ABSTRACT

LIE ALGEBRA REPRESENTATION THEORY

by

Erica Clark

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Chair: Dr. Chris Jantzen

Major Department: Mathematics

We give a brief introduction to structure theory of Lie algebras, followed by representation theory. This thesis culminates in the presentation of the Theorem of the Highest Weight for a Lie algebra.

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Erica Clark

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Erica Clark

APPROVED BY:

DIRECTOR OF THESIS:

Dr. Chris Jantzen

COMMITTEE MEMBER:

Dr. Chal Benson

COMMITTEE MEMBER:

Dr. Gail Ratcliff

COMMITTEE MEMBER:

Dr. Heather Ries

CHAIR OF THE DEPARTMENT
OF MATHEMATICS:

Dr. Johannes Hattingh

DEAN OF THE
GRADUATE SCHOOL:

Dr. Paul Gemperline

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Chapter 1: Introduction

1.1 Results

This thesis leads to the presentation of the Theorem of the Highest Weight, which states that there is a bijection between the dominant integral elements of the dual-space for the Cartan subalgebra for a semi-simple Lie algebra and the equivalence classes of the Lie algebra's finite-dimensional irreducible representations. Throughout the thesis, we build the necessary structure theory and representation theory elements to understand this major result. We further motivate this theorem, with the concrete example and building block of the general proof, by construction of the Theorem of the Highest Weight for $\mathfrak{sl}(2, \mathbb{C})$.

1.2 History of Lie Algebras

Lie algebras are closely connected to Lie groups, which have a structure which allows for differentiation and with differentiable group operations. Lie groups have applications in a wide array of fields; they are useful for many areas in math such as number theory, and sciences such as chemistry [4]. However, often the Lie group material is useful to another area in math, which is in turn useful to an application.

Sophus Lie was a Norwegian mathematician, and who is most generally known for the theory which we explore in this thesis. However, he was aided by contemporaries as well as those who built on his work. Lie began his research, originating theory and background for Lie algebras in 1873 ([1] p. 1), however the first time the phrase "Lie algebra" was used was in 1934 by Weyl ([2] p. 422). Additionally, Cartan is credited with the Cartan subalgebra, which is heavily discussed in this thesis.

Although we do not discuss Lie groups in this thesis, there is a strong correlation between Lie groups and Lie algebras. In fact, many of the applications from this theory are most directly related to Lie groups. The history of representations in general is discussed in the first chapters of [7].

Etingof et al. [5] has made some general remarks about historical facts in this area, which may be of interest. The reader may also consult Borel [1], or Bourbaki [2] [3], for more historical notes.

1.3 Brief Description of the Sections

Under the assumption that the reader has a general knowledge of linear and abstract algebra, in Chapter 2 we begin by introducing the definition of a Lie algebra, along with the basic properties that follow. Additionally, we provide a general introduction to bilinear forms.

In Chapter 3, we discuss the structure theory of Lie algebras. Importantly, we introduce root systems and root spaces, in part to motivate the idea of weights. Chapter 4 is a discussion of Lie algebra representations, which is interrelated with the roots discussed in the previous chapter.

Finally, in Chapter 5 we discuss the main result of this thesis, which is the Theorem of the Highest Weight.

1.4 Research Approach and Presentation

The areas in mathematics which involve Lie algebras are vast with many rabbit holes and potential directions. As discussed in Section 1.2, Lie groups are highly applicable in this field. Acknowledging the applications of Lie groups, as well as the importance of structure theory, we primarily focus on concepts which are necessary for representations of Lie algebras. This approach is much more streamlined than that in most textbooks, which attempt to develop the topics which we do not include.

The emphasis here is to introduce the notion of a Lie algebra with its basic properties and identify common concrete examples of Lie algebras, in order to present representation theory. We generally approach the theory first by motivational examples, and then by presentation of the general results.

Chapter 2: Introduction to Lie Algebras

2.1 Properties of a Lie Algebra

We begin with a general definition of a Lie algebra:

Definition 2.1. [8] A Lie algebra $\mathfrak{g}$ is a non-associative algebra over a field, whose product is called the Lie bracket, denoted $[X, Y]$. The Lie bracket is alternating; i.e., for all $X \in \mathfrak{g}$, $[X, X] = 0$. Additionally, it satisfies the Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0,$$

for all $X, Y, Z \in \mathfrak{g}$.

Since the Lie bracket is bilinear and alternating, for all $X, Y \in \mathfrak{g}$

$$0 = [X + Y, X + Y] = [X, X] + [Y, X] + [X, Y] + [Y, Y],$$

so additionally

$$[X, Y] = -[Y, X].$$

Throughout this thesis, we study finite-dimensional Lie algebras. Additionally, the focus here is on Lie algebras over the field $\mathbb{C}$. However, in Chapter 2, we begin by working over a general field F.

There are many well-known Lie algebras that we refer to throughout this thesis. To begin with, we introduce a Lie algebra that is most likely be familiar to the reader, $\mathfrak{gl}(n, F)$.

Example 2.2. Define the general linear Lie algebra as

$$\mathfrak{gl}(n, F) = \{n \times n \text{ matrices with entries from a field } F\}$$

with Lie bracket given by

$$[X, Y] = XY - YX.$$

In the definition above for the Lie bracket, the terms XY and YX denote standard matrix multiplication, which is generally defined for elements of $\mathfrak{gl}(n, F)$. However, in the Lie algebra, the product is given by the Lie bracket, not standard matrix multiplication.

The set $\mathfrak{gl}(n, F)$ with the defined product is indeed a Lie algebra. The product $[X, Y]$ is an alternating function which satisfies the Jacobi identity. First we show that it is an alternating function:

$$\begin{aligned} [X, X] &= XX - XX \\ &= 0. \end{aligned}$$

It remains to show that it satisfies the Jacobi identity which we calculate below:

$$\begin{aligned} &[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] \\ &= [X, YZ - ZY] + [Y, ZX - XZ] + [Z, XY - YX] \\ &= X(YZ - ZY) - (YZ - ZY)X \\ &+ Y(ZX - XZ) - (ZX - XZ)Y \\ &+ Z(XY - YX) - (XY - YX)Z \\ &= XYZ - XZY - YZX + ZYX \end{aligned}$$

$$+YZX - YXZ - ZXY + XZY$$
$$+ZXY - ZYX - XYZ + YXZ$$
$$= 0.$$

Definition 2.3. ([8]) Given a Lie algebra $\mathfrak{g}$, a Lie subalgebra is a subspace of $\mathfrak{g}$ which is closed under the Lie bracket.

We examine now a Lie subalgebra of $\mathfrak{gl}(n, F)$.

Example 2.4. We define the special linear Lie algebra as

$$\mathfrak{sl}(n, F) = \{X \in \mathfrak{gl}(n, F) \,|\, \text{trace}(X) = 0\}.$$

By definition, $\mathfrak{sl}(n, F)$ is a subspace of $\mathfrak{gl}(n, F)$, inheriting the Lie bracket defined as $[X, Y] = XY - YX$. (This Lie bracket holds for all elements of $\mathfrak{gl}(n, F)$, therefore for elements of $\mathfrak{sl}(n, F) \subset \mathfrak{gl}(n, F)$, the necessary conditions for the Lie bracket still hold.) However, we need