transformation on a vector space, the result is nontrivial, particularly when the vector space is infinite dimensional. The Spectral Theorem addresses this problem by extending the definition of $f(A)$ from polynomials to a broader class of functions. The Spectral Theorem is a fundamental result in operator theory and more generally C^*-algebra theory, with applications to Fourier analysis and abstract harmonic analysis in general. In this thesis, we fully develop the spectral theorem for the special case when A is a self-adjoint operator on a Hilbert space. We start with only the axioms of Hilbert spaces and develop the theory of operators on these spaces. The development of the general case for elements of some commutative C^*-algebra is outlined, and it is shown how our development may be obtained as a corollary of this more general theory.

The spectral theorem originally arose in the context of operator theory where the proof depended on well known results from measure theory, namely the Riesz Representation Theorem, and direct calculation. Later a generalization of the spectral theorem in the context of C^*-algebras arose using Gelfand Theory. In this thesis a merging of the two approaches is presented in the special case of the algebra generated by a single self-adjoint operator.

In Chapter 2 the necessary background knowledge concerning Hilbert spaces and operators is developed. In section 2.1 the Spectral Theorem in finite dimensional spaces is discussed as motivation for the infinite dimensional analogue. In section

2.2 basic facts and definitions for Hilbert spaces are reviewed, and examples are given, including $l^2(\mathbb{Z})$. In section 2.3 the properties of subspaces and orthogonality in Hilbert spaces are reviewed. In particular, the direct sum of a subspace and its orthogonal complement is introduced. Section 2.4 contains characterizations of bounded (continuous) operators and their inverses.

In section 2.5 the adjoint of an operator is defined, and its existence is proved as a consequence of the Riesz Representation Theorem. The example of $l^2(\mathbb{Z})$ is continued with a proof that the left and right shift operators are mutual adjoints. In section 2.6 orthogonal projections are introduced and developed. The useful characterization that an operator is orthogonal if and only if it is both idempotent and self-adjoint is presented. In section 2.7 the basics of spectra are investigated. Characterizations of the spectrum of special classes of operators, especially self-adjoint, are developed, and fundamental results, such as the fact that the spectrum is compact and non-empty, are proved. It is shown that the spectrum of the left and right shift operators on $l^2(\mathbb{Z})$ is the unit circle. The spectral radius is defined and investigated as well as how polynomials interact with the spectrum.

In Chapter 3 the Spectral Theorem for Self-Adjoint Operators is developed, culminating in an application computing the spectral measure for the right shift operator on $l^2(\mathbb{Z})$. In section 3.1 the functional calculus for continuous functions on the spectrum is developed by extending the natural definition of applying a polynomial to an operator. It is shown that this definition depends only on how the polynomial — and hence the continuous function — behaves on the spectrum, naturally highlighting why the spectrum is important. In section 3.2 the Riesz Representation Theorem is used to express the functional calculus in terms of integration with respect to a measure on the spectrum. This definition is then extended to include bounded measurable functions by relaxing the topologies in each space. It is proved that this association is a $*$-homomorphism

In section 3.3 the Spectral Theorem for Self-Adjoint Operators is presented. It is shown that the measures obtained in section 3.2 give rise to a projection valued measure. Then the general theory of integration with respect to a projection valued measure is developed, and the Spectral Theorem is proved for the case of self-adjoint operators. In section 3.4 the Spectral Theorem for Normal Operators is presented. In section 3.5 the development of the spectral theorem in the context of C^*-algebras is outlined. Further, it is shown that our result may be obtained as a corollary of this more general theory. In section 3.6 the Spectral Theorem is worked out in the case of the right shift operator on $l^2(\mathbb{Z})$. The spectral measure is explicitly computed, and the Spectral Theorem's statement verified. It is shown that this example is a special case of a general result in harmonic analysis on locally compact abelian groups.

CHAPTER 2: Hilbert Spaces

2.1 Motivation in Finite Dimensions

Consider a linear map A on an n-dimensional complex vector space V. Then we may express A as an $n \times n$ complex matrix. From finite dimensional linear algebra we know that if A is Hermitian it is diagonalizable with the diagonal entries being the eigenvalues of A. For example, consider a 3×3 matrix A with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Then using the appropriate choice of basis we have

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This expression in terms of coordinates is equivalent to writing the linear map A as a linear combination of the projections P_{λ_i} onto the eigenspace of λ_i for $i = 1, 2, 3$. That is,

$$A = \lambda_1 P_{\lambda_1} + \lambda_2 P_{\lambda_2} + \lambda_3 P_{\lambda_3}.$$

Since the product of two diagonal matrices is a diagonal matrix, if we consider the set of all functions f whose domain includes $\lambda_1, \lambda_2, \lambda_3$, we see that we can define

$$f(A) = \begin{bmatrix} f(\lambda_1) & 0 & 0 \\ 0 & f(\lambda_2) & 0 \\ 0 & 0 & f(\lambda_3) \end{bmatrix}$$

in a way that behaves well with respect to addition and multiplication of functions. That is $(f + g)(A) = f(A) + g(A)$ and $(fg)(A) = f(A)g(A)$. Equivalently,

$$f(A) = f(\lambda_1)P_{\lambda_1} + f(\lambda_2)P_{\lambda_2} + f(\lambda_3)P_{\lambda_3}.$$

In general, if a self-adjoint operator A on an n-dimensional space has eigenvalues $\lambda_1, \ldots, \lambda_m$, where $m \leq n$ and $P_{\lambda_1}, \ldots, P_{\lambda_m}$ are the projections onto the corresponding eigenspaces, then we have

$$A = \sum_{i=1}^{m} \lambda_i P_{\lambda_i}.$$

If a function f is defined for each λ_i then we may define

$$f(A) = \sum_{i=1}^{m} f(\lambda_i) P_{\lambda_i}.$$

The Spectral Theorem for Self-Adjoint Operators extends this result to infinite dimensional spaces.

2.2 Notation and Assumptions

The rest of this chapter is influenced by the treatments in [11] and [2].

Definition 2.1. Let V be a complex vector space. A Hermitian inner product on V is a function that assigns, to every ordered pair of vectors x and y in V, a complex number $\langle x, y \rangle$ so that for all $x, y, z \in V$ and $\alpha, \beta \in \mathbb{C}$:

1. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$;
2. $\langle x, \alpha y + \beta z \rangle = \overline{\alpha} \langle x, y \rangle + \overline{\beta} \langle x, z \rangle$;
3. $\langle x, x \rangle \geq 0$ with $\langle x, x \rangle = 0$ if and only if $x = 0$;
4. $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

By inner product, we mean a Hermitian inner product. If a space V has an inner product, it is well known that the inner product gives a natural norm $\|x\| = \sqrt{\langle x, x \rangle}$, which further gives a metric d on V defined by $d(x, y) = \|x - y\|$.

Definition 2.2. A Hilbert space $\mathcal{H}$ is a complex vector space together with an inner product $\langle \ , \ \rangle$ so that $\mathcal{H}$ is complete with respect to the metric induced by the inner product.

When we write $\mathcal{H}$ we mean a separable Hilbert space, that is, a Hilbert space with a countable dense subset. This is equivalent to $\mathcal{H}$ having a countable basis.

The following proposition may be found in [2].

Proposition 2.3. *The Cauchy-Schwarz inequality holds for any Hilbert space $\mathcal{H}$, that is, $|\langle x, y\rangle| \leq \|x\|\|y\|$ for all $x, y \in \mathcal{H}$.*

Notice that with respect to this topology the inner product $\langle , \rangle$ on $\mathcal{H}$ is continuous as a map from $\mathcal{H} \times \mathcal{H}$ to $\mathbb{C}$ for any Hilbert space $\mathcal{H}$.

Example 2.4. We see that $\mathbb{C}^n$ is a Hilbert space under the usual product $\langle x, y\rangle = \sum_{i=1}^n x_i \overline{y_i}$ where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$.

Example 2.5. Denote by $l^2(\mathbb{Z})$ the space of all square summable sequences $\alpha = (\alpha_k)_{k\in\mathbb{Z}}$ where $\alpha_k \in \mathbb{C}$ for all $k \in \mathbb{Z}$ and

$$\sum_{k\in\mathbb{Z}} |\alpha_k|^2 < \infty.$$

Then under the inner product

$$\langle \alpha, \beta\rangle = \sum_{k\in\mathbb{Z}} \alpha_k \overline{\beta_k},$$

$l^2(\mathbb{Z})$ is a Hilbert space.

Example 2.6. Let μ be Lebesgue measure on $[0, 1]$ and let $L^2([0, 1])$ be the set of all complex valued functions on $[0, 1]$ that are square-integrable with respect to Lebesgue measure. That is,

$$L^2(\mathbb{R}) = \{f : [0,1] \to \mathbb{C} : \int |f(x)|^2 \, d\mu < \infty\}.$$

It is well known that $L^2([0,1])$ is a Hilbert space with respect to the usual inner product

$$\langle f, g \rangle = \int f\overline{g} \, d\mu.$$

2.3 Subspaces and Orthogonality

Definition 2.7. A subset $W \subset \mathcal{H}$ is a subspace of $\mathcal{H}$ if W is a closed vector subspace of $\mathcal{H}$. That is, $W \subset \mathcal{H}$ is a subspace of $\mathcal{H}$ if W is a Hilbert space under the same operations as $\mathcal{H}$.

Given two subspaces W_1, W_2 it is natural to investigate the sum of those two subspaces $W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$.

Definition 2.8. If $W_1, W_2 \subset \mathcal{H}$ are subspaces satisfying

1. $W_1 \cap W_2 = \{0\}$;

2. $\mathcal{H} = \overline{W_1 + W_2}$;

then we say that $\mathcal{H}$ is the direct sum of W_1, W_2 denoted

$$\mathcal{H} = W_1 \oplus W_2.$$

If $\mathcal{H}$ is the direct sum of two subspaces W_1, W_2 we can think of $\mathcal{H}$ as being decomposed into the two smaller Hilbert spaces W_1, W_2.

The inner product allows us to generalize the notion of orthogonality in $\mathbb{R}^n$ to arbitrary Hilbert spaces.

Definition 2.9. Let $x, y \in \mathcal{H}$ with $x, y \neq 0$, we say that x, y are orthogonal (or perpendicular) if $\langle x, y\rangle = 0$. In this case, we write $x \perp y$.

Lemma 2.10. *If $x, y \in \mathcal{H}$ with $x \perp y$ then*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

Proof. Since $x \perp y$, we have

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&= \langle x, x, \rangle + \langle y, y\rangle \\
&= \|x\|^2 + \|y\|^2.
\end{aligned}$$

□

Notice the similarity between Lemma 2.10 and the Pythagorean Theorem in $\mathbb{R}^n$.

Definition 2.11. If $W \subset \mathcal{H}$ is a subspace, we define the orthogonal complement $W^\perp$ by

$$W^\perp = \{v \in \mathcal{H} : \langle w, v\rangle = 0 \text{ for all } w \in W\}.$$

Lemma 2.12. *If W is a subspace of $\mathcal{H}$, then $W^\perp$ is a subspace.*

Proof. If $x, y \in W^\perp$ and $\alpha \in \mathbb{C}$ then for all $w \in W$ we have

$$\langle w, \alpha x + y\rangle = \overline{\alpha}\langle w, x\rangle + \overline{\alpha}\langle w, y\rangle = 0.$$

Hence $\alpha x + y \in W^\perp$ and $W^\perp$ is a vector subspace. Let (x_n) be a Cauchy sequence in $W^\perp$. Then there exists $x \in \mathcal{H}$ so that $x = \lim x_n$. Therefore by the continuity of the inner product for all $w \in W$, we have

$$\langle w, x \rangle = \langle w, \lim x_n \rangle = \lim \langle w, x_n \rangle = 0$$

and so $x \in W^\perp$. Therefore $W^\perp$ is closed and thus is a subspace of $\mathcal{H}$. □

Proposition 2.13. *If W is a subspace of $\mathcal{H}$ then*

$$\mathcal{H} = W \oplus W^\perp = W + W^\perp.$$

Proof. If $w \in W \cap W^\perp$ then we have $\langle w, w \rangle = 0$, and hence $w = 0$.

It is clear that $W + W^\perp$ is a vector subspace of $\mathcal{H}$. We now verify that $W + W^\perp$ is closed. Let $z_n = w_n + x_n$ be a Cauchy sequence, where $w_n \in W, x_n \in W^\perp$. Then for $n, m \in \mathbb{N}$ we have

$$\begin{aligned} \|z_n - z_m\|^2 &= \|(w_n - w_m) + (x_n - x_m)\|^2 \\ &= \|w_n - w_m\|^2 + \|x_n - x_m\|^2 \end{aligned}$$

by Lemma 2.10. Thus $(w_n), (x_n)$ are Cauchy sequences. Hence there exists $w \in W$, $x \in W^\perp$ so that $\lim w_n = w, \lim x_n = x$. Therefore

$$\lim z_n = w + x \in W + W^\perp,$$

hence $W + W^\perp$ is closed.

To see that $W + W^\perp = \mathcal{H}$ consider $x \in (W + W^\perp)^\perp$. Then $\langle w, x \rangle = 0$ for all $w \in W$ so $x \in W^\perp$. On the other hand we have $\langle y, x \rangle = 0$ for all $y \in W^\perp$ so $x \in (W^\perp)^\perp$. Hence $x \in W^\perp \cap (W^\perp)^\perp = \{0\}$ so $x = 0$. Therefore $W + W^\perp = \mathcal{H}$ as needed. □

2.4 Basics of Operators

Linear transformations are the natural functions to consider on a vector space. However to guarantee continuity on infinite dimensional Hilbert spaces we need a more restrictive condition.

Definition 2.14. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces with norms $\|\cdot\|_1, \|\cdot\|_2$. A linear transformation $A : \mathcal{H}_1 \to \mathcal{H}_2$ is bounded if and only if there exists $C > 0$ so that

$$\|Ax\|_2 \leq C\|x\|_1$$

for all $x \in \mathcal{H}$.

If a linear transformation A is bounded we may define the norm of A, $\|A\|$ as follows.

Definition 2.15. Given a bounded linear transformation $A : \mathcal{H}_1 \to \mathcal{H}_2$ we define the operator norm

$$\|A\|_{\text{op}} = \inf\{C > 0 : \|Ax\|_2 \leq C\|x\|_1 \text{ for all } x \in \mathcal{H}\}.$$

When the context is clear we write $\|A\|$ for $\|A\|_{\text{op}}$.

Lemma 2.16. *Let A be a linear transformation, then for all $x, y \in \mathcal{H}$ we have*

$$4\langle Ax, y\rangle = \langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle + i\langle A(x+iy), x+iy\rangle - i\langle A(x-iy), x-iy\rangle.$$

Proof. Direct calculation of the right-hand side yields the desired result. $\square$

Proposition 2.17. *Let* $A : \mathcal{H}_1 \to \mathcal{H}_2$ *be a bounded linear transformation. Then*

$$\|A\| = \sup\{\|Ax\|_2 : \|x\|_1 = 1\}.$$

Further if $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ *then*

$$\|A\| = \sup\{|\langle Ax, y\rangle| : \|x\| = \|y\| = 1\}.$$

Proof. Let $r = \sup\{\|Ax\|_2 : \|x\|_1 = 1\}$. Then for all $x \in \mathcal{H}_1$ with $\|x\|_1 = 1$ we have

$$\|Ax\|_2 \leq \|A\|\|x_1\| = \|A\|$$

so $r \leq \|A\|$. On the other hand for any $x \in \mathcal{H}_1$ we have

$$\|Ax\|_2 = \left\| \|x\|_1 A\left(\frac{x}{\|x\|_1}\right)\right\|_2 \leq r\|x\|_1,$$

hence $\|A\| \leq r$ and

$$\|A\| = \sup\{\|Ax\|_2 : \|x\|_1 = 1\}.$$

If $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ then we claim that

$$\|Ax\| = \sup\{|\langle Ax, y\rangle : \|y\| = 1\}$$

for all $x \in \mathcal{H}$. Indeed by the Cauchy-Schwarz inequality we have

$$|\langle Ax, y\rangle| \leq \|Ax\|\|y\| = \|Ax\|$$

for $\|y\| = 1$. On the other hand for $Ax \neq 0$ we have

$$\left\langle Ax, \frac{Ax}{\|Ax\|} \right\rangle = \|Ax\|.$$

Therefore

$$\|A\| = \sup\{\|Ax\| : \|x\| = 1\} = \sup\{|\langle Ax, y\rangle| : \|x\| = \|y\| = 1\}.$$

□

The importance of studying bounded linear transformations is evident from the following proposition.

Proposition 2.18. *A linear function $A : \mathcal{H}_1 \to \mathcal{H}_2$ is continuous if and only if it is bounded.*

Proof. Let A be continuous and assume that A is unbounded. Then for each $n \in \mathbb{N}$ there exists a unit vector $x_n \in \mathcal{H}_1$ so that $\|A(x_n)\|_2 \geq n$. Then $\frac{1}{n}x_n \to 0$ in $\mathcal{H}_1$ so $A\left(\frac{1}{n}x_n\right) \to A(0) = 0$ in $\mathcal{H}_2$. But for each n

$$\left\|A\left(\frac{1}{n}x_n\right)\right\|_2 = \frac{1}{n}\|Ax_n\|_2 \geq 1$$

a contradiction. Therefore A must be bounded.

Conversely, suppose that $A \neq 0$ is bounded. Then given $\varepsilon > 0$ let $\delta = \frac{\varepsilon}{\|A\|}$. Then for all $x, y \in \mathcal{H}_1$ such that $\|x - y\|_1 \leq \delta$ we have

$$\|Ax - Ay\|_2 = \|A(x - y)\| \leq \|A\|\|x - y\|_1 \leq \varepsilon.$$

Therefore A is continuous as needed. □

Definition 2.19. An operator A on a Hilbert space $\mathcal{H}$ is a bounded linear transformation $A : \mathcal{H} \to \mathcal{H}$ and the set of all operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$.

Example 2.20. If $\mathcal{H}$ is finite dimensional then every linear transformation $A : \mathcal{H} \to \mathcal{H}$ is an operator. Indeed let $\{e_1, \dots, e_n\}$ be an orthonormal basis for $\mathcal{H}$. Then for all $x = \sum_{i=1}^n \alpha_i e_i \in \mathcal{H}$ with $\|x\| = 1$ we have

$$\|Ax\| = \left\| \sum_{i=1}^n \alpha_i A e_i \right\| \leq \sum_{i=1}^n |\alpha_i| \|Ae_i\| \leq \sum_{i=1}^n \|Ae_i\| < \infty$$

since each $|\alpha_i| \leq 1$. Hence A is bounded with $\|A\| \leq \sum_{i=1}^n \|Ae_i\|$.

Definition 2.21. Let $\varphi : [0,1] \to \mathbb{C}$ be a function. Then we write

$$\|\varphi\|_\infty = \sup\{|\varphi(x)| : x \in [0,1]\}$$

if the supremum exists.

Example 2.22. Example from [10]. Consider $\mathcal{H} = L^2([0,1])$. Then let $\varphi : [0,1] \to \mathbb{C}$ be continuous and define $M_\varphi : L^2([0,1]) \to L^2([0,1])$ by $M_\varphi(f) = \varphi f$. Then M_φ is bounded with $\|M_\varphi\| = \|\varphi\|_\infty$. Indeed, we have $|\varphi(x)| \leq \|\varphi\|_\infty$ so

$$\|M_\varphi(f)\|^2 = \int_0^1 |\varphi(x)|^2 |f(x)|^2 \, d\mu \leq \|\varphi\|_\infty^2 \int_0^1 |f(x)|^2 \, d\mu = \|\varphi\|_\infty^2 \|f\|^2.$$

Hence we have that M_φ is bounded with $\|M_\varphi\| \leq \|\varphi\|_\infty$. Since φ continuous and $[0,1]$ is compact there exists $x_0 \in [0,1]$ so that $\varphi(x_0) = \|\varphi\|_\infty$. Suppose first that $x_0 \neq 0, 1$. Thus given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ so that $|x - x_0| < \delta_\varepsilon$ implies $|\varphi(x) - \varphi(x_0)| < \varepsilon$, that is $\varphi(x_0) - \varepsilon < \varphi(x)$. Let $\Delta_\varepsilon = \{x \in [0,1] : |x - x_0| < \delta_\varepsilon\}$ and define

$$f_\varepsilon = \frac{1}{\sqrt{2\delta_\varepsilon}} \chi_{\Delta_\varepsilon},$$

where $\chi_{\Delta_\varepsilon}$ is the characteristic function for Δ_ε. Then $\|f_\varepsilon\| = 1$ so we have

$$\|M_\varphi\|^2 \geq \|M_\varphi f_\varepsilon\|^2 = \frac{1}{2\delta_\varepsilon} \int_{\Delta_\varepsilon} |\varphi|^2 \, d\mu \geq (\varphi(x_0) - \varepsilon)^2.$$

Hence letting $\varepsilon \to 0$ we have

$$\|M_\varphi\| \geq \varphi(x_0) = \|\varphi\|_\infty$$

as needed. If $x_0 = 0, 1$ then the same process may be used but with

$$f_\varepsilon = \frac{1}{\sqrt{\delta_\varepsilon}} \chi_{\Delta_\varepsilon}.$$

Example 2.23. Consider $\mathcal{H} = l^2(\mathbb{Z})$ and define

$$R, L : \mathcal{H} \to \mathcal{H}$$

by

$$(R\alpha)_k = \alpha_{k-1}, (L\alpha)_k = \alpha_{k+1}$$

for each $\alpha = (\alpha_k)_{k\in\mathbb{Z}} \in \mathcal{H}$. We say that R and L are the right and left shift operators on $l^2(\mathbb{Z})$ respectively. Then both R and L are bounded linear operators of norm 1. Indeed for all $\alpha = (\alpha_k)_{k\in\mathbb{Z}}, \beta = (\beta_k)_{k\in\mathbb{Z}} \in \mathcal{H}$ and $\lambda \in \mathbb{C}$ we have

$$\begin{aligned}(R(\lambda\alpha + \beta))_k &= (\lambda\alpha + \beta)_{k-1} \\ &= \lambda\alpha_{k-1} + \beta_{k-1} \\ &= \lambda(R\alpha)_k + (R\beta)_k\end{aligned}$$

for each $k \in \mathbb{Z}$. Hence R is linear.

Now for all $\alpha = (\alpha_k)_{k\in\mathbb{Z}} \in \mathcal{H}$ we have

$$\begin{aligned}\|R\alpha\|^2 &= \sum_{k\in\mathbb{Z}} |(R\alpha)_k|^2 \\ &= \sum_{k\in\mathbb{Z}} |\alpha_{k-1}|^2 \\ &= \sum_{k\in\mathbb{Z}} |\alpha_k|^2 \\ &= \|\alpha\|^2\end{aligned}$$

and so R is a bounded with $\|R\| \leq 1$. Further if $\|\alpha\| = 1$ then from the above calculation we see that

$$\|R\alpha\| = \|\alpha\| = 1$$

so

$$\|R\| = \sup\{\|R\alpha\| : \|\alpha\| = 1\} = 1$$

as claimed. We similarly see that L is an operator with $\|L\| = 1$.

Definition 2.24. An operator $A \in \mathcal{B}(\mathcal{H})$ is invertible if A^{-1} exists and is in $\mathcal{B}(\mathcal{H})$.

Remark 2.25. Notice that if A is continuous and invertible then A^{-1} is continuous by the open mapping theorem. Therefore by Proposition 2.18 it is enough for and operator A to be invertible in the usual sense.

Proposition 2.26. *If $A \in \mathcal{B}(\mathcal{H})$ then A is invertible if and only if the range of A is dense and there exists $\alpha > 0-$ so that*

$$\|Ax\| \geq \alpha\|x\|$$

for all $x \in \mathcal{H}$.

Proof. If A is invertible, then A is onto so the range of A is certainly dense. Given $x \in \mathcal{H}$ we have

$$\|x\| = \|A^{-1}Ax\| \leq \|A^{-1}\|\|Ax\|$$

so taking $\alpha = \frac{1}{\|A^{-1}\|} > 0$ we have

$$\alpha\|x\| \leq \|Ax\|$$

for all $x \in \mathcal{H}$.

Conversely, suppose that the range of A is dense and there exists $\alpha > 0$ so that $\|Ax\| \geq \alpha\|x\|$ for all $x \in \mathcal{H}$. We first show that the range of A is closed. Let $\{Ax_n\}$ be a Cauchy sequence in the range of A. Then there exists some $y \in \mathcal{H}$ so that $\lim Ax_n = y$. Now for all $n, m \in \mathbb{N}$ we have

$$\|Ax_n - Ax_m\| = \|A(x_n - x_m)\| \geq \alpha\|x_n - x_m\|$$

so

$$\|x_n - x_m\| \to 0$$

as $n, m \to \infty$ and $\{x_n\}$ is a Cauchy sequence. Therefore there exists $x \in \mathcal{H}$ with $\lim x_n = x$. But since A is continuous we have $y = \lim Ax_n = Ax$ and so the range of A is closed, that is A is onto.

Now let $x_1, x_2 \in \mathcal{H}$ with $Ax_1 = Ax_2$. Then

$$0 = \|Ax_1 - Ax_2\| \geq \alpha\|x_1 - x_2\|$$

so $\|x_1 - x_2\| = 0$ and $x_1 = x_2$. Hence we have that A is one-to-one, and the inverse function A^{-1} exists. We now verify that $A^{-1} \in \mathcal{B}(\mathcal{H})$. The linearity of A^{-1} follows from that of A. For all $x \in \mathcal{H}$, we have

$$\|x\| = \|AA^{-1}x\| \geq \alpha\|A^{-1}x\|$$

so

$$\|A^{-1}x\| \leq \frac{1}{\alpha}\|x\|$$

and $A^{-1} \in \mathcal{B}(\mathcal{H})$ as desired. □

Corollary 2.27. *Suppose that*

$$inf\left\{\frac{\|Ax\|}{\|x\|} : \|x\| \neq 0\right\} = 0$$

then A is not invertible.

Proposition 2.28. *If $A, B \in \mathcal{B}(\mathcal{H})$, $\alpha \in \mathbb{C}$ and $\alpha A, A + B$ and $AB = A \circ B$ are all defined in the usual way then*

1. *$\alpha A \in \mathcal{B}(\mathcal{H})$ with $\|\alpha A\| = |\alpha|\|A\|$;*

2. *$A + B \in \mathcal{B}(\mathcal{H})$ with $\|A + B\| \leq \|A\| + \|B\|$;*

3. *$AB \in \mathcal{B}(\mathcal{H})$ with $\|AB\| \leq \|A\|\|B\|$.*

Proof. Clearly $\alpha A, A + B, AB$ are all linear.

1. For all $x \in \mathcal{H}$, we have

$$\|\alpha Ax\| = \|A(\alpha x)\| \leq \|A\|\|\alpha x\| = |\alpha|\|A\|\|x\|$$

and hence $\|\alpha A\| \leq |\alpha|\|A\|$. If $\alpha = 0$ then we have $|\alpha|\|A\| = 0 \leq 0 = \|\alpha A\|$.

Now for $\alpha \neq 0$ and $v \in \mathcal{H}$

$$\|Ax\| = \left\|\frac{\alpha}{\alpha}Ax\right\| = \frac{1}{|\alpha|}\|\alpha Ax\| \leq \frac{1}{|\alpha|}\|\alpha A\|\|x\|$$

so as x is arbitrary $\|A\| \leq \frac{1}{|\alpha|}\|\alpha A\|$ or equivalently $|\alpha|\|A\| \leq \|\alpha A\|$.

2. For all $x \in \mathcal{H}$, we have

$$\|(A+B)x\| = \|Ax+Bx\| \leq \|Ax\|+\|Bx\| \leq \|A\|\|x\|+\|B\|\|x\| = (\|A\|+\|B\|)\|x\|$$

and so $\|A + B\| \leq \|A\| + \|B\|$.

3. For all $x \in \mathcal{H}$, we have

$$\|ABx\| \leq \|A\|\|Bx\| \leq \|A\|\|B\|\|x\|$$

giving $\|AB\| \leq \|A\|\|B\|$ as needed.

□

It is clear that under these operations $\mathcal{B}(\mathcal{H})$ is a complex vector space. Further we see that $\mathcal{B}(\mathcal{H})$ is an algebra with unity, namely the identity map $I : \mathcal{H} \to \mathcal{H}$.

2.5 Adjoint of an Operator

Definition 2.29. A linear functional on $\mathcal{H}$ is a bounded (that is continuous) linear map from $\mathcal{H}$ to $\mathbb{C}$.

It is clear that the set of all linear functionals on a Hilbert space is itself a vector space. The following fundamental result shows that there is a one-to-one correspondence between $\mathcal{H}$ and the set of all bounded linear functionals on $\mathcal{H}$.

Theorem 2.30 (The Riesz Representation Theorem for Bounded Linear Functionals)**.** *A linear map f from $\mathcal{H}$ to $\mathbb{C}$ is bounded if and only if there exists $y \in \mathcal{H}$ so that*

$$f(x) = \langle x, y \rangle$$

for all $x \in \mathcal{H}$. Further if such a y exists then it is unique and $\|f\|_{op} = \|y\|$.

Proof. If f is identically zero then we may take $y = 0$ and the results are trivial. Hence we now assume that f is nonzero.

First assume that f is bounded. Observe that $\mathrm{Ker}(f)$ has co-dimension 1 in $\mathcal{H}$. Now choose $w \in (\mathrm{Ker}(f))^{\perp}$ with $\|w\| = 1$ and let $\lambda = f(w)$. Note that since $w \notin \mathrm{Ker}(f)$ we have $\lambda \neq 0$. Then for each $x \in \mathcal{H}$, since $\mathrm{Ker}(f)$ is a subspace, we have

$$x = \alpha w + x'$$

where $x' \in \mathrm{Ker}(f)$ by Proposition 2.13. Further we see that $\alpha = \langle x, w \rangle$. Hence

$$\begin{aligned} f(x) &= \alpha f(x) + f(x') \\ &= \alpha f(x) \\ &= \langle x, w \rangle \lambda \\ &= \langle x, \overline{\lambda} w \rangle. \end{aligned}$$

Thus we may take $y = \overline{\lambda} w$.

Conversely, if for some $y \in \mathcal{H}$ we have the function $f : \mathcal{H} \to \mathbb{C}$ given by $f(x) = \langle x, y \rangle$ for all $x \in \mathcal{H}$ then we see that

$$|f(x)| = ||\langle x, y \rangle| \leq \|x\| \|y\|$$

and so f is bounded as needed.

Now suppose that there exist $y_1, y_2 \in \mathcal{H}$ so that

$$f(x) = \langle x, y_1 \rangle = \langle x, y_2 \rangle$$

for all $x \in \mathcal{H}$. Then $\langle x, y_1 - y_2 \rangle = 0$ for all $x \in \mathcal{H}$ and so $y_1 - y_2 = 0$. That is $y_1 = y_2$ as desired.

Finally, for all $x \in \mathcal{H}$ we have

$$|f(x)| = |\langle x, y \rangle| \leq \|x\| \|y\|$$

so $\|f\|_{\text{op}} \leq \|y\|$. On the other hand, we have

$$\|y\|^2 = \langle y, y \rangle = |\langle y, y \rangle| = f(y) \leq \|f\|_{\text{op}} \|y\|$$

so

$$\|y\| \leq \|f\|_{\text{op}}.$$

Therefore $\|f\|_{\text{op}} = \|y\|$ as claimed. $\square$

The following lemma may be found in [11].

Lemma 2.31. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then $A = B$ if and only if $\langle Ax, y \rangle = \langle Bx, y \rangle$ for all $x, y \in \mathcal{H}$.*

Definition 2.32. A map $\varphi : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ that is linear in the first coordinate and conjugate linear in the second coordinate is called a sesquilinear form on $\mathcal{H}$.

Notice that an inner product is a sesquilinear form.

Theorem 2.33. *Let φ be a continuous sesquilinear form on $\mathcal{H}$ so that there exists $C > 0$ with*

$$|\varphi(x, y)| \leq C\|x\|\|y\|$$

for all $x, y \in \mathcal{H}$. Then there exists a unique operator $A \in \mathcal{B}(\mathcal{H})$ so that

$$\varphi(x, y) = \langle Ax, y\rangle$$

for all $x, y \in \mathcal{H}$. In addition, $\|A\| \leq C$.

Proof. Fix $x \in \mathcal{H}$ and consider the bounded linear function

$$\mathcal{H} \to \mathbb{C}$$

given by

$$y \mapsto \overline{\varphi(x, y)}.$$

Then by the Riesz Representation Theorem 2.30 there exists $z \in \mathcal{H}$ with

$$\overline{\varphi(x, y)} = \langle y, z\rangle.$$

Define $A : \mathcal{H} \to \mathcal{H}$ by $Ax = z$. Then

$$\langle Ax, y\rangle = \overline{\langle y, Ax\rangle} = \varphi(x, y)$$

for all $x, y \in \mathcal{H}$.

Now given $x_1, x_2, y \in \mathcal{H}$ and $\alpha \in \mathbb{C}$ we have

$$\langle A(\alpha x_1 + x_2), y\rangle = \varphi(\alpha x_1 + x_2, y)$$

$$= \alpha\varphi(x_1, y) + \varphi(x_2, y)$$

$$= \alpha\langle Ax_1, y\rangle + \langle Ax_2, y\rangle$$

$$= \langle \alpha Ax_1 + Ax_2, y\rangle.$$

Hence by Lemma 2.31 we have that

$$A(\alpha x_1 + x_2) = \alpha Ax_1 + Ax_2$$

so A is linear.

Now for all $x \in \mathcal{H}$ we have

$$\|Ax\|^2 = |\langle Ax, Ax\rangle|$$

$$= |\varphi(x, Ax)|$$

$$\leq C\|x\|\|Ax\|$$

and so

$$\|Ax\| \leq C\|x\|.$$

Hence $A \in \mathcal{B}(\mathcal{H})$ with $\|A\| \leq C$, as desired. □

Corollary 2.34. *Given an operator $A \in \mathcal{B}(\mathcal{H})$, there is a unique operator $A^* \in \mathcal{B}(\mathcal{H})$ so that*

$$\langle Ax, y\rangle = \langle x, A^*y\rangle$$

for all $x, y \in \mathcal{H}$. We call A^* the adjoint of A and further note that $(A^*)^* = A$ and $\|A^*\| = \|A\|$.

Proof. Observe that the map

$$(y, x) \mapsto \langle y, Ax\rangle$$

is a sesquilinear form, so there exists a unique operator $A^* \in \mathcal{B}(\mathcal{H})$ such that

$$\langle y, Ax\rangle = \langle A^*y, x\rangle,$$

that is,

$$\langle Ax, y\rangle = \langle x, A^*y\rangle.$$

Further, we see that

$$|\langle y, Ax\rangle| \leq \|y\|\|A\|\|x\|$$

and so by Theorem 2.33,

$$\|A^*\| \leq \|A\|.$$

It is clear that $(A^*)^* = A$ and so the same argument shows that

$$\|A\| \leq \|A^*\|,$$

as needed. □

Example 2.35. The adjoint of the left shift operator L on $H = l^2(\mathbb{Z})$ is the right shift operator R.

Proof. For $\alpha = (\alpha_k)_{k\in\mathbb{Z}}, \beta = (\beta_k)_{k\in\mathbb{Z}} \in \mathcal{H}$, we have

$$\begin{aligned}\langle L\alpha, \beta\rangle &= \sum_{k\in\mathbb{Z}} (L\alpha)_k \overline{\beta_k} \\ &= \sum_{k\in\mathbb{Z}} \alpha_{k+1}\overline{\beta_k} \\ &= \sum_{k\in\mathbb{Z}} \alpha_k \overline{\beta_{k-1}} \\ &= \sum_{k\in\mathbb{Z}} \alpha_k \overline{(R\beta)_k}\end{aligned}$$

$$= \langle \alpha, R\beta \rangle$$

thus $L^* = R$. Notice that this also gives $R^* = L$. □

Proposition 2.36. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and let $\alpha \in \mathbb{C}$. Then the following hold:*

1. $(\alpha A)^* = \overline{\alpha} A^*$
2. $(A + B)^* = A^* + B^*$
3. $(AB)^* = B^* A^*$
4. *If A is invertible then $(A^{-1})^* = (A^*)^{-1}$.*

Proof. We prove (2) and (3), the others are similar.

2. For all $x, y \in \mathcal{H}$ we have

$$\langle (A + B)x, y \rangle = \langle Ax, y \rangle + \langle Bx, y \rangle = \langle x, A^* y \rangle + \langle x, B^* y \rangle = \langle x, (A^* + B^*)y \rangle.$$

Therefore $(A + B)^* = A^* + B^*$ as claimed.

3. For all $x, y \in \mathcal{H}$ we have

$$\langle ABx, y \rangle = \langle Bx, A^* y \rangle = \langle x, B^* A^* y \rangle.$$

Therefore $(AB)^* = B^* A^*$.

□

The following proposition may be found in [12].

Proposition 2.37. *If $A \in \mathcal{B}(\mathcal{H})$ then $\|A^* A\| = \|A\|^2$.*

Proof. We have that $\|A^*A\| \leq \|A^*\|\|A\| = \|A\|^2$ since $\|A^*\| = \|A\|$ (Corollary 2.34). Conversely, we have

$$\begin{aligned}\|A\|^2 &= \sup_{\|x\|=1} \|Ax\|^2 \\ &= \sup_{\|x\|=1} \langle Ax, Ax\rangle \\ &= \sup_{\|x\|=1} \langle A^*Ax, x\rangle \\ &\leq \|A^*A\|\end{aligned}$$

since $|\langle A^*Ax, x\rangle| \leq \|A^*A\|\|x\|^2 = \|A^*A\|$ for $\|x\| = 1$. □

Definition 2.38. We say that an operator A is **Hermitian** or **self-adjoint** if $A = A^*$ and that A is **unitary** if $U^* = U^{-1}$, further we say that A is **normal** if $A^*A = AA^*$.

Definition 2.39. An operator A is positive, denoted $A \geq 0$, if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$.

Notice that the identity operator I is Hermitian. We need a few elementary properties of adjoints.

2.6 Projections

We now consider a very special type of operator, the orthogonal projection.

Definition 2.40. Given a subspace $W \subset \mathcal{H}$, we define the orthogonal projection P_W onto W as follows. For each $x \in \mathcal{H}$, write

$$x = w + y$$

for the unique elements $w \in W, y \in W^{\perp}$. This is possible by Proposition 2.13. Then define

$$P_W x = w.$$

Proposition 2.41. *For any subspace $W \subset \mathcal{H}$, $P_W \in \mathcal{B}(\mathcal{H})$. If $W \neq \{0\}$ then $\|P_W\| = 1$.*

Proof. Consider any $x = w + y \in W + W^{\perp} = \mathcal{H}$. Since $\|x\|^2 = \|w\|^2 + \|y\|^2$, we have

$$\|P_W x\| = \|w\| \leq \|x\|$$

so $P_W \in \mathcal{B}(\mathcal{H})$ with $\|P_W\| \leq 1$. However, for any $w \in W$

$$\|P_W w\| = \|w\|$$

and so $\|P_W\| \geq 1$. □

Definition 2.42. An operator A is idempotent if $A^2 = A$.

Proposition 2.43. *For every subspace W, P_W is idempotent and Hermitian.*

Proof. Given $x \in \mathcal{H}$, write $x = w + y$ with $w \in W, y \in W^{\perp}$. Then we have

$$P_W^2(x) = P_W(P_W(x)) = P_W(w) = w = P_W(x).$$

Hence P_W is idempotent. Now given $x_1, x_2 \in \mathcal{H}$, write $x_1 = w_1 + y_1, x_2 = w_2 + y_2$ with $w_1, w_2 \in W$ and $y_1, y_2 \in W^{\perp}$. Then we have

$$\begin{aligned} \langle P_W x_1, x_2 \rangle &= \langle w_1, x_2 \rangle \\ &= \langle w_1, w_2 \rangle \end{aligned}$$

$$= \langle x_1, w_2 \rangle$$
$$= \langle x_1, P_W x_2 \rangle.$$

Hence $P_W^* = P_W$ and P_W is Hermitian, as claimed. □

Notice that if P is the orthogonal projection onto a subspace W then we have

$$W = \text{Im}(P) = \{x \in \mathcal{H} : P(x) = x\}.$$

Thus we see that for each subspace $W \subset \mathcal{H}$ there is an associated idempotent, Hermitian operator, namely P_W.

Proposition 2.44. *If $P \in \mathcal{B}(\mathcal{H})$ is idempotent and Hermitian then P is the orthogonal projection onto the subspace Im(P) and*

$$Im(P)^\perp = Ker(P).$$

Proof. First, we show that $\text{Im}(P)^\perp = \text{Ker}(P)$. Given $y \in \text{Ker}(P)$, for all $w \in \text{Im}(P)$ we have

$$\langle w, y \rangle = \langle Pw, y \rangle = \langle w, Py \rangle = 0$$

since $Py = 0$. Hence $y \in \text{Im}(P)^\perp$ and $\text{Ker}(P) \subset \text{Im}(P)^\perp$. Conversely, given $y \in \text{Im}(P)^\perp$ for all $x \in \mathcal{H}$ we have

$$0 = \langle Px, y \rangle = \langle x, Py \rangle.$$

Since x is arbitrary we have $Py = 0$ and $y \in \text{Ker}(P)$, hence $\text{Im}(P)^\perp = \text{Ker}(P)$.

To see that P is an orthogonal projection onto $\text{Im}(P)$, let $w \in \text{Im}(P)$ and $x \in \mathcal{H}$.

We have

$$\begin{aligned}\langle w, x - Px\rangle &= \langle w, x\rangle - \langle w, Px\rangle \\ &= \langle w, x\rangle - \langle Pw, x\rangle \\ &= 0.\end{aligned}$$

Thus $x - Px \in \mathrm{Im}(P)^{\perp}$ for all $x \in \mathcal{H}$. Hence for all $x \in \mathcal{H}$ we have

$$x = Px + (x - Px) \in \mathrm{Im(P)} \oplus (\mathrm{Im(P)})^{\perp}$$

and we see that P is indeed the orthogonal projection onto the image of P. □

That is there is a one-to-one correspondence between idempotent Hermitian operators and subspaces.

2.7 The Spectrum of an Operator

Definition 2.45. Given $A \in \mathcal{B}(\mathcal{H})$ we define the spectrum of A to be

$$\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not invertible}\}$$

and the resolvent set of A as

$$\rho(A) = \mathbb{C} \backslash \sigma(A).$$

Notice that in finite dimensions the spectrum of an operator A is precisely the set of eigenvalues of A. Most of the proofs and examples in this section are from [10].

Example 2.46. Consider the Hilbert space $H = L^2([0, 1])$ and let φ be a continuous function on $[0, 1]$ with M_φ the multiplication operator on H introduced in Example

2.22. Then $\sigma(M_\varphi) = \varphi([0,1])$.

Proof. First, suppose that $\lambda = \varphi(x_0)$ for some $x_0 \in [0,1]$. Then, since φ is continuous, for each $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ so that $x \in [0,1]$ and $|x - x_0| < \delta_\varepsilon$ implies $|\varphi(x) - \lambda| < \varepsilon$. Let $a = \min\{(x_0 - \delta_\varepsilon), 0\}$ and $b = \max\{(x_0 + \delta_\varepsilon), 1\}$. Now there exists a non-zero function $f_\varepsilon \in H$ so that f_ε is supported on $[a,b]$. Hence we have

$$\begin{aligned}
\|(M_\varphi - \lambda)f_\varepsilon\|^2 &= \int |(\varphi(x) - \lambda)f_\varepsilon(x)|^2 \, dx \\
&= \int |\varphi(x) - \lambda|^2 |f_\varepsilon(x)|^2 \, dx \\
&\leq \varepsilon^2 \|f_\varepsilon\|^2.
\end{aligned}$$

Since ε is arbitrary we have that $M_\varphi - \lambda$ is not invertible by Corollary 2.27 and therefore $\lambda \in \sigma(M_\varphi)$.

Conversely, suppose that $\lambda \notin \varphi([0,1])$. Thus $\frac{1}{\varphi(x)-\lambda}$ is defined for all $x \in [0,1]$. Given $f \in H$, we have $(M_\varphi - \lambda)((\varphi - \lambda)^{-1} f) = f$ and $M_\varphi - \lambda$ is onto. Further since $[0,1]$ is compact — and hence closed — there must exist $\alpha > 0$ so that $|\varphi(x) - \lambda| > \alpha$ for all $x \in [0,1]$. Hence for any $f \in H$ we have

$$\|(M_\varphi - \lambda)f\|^2 = \int |\varphi(x) - \lambda|^2 |f(x)|^2 \, dx \geq \alpha \|f\|^2$$

so by Proposition 2.26 $M_\varphi - \lambda$ is invertible and $\lambda \notin \sigma(M_\varphi)$. Therefore $\sigma(M_\varphi) = \varphi([0,1])$ as needed. $\square$

Proposition 2.47. *Let $A \in \mathcal{B}(\mathcal{H})$, then*

$$\sigma(A^*) = \overline{\sigma(A)}.$$

Proof. We have

$$A^* - \lambda = (A - \overline{\lambda})^*.$$

Hence $A^* - \lambda$ is invertible if and only if $A - \overline{\lambda}$ is invertible. Therefore

$$\sigma(A^*) = \overline{\sigma(A)}.$$

□

Corollary 2.48. *If $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint then*

$$\sigma(A) \subset \mathbb{R}.$$

Proof. Indeed

$$\sigma(A) = \overline{\sigma(A^*)} = \overline{\sigma(A)}$$

hence

$$\sigma(A) \subset \mathbb{R}.$$

□

The following proposition may be found in [1].

Proposition 2.49. *If A is a positive operator then $\sigma(A) \subset \{x \in \mathbb{R} : x \geq 0\}$.*

Proof. First, notice that since A is positive, that is $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$, we have

$$\langle Ax, x\rangle = \overline{\langle Ax, x\rangle} = \langle x, Ax\rangle$$

for all $x \in \mathcal{H}$. Then using Lemma 2.16, for all $x, y \in \mathcal{H}$ we have

$$\begin{aligned} 4\langle Ax, y\rangle &= \langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle + i\langle A(x+iy), x+iy\rangle - i\langle A(x-iy), x-iy\rangle \\ &= \langle x+y, A(x+y)\rangle - \langle x-y, A(x-y)\rangle + i\langle x+iy, A(x+iy)\rangle - i\langle x-iy, A(x-iy)\rangle \\ &= 4\langle x, Ay\rangle. \end{aligned}$$

Thus

$$\langle Ax, y\rangle = \langle x, Ay\rangle$$

and A is self-adjoint. Hence by Corollary 2.48 we have $\sigma(A) \subset \mathbb{R}$.

For all $x \in \mathcal{H}$, we have

$$\begin{aligned} -\lambda\|x\|^2 &= \langle -\lambda x, x\rangle \\ &\leq \langle -\lambda x, x\rangle + \langle Ax, x\rangle \\ &= \langle (A-\lambda)x, x\rangle \\ &\leq \|(A-\lambda)x\|\|x\|. \end{aligned}$$

Hence for $\|x\| \neq 0$ we have

$$\|(A-\lambda)x\| \geq -\lambda\|x\|$$

and so the relation holds for all $x \in \mathcal{H}$. From the proof of Proposition 2.26 we see that $A-\lambda$ is one-to-one for all $\lambda < 0$. Let $y \in \mathcal{H}$ be perpendicular to the range of $A-\lambda$. Then for all $x \in \mathcal{H}$, we have

$$\begin{aligned} 0 &= \langle (A-\lambda)x, y\rangle \\ &= \langle x, (A-\lambda)^* y\rangle \\ &= \langle x, (A-\lambda)y\rangle, \end{aligned}$$

that is $(A-\lambda)y = 0$. Thus since $A-\lambda$ is one-to-one we have $y = 0$ and the range of $A-\lambda$ is dense. Hence by Proposition 2.26 we have that $A-\lambda$ is invertible so $\lambda \notin \sigma(A)$. Therefore $\sigma(A) \subset \{x \in \mathbb{R} : x \geq 0\}$ as claimed. □

Lemma 2.50. *If $A \in \mathcal{B}(\mathcal{H})$ then $\sigma(A)$ is bounded by $\|A\|$.*

Proof. Suppose $\lambda > \|A\|$. Then $\left\|\frac{1}{\lambda}A\right\| < 1$, so

$$\left(1 - \frac{1}{\lambda}A\right)^{-1} = \sum_{i=0}^{\infty} \frac{1}{\lambda^n} A^n$$

with convergence in norm. Thus since $A-\lambda = -\lambda\left(1 - \frac{1}{\lambda}A\right)$, we have $A-\lambda$ invertible, that is, $\lambda \notin \sigma(A)$. Therefore if $\lambda \in \sigma(A)$ then $|\lambda| \leq \|A\|$, as claimed. □

Definition 2.51. Given $A \in \mathcal{B}(\mathcal{H})$ we define the spectral radius $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$.

Lemma 2.52. *Given $\lambda_0 \in \rho(A)$, there exists an open set U containing λ_0 so that if $\lambda \in U$ then $A-\lambda$ is invertible. Further $(A-\lambda)^{-1}$ is given by a convergent power series about λ_0.*

Proof. For $\lambda_0 \in \rho(A)$ we have that $(A-\lambda_0)$ is invertible. Now let

$$U = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \frac{1}{\|(A-\lambda_0)^{-1}\|}\}.$$

Then for $\lambda \in U$,

$$A - \lambda = (A-\lambda_0)(1 - (\lambda-\lambda_0)(A-\lambda_0)^{-1})$$

with

$$(1 - (\lambda-\lambda_0)(A-\lambda_0)^{-1})^{-1} = \sum_{n=0}^{\infty} (\lambda-\lambda_0)^n \left((A-\lambda_0)^{-1}\right)^n$$

which converges since $\|(\lambda-\lambda_0)(A-\lambda_0)^{-1}\| < 1$. Thus $A-\lambda$ is invertible and $(A-\lambda)^{-1}$ is given by a convergent powers series about λ_0. □

Corollary 2.53. *For each $A \in \mathcal{B}(\mathcal{H})$, $\sigma(A)$ is a compact subset of $\mathbb{C}$.*

Proof. Putting together the preceding two lemmas we have that $\sigma(A)$ is bounded and closed, therefore by the Heine-Borel Theorem it is compact. □

Proposition 2.54. *If A is unitary then $\sigma(A)$ and $\sigma(A^{-1}) = \sigma(A^*)$ are subsets of the unit circle in $\mathbb{C}$.*

Proof. First, since $\|A\| = 1$ and $\|A^*\| = \|A^{-1}\| = 1$ we have that the spectra of A and A^* are contained in the unit disk. Now for all $\lambda \neq 0$, we have

$$A - \lambda = -A(A^{-1} - \lambda^{-1})\lambda$$

and so

$$(A - \lambda)^{-1} = -\lambda^{-1}(A^{-1} - \lambda^{-1})^{-1}A^{-1}.$$

Thus we see that $\lambda \in \sigma(A)$ if and only if $\lambda^{-1} \in \sigma(A^{-1})$. Since both $\sigma(A), \sigma(A^{-1})$ are contained in the unit disk this is only possible if both $\sigma(A)$ and $\sigma(A^{-1})$ are contained in the unit circle. □

Example 2.55. Consider $\mathcal{H} = l^2(\mathbb{Z})$ with L, R the left and right shift operators introduced in Example 2.23. Then

$$\sigma(L) = \sigma(R) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

Proof. Indeed, since $LL^* = LR = I$ we have that L, R are unitary and so $\sigma(L), \sigma(R)$ are contained in the unit circle. We now show that the unit circle is contained in

$\sigma(R)$; the case for L is similar. Let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Then for each $k \in \mathbb{N}$, consider the vector $v^k \in \mathcal{H}$ given by

$$v^k = \sum_{n=0}^{k} \lambda^{-n} e_n.$$

Then

$$Rv^k = \sum_{n=0}^{k} \lambda^{-n} e_{n-1} = \sum_{n=1}^{k+1} \lambda^{-n+1} e_n$$

and

$$\lambda v^k = \sum_{n=0}^{k} \lambda^{-n+1} e_n.$$

Hence we see that

$$(R - \lambda)v^k = \lambda^{-k} e_{k+1} - \lambda e_0$$

and so

$$\|(R - \lambda)v^k\| = \sqrt{2}.$$

However

$$\|v^k\| = \sqrt{k+1}.$$

Thus

$$\frac{\|(R-\lambda)v^k\|}{\|v^k\|} = \frac{\sqrt{2}}{\sqrt{k+1}} \to 0$$

as $k \to \infty$. By Corollary 2.27 we see that $R - \lambda$ is not invertible, that is $\lambda \in \sigma(R)$. $\square$

Definition 2.56. Let $A \in \mathcal{B}(\mathcal{H})$, then we define the resolvent function

$$R : \rho(A) \to \mathcal{B}(\mathcal{H})$$

by

$$\lambda \mapsto R_\lambda = (A - \lambda)^{-1}.$$

Notice that by the computation in the proof of Lemma 2.52 we see that

$$\lambda \mapsto R_\lambda$$

is analytic on $\rho(A)$.

Proposition 2.57. *Let $A \in \mathcal{B}(\mathcal{H})$ with $\mathcal{H} \neq \{0\}$. Then $\sigma(A) \neq \emptyset$.*

Proof. Assume by way of contradiction that $\sigma(A) = \emptyset$. Then R_λ is defined for all $\lambda \in \mathbb{C}$. For all $x, y \in \mathcal{H}$ consider the function

$$f_{x,y} : \mathbb{C} \to \mathbb{C}$$

defined by

$$f_{x,y}(\lambda) = \langle R_\lambda x, y \rangle.$$

Since for each $\lambda_0 \in \rho(A) = \mathbb{C}$ there exists an open set U with $R|_U$ given by a convergent power series around λ_0 it follows that $f_{x,y}$ is analytic on U. That is, $f_{x,y}$ is entire for each $x, y \in \mathcal{H}$. Let $\lambda > \|A\|$. Then since $\|\lambda^{-1} A\| < 1$, we have

$$\begin{aligned}
\|R_\lambda\| &= |\lambda|^{-1} \|(1 - \lambda^{-1} A)^{-1}\| \\
&= |\lambda|^{-1} \left\| \sum_{n=0}^{\infty} (\lambda^{-1} A)^n \right\| \\
&\leq |\lambda|^{-1} \sum_{n=0}^{\infty} \|\lambda^{-1} A\|^n \\
&= |\lambda|^{-1} \frac{1}{1 - \|\lambda^{-1} A\|} \\
&= \frac{1}{|\lambda| - \|A\|} \\
&\to 0
\end{aligned}$$

as $\lambda \to \infty$. We claim that $f_{x,y}$ is bounded for each x, y. Indeed for all $\lambda \in \mathbb{C}$ we have

$$|f_{x,y}(\lambda)| = |\langle R_\lambda x, y\rangle| \leq \|R_\lambda\|\|x\|\|y\| \to 0$$

as $\lambda \to \infty$. Thus $f_{x,y}$ is bounded and entire and so by Liouville's Theorem is constant. Since $f_{x,y} \to 0$ as $|\lambda| \to \infty$ we must have

$$0 = f_{x,y}(\lambda) = \langle R_\lambda x, y\rangle$$

for all $\lambda \in \mathbb{C}$ and $x, y \in \mathcal{H}$. However this implies that $R_\lambda x = 0$ for all $x \in \mathcal{H}$, a contradiction since R_λ is invertible. Therefore $\sigma(A) \neq \emptyset$.

□

Proposition 2.58. *For all $A \in \mathcal{B}(\mathcal{H})$ we have*

$$r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}.$$

Proof. Let $A \in \mathcal{B}(\mathcal{H})$ and consider $f(z) = -R_{z^{-1}}$, defined for all $z \in \mathbb{C}\backslash\{0\}$ with $z^{-1} \in \rho(A)$. In particular, for $|z| < \|A\|^{-1}$ we have $|z|^{-1} > \|A\|$ so $f(z)$ satisfies

$$f(z) = (z^{-1} - A)^{-1} = z(1 - zA)^{-1} = z\sum_{n=0}^{\infty}(zA)^n.$$

Notice that we may extend f analytically to include $z = 0$ by defining $f(0) = 0$, so f is given by a power series centered at 0. Then from complex analysis, we recall that the radius of convergence R for this series is the distance from 0 to the nearest singularity, that is, the distance to $\sigma(A)$. Then we have

$$R = \inf\{|\lambda| : \lambda^{-1} \in \sigma(A)\}$$

$$= \inf\{|\lambda|^{-1} : \lambda \in \sigma(A)\}$$
$$= \frac{1}{r(A)}.$$

Further by Hadamard's Theorem we have

$$R^{-1} = \limsup_{n\to\infty} \|A^n\|^{\frac{1}{n}}.$$

Thus $r(A) = \limsup \|A^n\|^{\frac{1}{n}}$.

Now for each $n \in \mathbb{N}$, we have

$$A^n - \lambda^n = (A - \lambda) \sum_{i=0}^{n-1} A^i \lambda^{n-1-i}.$$

Hence if $A^n - \lambda^n$ is invertible then $A - \lambda$ is invertible, or equivalently if $\lambda \in \sigma(A)$ then $\lambda^n \in \sigma(A^n)$. Therefore by Lemma 2.50 we have

$$|\lambda|^n = |\lambda^n| \leq \|A^n\|$$

for all $\lambda \in \sigma(A)$ so $|\lambda| \leq \|A^n\|^{\frac{1}{n}}$. Thus $r(A) \leq \|A^n\|^{\frac{1}{n}}$ for all $n \in \mathbb{N}$. Therefore $r(A) \leq \liminf_{n\to\infty} \|A^n\|^{\frac{1}{n}}$. Hence we have

$$\limsup_{n\to\infty} \|A^n\|^{\frac{1}{n}} = r(A) \leq \liminf_{n\to\infty} \|A^n\|^{\frac{1}{n}}$$

so $\lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$ exists and is equal to $r(A)$. □

Notice that in Example 2.46, we have $\sigma(M_\varphi) = \varphi([0,1])$, so certainly $\sigma(M_\varphi)$ is bounded by $\|M_\varphi\| = \|\varphi\|_\infty$. However, even more holds: we have $\|M_\varphi\| = \sup\{|\lambda| : \lambda \in \sigma(M_\varphi)\}$. This useful property generalizes to any normal operator.

The following proof may be found in [1].

Proposition 2.59. *If $A \in \mathcal{B}(\mathcal{H})$ is normal then*

$$r(A) = \|A\|.$$

Proof. First assume that A is self-adjoint. Then we have

$$\|A\|^2 = \|AA^*\| = \|A^2\|$$

and

$$\|A\|^4 = \left(\|A\|^2\right)^2 = \|A^2\|^2 = \|A^2(A^2)^*\| = \|A^4\|.$$

Continuing this we see that for each $n \in \mathbb{N}$ we have

$$\|A\|^{2^n} = \|A^{2^n}\|$$

and hence that

$$r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|A^{2^n}\|^{\frac{1}{2^n}} = \|A\|.$$

Now using the normality of A we have

$$\begin{aligned}\|(AA^*)^{2^n}\| &= \|A^{2^n}(A^*)^{2^n}\| \\ &= \|A^{2^n}(A^{2^n})^*\| \\ &= \|A^{2^n}\|^2.\end{aligned}$$

If A is normal then AA^* is self-adjoint so

$$\begin{aligned}\|A\|^2 &= \|AA^*\| \\ &= r(AA^*)\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \|(AA^*)^{2^n}\|^{\frac{1}{2^n}} \\
&= \lim_{n\to\infty} \left(\|A^{2^n}\|^2\right)^{\frac{1}{2^n}} \\
&= \lim_{n\to\infty} \left(\|A^{2^n}\|^{\frac{1}{2^n}}\right)^2 \\
&= r(A)^2.
\end{aligned}
$$

Therefore $\|A\| = r(A)$ as claimed. □

Definition 2.60. Let $\mathcal{P}(\mathbb{C})$ be the set of all polynomials with complex coefficients viewed as functions on $\mathbb{C}$.

Notice that given an operator $A \in \mathcal{B}(\mathcal{H})$ and a polynomial $p(x) = \sum_{i=0}^{n} a_i x^i \in \mathcal{P}(\mathbb{C})$, we may define $p(A) \in \mathcal{B}(\mathcal{H})$ in the obvious way, namely

$$p(A) = \sum_{i=0}^{n} a_i A^i,$$

with the usual understanding that $A^0 = I$.

Proposition 2.61. *Let $A \in \mathcal{B}(\mathcal{H})$ and $p(x) \in \mathcal{P}(\mathbb{C})$. Then $\sigma(p(A)) = p(\sigma(A))$.*

Proof. Suppose that $p(x)$ has degree n, $\lambda \in \mathbb{C}$ and consider the polynomial $p(x) - \lambda$. Then we may factor $p(x) - \lambda$ as

$$p(x) - \lambda = a \prod_{j=1}^{n} (x - \lambda_j)$$

where $a, \lambda_j \in \mathbb{C}$ for $j = 1, \ldots, n$. Then we have

$$p(A) - \lambda = a \prod_{j=1}^{n} (A - \lambda_j).$$

Hence we see that $p(A) - \lambda$ is invertible if and only if $A - \lambda_j$ is invertible for each

j. In particular, we have $\lambda \in \sigma(p(A))$ if and only if $A - \lambda_j$ is not invertible for some $j = 1, \ldots, n$, that is, if and only if $\lambda_j \in \sigma(A)$ for some j. But in this case we have

$$p(\lambda_j) - \lambda = 0$$

so

$$\lambda = p(\lambda_j) \in p(\sigma(A)).$$

Therefore $\sigma(p(A)) = p(\sigma(A))$ as claimed. $\square$

Proposition 2.62. *Let $A \in \mathcal{B}(\mathcal{H})$ be a self adjoint operator and let $p(x) \in \mathbb{C}[x]$. Then $p(A) \in \mathcal{B}(\mathcal{H})$ is normal and*

$$\|p(A)\|_{op} = \sup\{|p(\lambda)| : \lambda \in \sigma(A)\}.$$

Proof. Let $p(x) = \sum_{i=0}^{n} a_i x^i$ with $a_i \in \mathbb{C}$ for $i = 0, \ldots, n$. Then using the properties of an adjoint, we see that

$$\begin{aligned}
(p(A))^* p(A) &= \left(\sum_{i=0}^{n} \overline{a_i} A^i\right)\left(\sum_{j=0}^{n} a_j A^j\right) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n} \overline{a_i} a_j A^i A^j \\
&= \sum_{j=0}^{n}\sum_{i=0}^{n} a_j A^j \overline{a_i} A^i \\
&= \left(\sum_{j=0}^{n} a_j A^j\right)\left(\sum_{i=0}^{n} \overline{a_i} A^i\right) \\
&= p(A)(p(A))^*.
\end{aligned}$$

Thus $p(A)$ is normal and

$$\|p(A)\| = r(p(A)).$$

However from Proposition 2.61

$$r(p(A)) = \sup\{|\lambda| : \lambda \in \sigma(p(A))\} = \sup\{|p(\lambda)| : \lambda \in \sigma(A)\}$$

so

$$\|p(A)\| = \sup\{|p(\lambda)| : \lambda \in \sigma(A)\}$$

as claimed. $\square$

CHAPTER 3: Spectral Theorem for Self-Adjoint Operators

Throughout this chapter we take $A \in \mathcal{B}(\mathcal{H})$ to be self-adjoint unless otherwise noted and we write $\sigma(A) = \Sigma$.

Definition 3.1. A $*$-homomorphism (respectively, $*$-isomorphism) is a map $\varphi : B(\mathbb{C}) \to \mathcal{B}(\mathcal{H})$ so that φ is an algebra homomorphism (respectively, isomorphism) and

$$\varphi\left(\overline{f}\right) = \varphi(f)^*.$$

3.1 Functional Calculus for Continuous Functions

We have seen that given an operator A and a polynomial p we can naturally define the operator $p(A)$ in a way that behaves well with respect to the spectrum. In this section, we generalize this result to arbitrary continuous functions on the spectrum in the case that A is a self-adjoint operator.

Definition 3.2. Let $\mathcal{P}(\Sigma)$ denote the set of all polynomials in $\mathcal{P}(\mathbb{C})$ restricted to Σ. We define the norm $\|\cdot\|_\infty$ on $\mathcal{P}(\Sigma)$ in the usual way and write

$$\|p\|_\Sigma = \|p|_\Sigma\|_\infty = \sup_{\lambda \in \Sigma} |p(\lambda)|.$$

We show that our definition of $p(A)$ depends only on $p|_\Sigma$ so that our map from $\mathcal{P}(\mathbb{C})$ to $\mathcal{B}(\mathcal{H})$ restricts to a well defined map from $\mathcal{P}(\Sigma)$ to $\mathcal{B}(\mathcal{H})$.

Proposition 3.3. *Let $p \in \mathcal{P}(\mathbb{C})$. Then $p(A) = 0$ if and only if $p|_\Sigma \equiv 0$.*

Proof. Note that $p(A) = 0$ if and only if $\|p(A)\| = 0$. But

$$\|p(A)\| = \sup\{|p(\lambda)| : \lambda \in \Sigma\},$$

so $\|p(A)\| = 0$ if and only if $p(\lambda) = 0$ for all $\lambda \in \Sigma$, giving the desired result. $\square$

Corollary 3.4. *If $p, q \in \mathcal{P}(\mathbb{C})$ then $p|_\Sigma = q|_\Sigma$ if and only if $p(A) = q(A) \in \mathcal{B}(\mathcal{H})$.*

Proof. Indeed $p - q$ is a polynomial defined on $\mathbb{C}$. By the previous proposition we have

$$(p - q)|_\Sigma = 0$$

if and only if

$$p(A) - q(A) = 0$$

giving the desired result. $\square$

Hence we now have an injection

$$\varphi : \mathcal{P}(\Sigma) \hookrightarrow \mathcal{B}(\mathcal{H})$$

given by

$$\varphi(p) = p(A).$$

Viewing $\mathcal{P}(\Sigma)$ as a normed algebra under the norm $\|\cdot\|_\Sigma$, we also have that φ preserves norms. We summarize:

Proposition 3.5. *The map $\varphi : \mathcal{P}(\Sigma) \to \mathcal{B}(\mathcal{H})$ is an isometry and a $*$-isomorphism onto its image in $\mathcal{B}(\mathcal{H})$.*

Definition 3.6. Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint. We define $\langle A \rangle$ to be the closed subalgebra generated by A and I. In particular, $\langle A \rangle$ contains $p(A)$ for every polynomial $p \in \mathcal{P}(\Sigma)$.

Proposition 3.7. *The map φ extends uniquely to an isometric $*$-isomorphism $\varphi : C(\Sigma) \to \langle A \rangle$. For $f \in C(\Sigma)$, we denote $\varphi(f) = f(A)$.*

Proof. Since Σ is compact and the polynomials separate points, it follows from the Stone-Weierstrass theorem that $\mathcal{P}(\Sigma)$ is dense in the set $C(\Sigma)$ of continuous functions on Σ. Since $\varphi : \mathcal{P}(\Sigma) \to \mathcal{B}(\mathcal{H})$ is an isometry we may extend φ uniquely by continuity to

$$\varphi : C(\Sigma) \to \langle A \rangle.$$

That is, given $f \in C(\Sigma)$ there exists a sequence of polynomials $\{p_n\} \subset \mathcal{P}(\Sigma)$ so that $p_n \to f$ in the $\|\cdot\|_\infty$ norm. Then we define $f(A) = \lim p_n(A) \in \langle A \rangle$ where convergence here is in the operator norm. That φ is still an isomorphism preserving norms follows directly since convergence is uniform in $C(\Sigma)$. □

This association of continuous maps on Σ with operators in $\langle A \rangle \subset \mathcal{B}(\mathcal{H})$ is known as the functional calculus of the operator.

3.2 Extending the Functional Calculus

The development in this section is inspired by [8] and [4].

We have the injection

$$C(\Sigma) \to \mathcal{B}(\mathcal{H})$$

given by

$$f \mapsto f(A).$$

We wish to extend this to define $f(A)$ whenever $f \in B(\Sigma)$ is bounded and measurable. Notice that $C(\Sigma)$ is dense in $B(\Sigma)$ under the topology of pointwise bounded convergence. That is, for every $f \in \mathcal{B}(\Sigma)$, there exists a sequence (f_n) with each $f_n \in C(\Sigma)$ with a uniform bound and $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in \Sigma$.

Definition 3.8. Let X be a compact Hausdorff space. Then $\text{Radon}(X)$ is the normed space of signed Radon measures on X under the total variation norm, and $C(X)$ is the linear space of continuous real valued functions on X under the maximum norm.

The following result may be found in [6] page 464.

Theorem 3.9 (Riesz Representation Theorem for the Dual of $C(X)$)**.** *Define the operator $T : Radon(X) \to [C(X)]^*$ by setting, for $\nu \in Radon(X)$, and $f \in C(X)$,*

$$T_\nu(f) = \int_X f \; d\nu$$

Then T is a linear isometric isomorphism of $Radon(X)$ onto $[C(X)]^*$.

Corollary 3.10. *Let $x, y \in \mathcal{H}$. Then there exists a unique regular finite Borel measure $\mu_{x,y}$ on Σ so that*

$$\langle f(A)x, y\rangle = \int f \; d\mu_{x,y}$$

for all $f \in C(\Sigma)$. Further we have

$$\mu_{x,y}(M) \leq \|x\|\|y\|$$

for all Borel sets $M \subset \Sigma$.

Proof. For each $f \in C(\Sigma)$, we have

$$|\langle f(A)x, y\rangle| \leq \|f(A)\|\|x\|\|y\| = \|f\|_\Sigma\|x\|\|y\| < \infty.$$

So the map $F : C(\Sigma) \to \mathbb{C}$, given by $f \mapsto \langle f(A)x, y\rangle$ is a continuous linear functional. Since Σ is compact and Hausdorff, the Riesz Representation Theorem (Theorem 3.9) implies that there exists a unique regular finite Borel measure $\mu_{x,y}$ on Σ so that

$$\langle f(A)x, y\rangle = \int f \; d\mu_{x,y}$$

for all $f \in C(\Sigma)$. Further this measure satisfies $\mu_{x,y}(\Sigma) = \|F\| \leq \|x\|\|y\|$. Therefore for every Borel set $M \subset \Sigma$ we have

$$\mu_{x,y}(M) \leq \mu_{x,y}(\Sigma) \leq \|x\|\|y\|$$

as needed. □

Proposition 3.11. *For each Borel set $M \subset \Sigma$, the map $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$ given by $(x, y) \mapsto \mu_{x,y}(M)$ is linear in the first coordinate and conjugate linear in the second coordinate. Further we have*

$$\mu_{x,y}(M) = \overline{\mu_{y,x}(M)}.$$

Proof. Let $x_1, x_2, y \in \mathcal{H}$ and $\lambda \in \mathbb{C}$. Then for all $f \in C(\Sigma)$ we have

$$\begin{aligned}\int f \ d\mu_{\lambda x_1 + x_2, y} &= \langle f(A)(\lambda x_1 + x_2), y\rangle \\ &= \lambda \langle f(A)x_1, y\rangle + \langle f(A)x_2, y\rangle \\ &= \lambda \int f \ d\mu_{x_1,y} + \int f \ d\mu_{x_2,y} \\ &= \int f \ d(\lambda\mu_{x_1,y} + \mu_{x_2,y}).\end{aligned}$$

Therefore, since f is arbitrary, we have

$$\mu_{\lambda x_1 + x_2, y}(M) = \lambda\mu_{x_1,y}(M) + \mu_{x_2,y}(M)$$

as needed. Similarly we see that the map is conjugate linear in the second coordinate.

Let $f \in C(\Sigma)$. Then since $f \mapsto f(A)$ is a $*$-homomorphism we have

$$(f(A))^* = \overline{f}(A).$$

Thus

$$\begin{aligned}\int f \; d\mu_{x,y} &= \langle f(A)x, y\rangle \\ &= \langle x, (f(A))^* y\rangle \\ &= \overline{\langle (f(A))^* y, x\rangle} \\ &= \overline{\int \overline{f} \; d\mu_{y,x}} \\ &= \int f \; d\overline{\mu_{y,x}}.\end{aligned}$$

As f is arbitrary, we then have

$$\mu_{x,y}(M) = \overline{\mu_{y,x}(M)}$$

as needed. □

Given a function $f \in B(\Sigma)$, for all $x, y \in \mathcal{H}$ we see

$$\left| \int f \; d\mu_{x,y} \right| \leq \|f\|_\Sigma \mu_{x,y}(\Sigma) \leq \|f\|_\Sigma \|x\| \|y\|.$$

Hence we see that the map

$$(x, y) \mapsto \int f \; d\mu_{x,y}$$

is a bounded sesquilinear form. By the Riesz Representation Theorem (Theorem

2.30) there exists an operator $f(A) \in \mathcal{B}(\mathcal{H})$ so that

$$\langle f(A)x, y\rangle = \int f \; d\mu_{x,y}.$$

Proposition 3.12. *If (f_n) is a sequence of functions in $B(\Sigma)$ that have a uniform bound and $f_n \to f$ pointwise then $f_n(A) \to f(A)$ in the weak operator topology.*

Proof. Since (f_n) is uniformly bounded it follows from the Dominated Convergence Theorem that

$$\langle f_n(A)x, y\rangle = \int f_n \; d\mu_{x,y} \to \int f \; d\mu_{x,y}$$

for all $x, y \in \mathcal{H}$. Therefore

$$\langle f_n(A)x, y\rangle \to \langle f(A)x, y\rangle$$

for all $x, y \in \mathcal{H}$ as needed. □

Proposition 3.13. *The map*

$$B(\Sigma) \to \mathcal{B}(\mathcal{H})$$

given by

$$f \mapsto f(A)$$

as above is as $*$-homomorphism that extends the functional calculus.

Proof. It is clear that the definition of $f(A)$ agrees with the functional calculus when f is continuous, we now verify that this association is a homomorphism.

By linearity of integration we have that $f \mapsto f(A)$ is linear. We need to show that $(f(A))^* = \overline{f}(A)$ and that $(fg)(A) = f(A)g(A)$ for all $f, g \in B(\Sigma)$.

Let $f \in B(\Sigma)$ and $x, y \in \mathcal{H}$. Then we have

$$\begin{aligned}
\langle \overline{f}(A)x, y\rangle &= \int \overline{f}\ d\mu_{x,y} \\
&= \int \overline{f}\ d\overline{\mu_{y,x}} \\
&= \overline{\int f\ d\mu_{y,x}} \\
&= \overline{\langle f(A)y, x\rangle} \\
&= \langle (f(A))^* x, y\rangle.
\end{aligned}$$

since this holds for all x, y we have $\overline{f}(A) = (f(A))^*$ as needed.

For the second claim,

$$(fg)(A) = f(A)g(A)$$

for all $f, g \in C(\Sigma)$. Thus given $f, g \in B(\Sigma)$ there exists uniformly bounded sequences $(f_n), (g_m)$ with $f_n, g_m \in C(\Sigma)$ for all n, m so that $f_n \to f, g_m \to g$ pointwise as $n, m \to \infty$. Therefore by Proposition 3.12, by taking limits in m and then n respectively we have

$$\begin{aligned}
\langle (f_n g_m)(A)x, y\rangle &= \langle f_n(A) g_m(A)x, y\rangle \\
&\to \langle f_n(A) g(A)x, y\rangle \\
&\to \langle f(A) g(A)x, y\rangle
\end{aligned}$$

for all $x, y \in \mathcal{H}$. But $f_n g_m \to fg$ so

$$\langle (f_n g_m)(A)x, y\rangle \to \langle (fg)(A)x, y\rangle.$$

Hence

$$\langle (fg)(A)x, y\rangle = \langle f(A)g(A)x, y\rangle$$

for all $x, y \in \mathcal{H}$ and $(fg)(A) = f(A)g(A)$. □

3.3 The Spectral Theorem

Recall that $C(\Sigma)$ is dense in $B(\Sigma)$ under pointwise bounded convergence. However, simple functions supported on Borel sets are dense in $B(\Sigma)$ under uniform convergence. That is for each $f \in B(\Sigma)$ there exists a sequence of simple functions (f_n) on Σ that converges uniformly to f.

Definition 3.14. An $\mathcal{H}$-projection valued measure on a Σ is a map P from Borel subsets of Σ to $\mathcal{B}(\mathcal{H})$ so that the following hold for all Borel sets $M, N \subset \Sigma$.

1. Each $P(M)$ is an orthogonal projection.

2. $P(\emptyset) = 0$ and $\mathcal{P}(\Sigma) = I$.

3. $P(M \cap N) = P(M)P(N)$

4. If $\{M_i\}_{i\in\mathbb{N}}$ are disjoint Borel subsets of Σ then

$$P\left(\bigcup_i M_i\right) = \sum_i P(M_i)$$

   where the sum converges in the strong operator topology.

Recall from the last section, we have the homomorphism from $B(\Sigma)$ to $B(\mathcal{H})$ given by $f \mapsto f(A)$, where

$$\langle f(A)x, y\rangle = \int_\Sigma f \; d\mu_{x,y}.$$

Definition 3.15. Define the map P from the set of all Borel subsets of Σ to $\mathcal{B}(\mathcal{H})$ by

$$P(M) = \chi_M(A)$$

for each Borel set $M \subset \Sigma$, where χ_M is the characteristic function of M.

Proposition 3.16. *The map P defined in Definition 3.15 is a projection valued measure.*

Proof. 1. Let $x, y \in \mathcal{H}$, then we have

$$\langle (P(M))^2 x, y\rangle = \int \chi_M^2 \, d\mu_{x,y} = \int \chi_M \, d\mu_{x,y} = \langle P(M)x, y\rangle$$

so $(P(M))^2 = P(M)$ and $P(M)$ is idempotent. Further we have $(P(M))^* = \overline{\chi_M}(A) = \chi_M(A) = P(M)$ so $P(M)$ is Hermitian. Therefore, by Proposition 2.44, $P(M)$ is the orthogonal projection onto its range.

2. For all $x, y \in \mathcal{H}$ we have

$$\langle P(\emptyset)x, y\rangle = \int \chi_\emptyset \, d\mu_{x,y} = 0$$

and

$$\langle P(\Sigma)x, y\rangle = \int \chi_\Sigma \, d\mu_{x,y} = \int 1 \, d\mu_{x,y} = \langle Ix, y\rangle$$

. Thus $P(\emptyset) = 0$ and $P(\Sigma) = I$ as desired.

3. We have that

$$\chi_{M\cap N} = \chi_M \chi_N$$

for every pair $M, N \subset \Sigma$ of Borel sets. Therefore for all $x, y \in \mathcal{H}$ we have

$$\begin{aligned}\langle P(M \cap N)x, y\rangle &= \int \chi_{M\cap N}\ d\mu_{x,y}\\ &= \int \chi_M \chi_N\ d\mu_{x,y}\\ &= \langle P(M)P(N)x, y\rangle\end{aligned}$$

since this holds for all $x, y \in \mathcal{H}$ we have $P(M) = P(N)$ as needed.

4. For each $n \in \mathbb{N}$ let $F_n = \bigcup_{i=1}^{n} M_i$ and let $F = \bigcup_{i=1}^{\infty} M_i$. Then since $\chi_{F_n} = \sum_{i=1}^{n} \chi_{M_i}$ we have

$$P(F_n) = \chi_{F_n}(A) = \sum_{i=1}^{n} \chi_{M_i}(A) = \sum_{i=1}^{n} P(M_i).$$

Since $\chi_{F_n} \to \chi_F$ pointwise and bounded we have that

$$\chi_{F_n}(A) = \sum_{i=1}^{n} P(M_i) \to \chi_F(A) = P(F)$$

weakly by Theorem 3.12.

Also for each $n \in \mathbb{N}$ we see that $F = F_n \cup (F \backslash F_n)$ where $F_n, (F \backslash F_n)$ are disjoint. Thus

$$P(F) = P(F_n) + P(F \backslash F_n)$$

and for all $x \in \mathcal{H}$,

$$\begin{aligned}\|P(F)x - P(F_n)x\|^2 &= \|P(F \backslash F_n)x\|^2\\ &= \langle P(F \backslash F_n)x, P(F \backslash F_n)x\rangle\\ &= \langle P(F \backslash F_n)x, x\rangle\end{aligned}$$

$$= \langle (P(F) - P(F_n))x, x\rangle$$
$$\to 0$$

as $n \to \infty$. Therefore $P(F_n) \to P(F)$ strongly.

□

Proposition 3.17. *For any projection valued measure P and each pair $x, y \in \mathcal{H}$, the map given by*

$$P_{x,y}(M) = \langle P(M)x, y\rangle$$

is a standard complex measure on Σ.

Now let P be any projection valued measure on Σ defined on the σ-algebra of Borel subsets of Σ.

Definition 3.18. Let f be a simple function on Σ. Then we may write

$$f = \sum_{i=1}^{n} \alpha_i \chi_{M_i},$$

where $\alpha_i \in \mathbb{C}$ and $M_i \subset \Sigma$ is measurable for each $i = 0, \ldots, n$ and the family (M_i) is pairwise disjoint. Then we define the integral of f with respect to P as

$$\int f \; dP = \sum_{i=1}^{n} \alpha_i P(M_i).$$

Using this definition we can develop integration of a bounded measurable function in the usual way. To begin, we need a technical result.

Lemma 3.19. *If $f \in B(\Sigma)$ is a simple function then*

$$\left\| \int f \; dP \right\|_{op} \leq \|f\|_\infty.$$

Proof. Write $f = \sum_{i=1}^{n} \alpha_i \chi_{M_i}$ where (M_i) is a disjoint family of Borel subsets of Σ. Then we see that

$$\|f\|_\infty = \sup_{x\in\Sigma} |f(x)| = \max_i |\alpha|_i.$$

Notice that $P(M_j)P(M_i) = P(M_i)P(M_j) = P(M_i \cap M_j) = 0$ by Proposition 3.16, so the range of $P(M_i)$ is orthogonal to the range of $P(M_j)$ for each $i \neq j$. Now for all $x \in \mathcal{H}$ we have

$$\begin{aligned}
\left\| \left(\int f \; dP \right) x \right\|^2 &= \| \sum_{i=1}^{n} \alpha_i P(M_i) x \|^2 \\
&\leq \|f\|_\infty \sum_{i=1}^{n} \|P(M_i)x\|^2 \\
&= \|f\|_\infty \| \sum_{i=1}^{n} P(M_i)x\|^2 \\
&= \|f\|_\infty \left\| P\left(\bigcup_{i=1}^{n} M_i \right) \right\|_{\text{op}} \\
&\leq \|f\|_\infty \|x\|.
\end{aligned}$$

since $\|P\left(\bigcup_{i=1}^{n} M_i\right)\|_{\text{op}} = 1$. Hence we have

$$\left\| \int f \; dP \right\|_{\text{op}} \leq \|f\|_\infty$$

, as claimed. ∎

Proposition 3.20. *Let $f \in B(\Sigma)$ and (f_n) be a sequence of simple functions (f_n) converging uniformly to f. There exists an operator, denoted $\int f \; dP$, so that $\int f_n \; dP$ converges to $\int f \; dP$ in norm.*

Proof. For each $n, m \in \mathbb{N}$, we have

$$\left\| \int f_n \, dP - \int f_m \, dP \right\| \leq \|f_n - f_m\|.$$

Thus since $\|f_n - f_m\| \to 0$ as $n, m \to \infty$, we have

$$\left(\int f_n \, dP \right)$$

is a Cauchy sequence in $\mathcal{B}(\mathcal{H})$ and so there exists an operator $\int f \, dP$ so that $\int f_n \, dP \to \int f \, dP$ in norm. $\square$

Remark 3.21. Notice that the definition of $\int f \, dP$ does not depend on the choice of sequence (f_n).

Theorem 3.22. *The map from $B(\Sigma)$ to $\mathcal{B}(\mathcal{H})$ given by*

$$f \mapsto \int f \, dP$$

is a $*$-homomorphism. Further for all $x, y \in \mathcal{H}$ we have

$$\left\langle \left(\int f \, dP \right) x, y \right\rangle = \int f \, dP_{x,y}.$$

Proof. Let $f \in B(\Sigma)$ and (f_n) be a sequence of simple functions so that $f_n \to f$ uniformly. Since

$$\left\| \int f_n \, dP \right\| \leq \|f_n\|$$

for all n, by taking limits we obtain

$$\left\| \int f \, dP \right\| \leq \|f\| < \infty,$$

so

$$\int f \, dP \in \mathcal{B}(\mathcal{H}).$$

We readily see that the map is linear just as for integration with respect to ordinary measures. Now if

$$f = \sum_{i=1}^{n} \alpha_i \chi_{M_i}, \quad g = \sum_{j=1}^{m} \beta_j \chi_{N_j}$$

are two simple functions we have

$$fg = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j \chi_{M_i} \chi_{N_j} = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j \chi_{M_i \cap N_j}.$$

Hence fg is a simple function and

$$\begin{aligned}
\int fg \, dP &= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_i P(M_i \cap N_j) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_i P(M_i) P(N_j) \\
&= \left(\sum_{i=1}^{n} \alpha_i P(M_i) \right) \left(\sum_{j=1}^{m} \beta_j P(N_j) \right) \\
&= \left(\int f \, dP \right) \left(\int g \, dP \right).
\end{aligned}$$

Now for each simple function $f = \sum_{i=1}^{n} \alpha_i \chi_{M_i}$, we have

$$\begin{aligned}
\left(\int f \, dP \right)^* &= \left(\sum_{i=1}^{n} \alpha_i P(M_i) \right)^* \\
&= \sum_{i=1}^{n} \overline{\alpha_i} P(M_i) \\
&= \int \overline{f} \, dP
\end{aligned}$$

and the map is a $*$-homomorphism as desired.

Finally for a simple function $f = \sum_{i=1}^{n} \alpha_i \chi_{M_i}$ and $x, y \in \mathcal{H}$, we have

$$\begin{aligned}\left\langle \left(\int f \, dP \right) x, y \right\rangle &= \left\langle \left(\sum_{i=1}^{n} \alpha_i P(M_i) \right) x, y \right\rangle \\ &= \sum_{i=1}^{n} \alpha_i \langle P(M_i)x, y \rangle \\ &= \sum_{i=1}^{n} \alpha_i P_{x,y}(M_i) \\ &= \int f \, dP_{x,y}.\end{aligned}$$

□

We now come to the fundamental result of this chapter.

Theorem 3.23 (The Spectral Theorem for Self-Adjoint Operators)**.** *Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator with $\Sigma = \sigma(A)$. Then there exists a unique spectral measure P so that*

$$A = \int_{\Sigma} \lambda \, dP.$$

Further, for any bounded measurable function f defined on the spectrum, we obtain $f(A) \in \mathcal{B}(\mathcal{H})$ defined by

$$f(A) = \int_{\Sigma} f(\lambda) \, dP.$$

Proof. The projection valued measure defined in Definition 3.15 is a spectral measure so that

$$f(A) = \int_{\Sigma} f \, dP$$

for all $f \in B(\Sigma)$. In particular, for $f(\lambda) = \lambda$ we have

$$A = \int_\Sigma \lambda \, dP,$$

establishing existence.

Now suppose that Q is any spectral measure on Σ such that

$$A = \int_\Sigma \lambda \, dQ.$$

Then since

$$\int_\Sigma \, dQ = P(\Sigma) = I$$

and the map $f \mapsto \int f \, dP$ is multiplicative, we see that for any polynomial $p \in \mathcal{P}(\Sigma)$ we have

$$\int_\Sigma p(\lambda) \, dQ = p(A).$$

Hence for all $x, y \in \mathcal{H}$, we have

$$\int_\Sigma p(\lambda) \, dQ_{x,y} = \langle p(A)x, y \rangle = \int_\Sigma p(\lambda) \, dP_{x,y}.$$

Thus since polynomials are dense in $C(X)$ we see that

$$\int_\Sigma f(\lambda) \, dQ_{x,y} = \int_\Sigma f(\lambda) \, dP_{x,y}$$

for all continuous functions f on Σ. Hence

$$Q_{x,y} = P_{x,y}$$

for all $x, y \in \mathcal{H}$. Thus for each Borel set $M \subset \Sigma$ we have

$$\langle Q(M)x, y\rangle = \int_\Sigma \chi_M \; dQ_{x,y} = \int_\Sigma \chi_M \; dP_{x,y} = \langle P(M)x, y\rangle$$

and hence $Q(M) = P(M)$. Therefore $P = Q$ and the spectral measure is unique, as claimed.

□

Given a measurable bounded function f on the spectrum Σ of A, the Spectral Theorem allows us to define

$$f(A) = \int_\Sigma f(\lambda) \; dP.$$

Notice however that some care must be taken here since the function must be defined on the spectrum. For example in order to define the logarithm or square root of an operator using the standard branch cut from complex analysis the spectrum must not contain any negative real numbers.

Example 3.24. Let A be a positive operator. Then the spectrum of A does not contain any negative real numbers by Proposition 2.49. Hence we may define

$$\sqrt{A} = \int_\Sigma \sqrt{\lambda} \; dP.$$

If further A is invertible, that is $0 \notin \Sigma$, then

$$\log(A) = \int_\Sigma \log(\lambda) \; dP$$

where $\sqrt{\lambda}, \log(\lambda)$ are the usual definitions for real numbers.

3.4 The Spectral Theorem for Normal Operators

The spectral theorem for self-adjoint operators can be extended to a spectral theorem for normal operators. The idea is illustrated in the following lemma.

Lemma 3.25. *Let $A \in \mathcal{B}(\mathcal{H})$ be a normal operator. Then there exists two unique commuting self-adjoint operators A_1, A_2 so that*

$$A = A_1 + iA_2.$$

Proof. Indeed, let

$$A_1 = \frac{1}{2}(A + A^*)$$

and

$$A_2 = \frac{1}{2i}(A - A^*).$$

Then both A_1, A_2 are self adjoint and

$$A_1 + iA_2 = \frac{1}{2}(A + A^* + A - A^*) = A$$

as needed. Uniqueness is verified by direct calculation. □

The following result may be found in [11].

Theorem 3.26 (The Spectral Theorem for Normal Operators)**.** *Let $A \in \mathcal{B}(\mathcal{H})$ be a normal operator and let $\Sigma = \sigma(A)$. Then there exists a unique projection valued measure P on Σ so that*

$$A = \int_\Sigma \lambda \, dP.$$

Further for any bounded measurable function f defined on the spectrum we obtain

$f(A) \in \mathcal{B}(\mathcal{H})$ defined by

$$f(A) = \int f(\lambda)\ dP.$$

3.5 C^*-algebra Approach

Ultimately this development is taking place in the broader context of Gelfand theory on C^*-algebras. We outline this development as performed in [8].

Definition 3.27. Let $\mathcal{A}$ be a C^*-algebra with identity I. Then for each $x \in \mathcal{A}$ we define the spectrum of x

$$\sigma(x) = \{\lambda \in \mathbb{C} : \lambda I - x \text{ is not invertible in } \mathcal{A}\}.$$

This definition is a natural extension of the spectrum of an operator. However, an alternative way of looking at the spectrum is necessary.

Definition 3.28. Let $\mathcal{A}$ be a commutative C^*-algebra with identity I. The spectrum of $\mathcal{A}$, $\Sigma(\mathcal{A})$, is the set of all non-zero multiplicative functionals on $\mathcal{A}$.

It follows from Tychonoff's Theorem that $\Sigma(\mathcal{A})$ is a compact Hausdorff space under the weak topology. The key is to relate these two different concepts of spectrum. In order to do this we need the following proposition.

Proposition 3.29. *The map $h \mapsto ker(h)$ is a bijection between $\Sigma(\mathcal{A})$ and the set of maximal ideals in $\mathcal{A}$.*

We are now in a position to consider the map that relates these concepts, the Gelfand transform.

Definition 3.30. The Gelfand transform is the map $\Gamma_{\mathcal{A}}$ from $\mathcal{A}$ to $C(\Sigma(\mathcal{A}))$ defined by

$$\Gamma_{\mathcal{A}}(x)(h) = \widehat{x}(h) = h(x)$$

for each $h \in \Sigma(\mathcal{A})$.

We write $\Sigma = \Sigma(\mathcal{A})$.

Theorem 3.31. *Let $\mathcal{A}$ be a commutative C^* algebra, then the following hold.*

1. *The Gelfand transform is a homomorphism with $\widehat{I} = 1$.*

2. *For all $x \in \mathcal{A}$, x is invertible if and only if $\widehat{x}$ is never zero.*

3. *For all $x \in \mathcal{A}$, $\widehat{x}(\Sigma) = \sigma(x)$.*

4. *For all $x \in \mathcal{A}$,*
$$\|\widehat{x}\|_\infty \leq \|x\|.$$

Proof. We outline the proof of statements 2 and 3 since these directly relate $\Sigma(A)$ and $\sigma(x)$.

For statement 2, suppose that x is not invertible. Then the ideal generated by x is a proper ideal and so is contained in some maximal ideal J. By Proposition 3.29 there exists $h \in \Sigma(\mathcal{A})$ with $ker(h) = J \ni x$. Therefore $h(x) = 0$, that is $\widehat{x}(h) = 0$. This line of reasoning may be reversed, hence we have that x is not invertible if and only if $\widehat{x}$ has a zero, which is equivalent to the desired result.

For statement 3, let $x \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. By the previous statement we see that $\lambda I - x$ is not invertible if and only if

$$\widehat{\lambda I - x} = \lambda - \widehat{x}$$

has a zero. That is if and only if $\lambda = \widehat{x}(h)$ for some $h \in \Sigma(\mathcal{A})$, giving $\widehat{x}(\Sigma) = \sigma(x)$. $\square$

We have a homomorphism $\Gamma_{\mathcal{A}} : \mathcal{A} \to C(\Sigma)$. In fact, however, $\Gamma_{\mathcal{A}}$ is onto.

Theorem 3.32. *If $\mathcal{A}$ is a C^*-algebra, then $\Gamma_{\mathcal{A}}(\mathcal{A})$ is dense in $C(\Sigma)$.*

Note that this is a consequence of the Stone-Weierstrass Theorem.

The following theorem may be found in [8].

Theorem 3.33 (The Gelfand-Naimark Theorem)**.** *If $\mathcal{A}$ is a commutative unital C^* algebra then the Gelfand Transform $\Gamma_{\mathcal{A}}$ is an isometric $*$-isomorphism from $\mathcal{A}$ to $C(\sigma(\mathcal{A}))$.*

The Gelfand-Naimark Theorem generalizes the result obtained in Proposition 3.7 for the special case of a self-adjoint operator.

In particular, consider any commutative C^*-algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ containing the identity operator I. Then for each $f \in C(\Sigma)$, we obtain an operator in $\mathcal{A}$ by taking the inverse Gelfand-transform. Then following the same construction as in sections 3.2 and 3.3 we obtain the following generalization of the Spectral Theorem for Self-Adjoint Operators.

Theorem 3.34 (Spectral Theorem)**.** *Let $\mathcal{A}$ be a commutative C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ containing I and let $\Sigma = \Sigma(\mathcal{A})$ be its spectrum. Then there is a unique regular projection valued measure P on Σ so that*

$$A = \int_{\Sigma} \widehat{A}\, dP$$

for all $A \in \mathcal{A}$.

The spectral theorem for self-adjoint operators, or indeed for normal operators, may be obtained as a corollary of Theorem 3.34 by considering the C^* algebra $\mathcal{A} = \langle I, A \rangle$ or $\mathcal{A} = \langle I, A, A^* \rangle$ if A self-adjoint or normal, respectively. The connection to our development is that in the case of A self adjoint, then $\Sigma(\mathcal{A})$ may be identified with $\sigma(A)$. Indeed since $\mathcal{A} = \langle I, A \rangle$ and for any $h \in \Sigma$ we must have $h(I) = 1$, we see that h is completely determined by $h(A)$. Now it can be shown that $A - \lambda I$ is not invertible if and only if there exists some $h \in \Sigma(\mathcal{A})$ so that

$$\lambda - h(A) = 0,$$

hence if and only if

$$h(A) = \lambda \in \sigma(A).$$

Therefore we identify $\Sigma(\mathcal{A})$ with $\sigma(A)$ via the bijective map

$$h \mapsto h(A).$$

3.6 Example, The Shift Operator on $l^2(\mathbb{Z})$

Let $\mathcal{H} = l^2(\mathbb{Z})$. We illustrate the spectral theorem by computing the spectral measure for the right shift operator R. By Example 2.55 R is unitary, hence normal. Thus the spectral theorem guarantees the existence of a spectral measure P on $\sigma(R) = \mathbb{T}$ where $\mathbb{T}$ is the unit circle in $\mathbb{C}$. In order to calculate this, we need some basic results of Fourier analysis summarized in the following theorem, see for example [7]. In order to simplify notation we integrate with respect to the measure

$$d\gamma = \frac{1}{2\pi} dx$$

where dx is Lebesgue measure on $[0, 2\pi]$. Here we identify $[0, 2\pi]$ with $\mathbb{T}$ in the usual way,

$$x \mapsto e^{ix}.$$

Theorem 3.35. *Let $\alpha \in l^2(\mathbb{Z})$, then the function $\widehat{\alpha}$ defined by*

$$\widehat{\alpha}(\gamma) = \sum_{n \in \mathbb{Z}} \alpha_n \gamma^n$$

is in $L^2(\mathbb{T})$. Further we have the inversion formula

$$\alpha_n = \int_{\mathbb{T}} \widehat{\alpha}(\gamma) \gamma^{-n} \, d\gamma$$

and Parseval's Theorem

$$\langle \alpha, \beta \rangle = \langle \widehat{\alpha}, \widehat{\beta} \rangle$$

for all $\alpha, \beta \in L^2(\mathbb{T})$.

For the remainder of this section, we take α, β to be arbitrary elements of $l^2(\mathbb{Z})$. We have

$$\begin{aligned}
\widehat{R\alpha}(\gamma) &= \sum_{n \in \mathbb{Z}} (R\alpha)_n \gamma^n \\
&= \sum_{n \in \mathbb{Z}} \alpha_{n-1} \gamma^n \\
&= \sum_{n \in \mathbb{Z}} \alpha_n \gamma^{n+1} \\
&= \gamma \widehat{\alpha}(\gamma).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\langle R\alpha, \beta \rangle &= \langle \widehat{R\alpha}, \widehat{\beta} \rangle \\
&= \int_{\mathbb{T}} \widehat{R\alpha}(\gamma) \overline{\widehat{\beta}(\gamma)} \, d\gamma \\
&= \int_{\mathbb{T}} \gamma \widehat{\alpha}(\gamma) \overline{\widehat{\beta}(\gamma)} \, d\gamma
\end{aligned}$$

and we see that the unique measure guaranteed by the Riesz Representation Theorem is given by

$$d\mu_{\alpha,\beta} = \widehat{\alpha}(\gamma)\overline{\widehat{\beta}(\gamma)}d\gamma.$$

For any Borel set $M \subset \mathbb{T}$ we then have the projection valued measure P given by

$$\langle P(M)\alpha, \beta\rangle = \int_{\mathbb{T}} \chi_M \widehat{\alpha}(\gamma)\overline{\widehat{\beta}(\gamma)}\, d\gamma.$$

Writing $P_M = P(M)$ we see that

$$\langle \widehat{P_M\alpha}, \widehat{\beta}\rangle = \int_{\mathbb{T}} \chi_M \widehat{\alpha}(\gamma)\overline{\widehat{\beta}(\gamma)}\, d\gamma,$$

in particular $\widehat{P_M\alpha}(\gamma) = \chi_M(\gamma)\widehat{\alpha}(\gamma)$. Thus P_M is the projection onto the set of all $\alpha \in l^2(\mathbb{Z})$ so that $\mathrm{supp}(\widehat{\alpha}) \subset M$. Therefore for each $M \subset \mathbb{T}$, the inversion formula gives

$$(P_M\alpha)_n = \int_{\mathbb{T}} \chi_M(\gamma)\widehat{\alpha}(\gamma)\gamma^{-n}d\gamma$$

for each $n \in \mathbb{Z}$.

Now $\widehat{R\alpha}(\gamma) = \gamma\widehat{\alpha}(\gamma)$ and

$$(R\alpha)_n = \int \gamma\widehat{\alpha}(\gamma)\gamma^{-n}\, d\gamma.$$

This defines the projection valued measure P on $\mathbb{T}$ so that

$$R = \int \gamma\, dP$$

as guaranteed by the spectral theorem for normal operators.

This example is a special case of a broader concept in harmonic analysis. Here $\mathbb{Z}$

is a group under addition and R generates the regular representation of $\mathbb{Z}$ on $l^2(\mathbb{Z})$. That is for $n \in \mathbb{Z}$ we represent n as the operator

$$R_n = R^n$$

on $l^2(\mathbb{Z})$ defined by

$$(R_n\alpha)_k = \alpha_{k-n}$$

for each $\alpha \in l^2(\mathbb{Z})$. Since $\mathbb{Z}$ is a locally compact abelian group, each irreducible representation is 1-dimensional (see [8]). Further we can identify the spectrum of the operator R with the dual group of $\mathbb{Z}$, that is, with $\mathbb{T}$. This approach depends on viewing the spectrum as a set of multiplicative functionals as used in the development of Gelfand theory. Here, in particular, we mean that any representation

$$p : \mathbb{Z} \to \mathbb{T}$$

has the form $p(n) = \gamma^n$ for some $\gamma \in \mathbb{T}$.

In general, the dual group of a locally compact abelian group G, denoted $\widehat{G}$, is the set of all continuous homomorphisms of G into $\mathbb{T}$ (viewed as group under the standard multiplication operation). In general, $\xi \in \widehat{G}$ extends to a $*$-representation of $L^1(G)$ on $\mathbb{C}$, by setting

$$\xi(f) = \int \xi(g)f(g)\ d\mu(g)$$

where $d\mu$ is Haar measure on G. With these identifications we have the following result that depends on the spectral theorem and generalizes the example above. This theorem may be found in [8]

Theorem 3.36. *Let G be a locally compact abelian group and let π be the regular representation of G on $L^2(G)$. That is,*

$$\big(\pi(h)(f)\big)(g) = f(h^{-1}g)$$

for all $f \in L^2(G)$ and $h \in G$. Then there exists a unique regular projection valued measure P on $\widehat{G}$ such that

$$\pi(g) = \int_{\widehat{G}} \xi(g)\ dP(\xi) \text{ for all } g \in G$$

and

$$\pi(f) = \int_{\widehat{G}} \xi(f)\ dP(\xi) \text{ for all } f \in L^1(G).$$

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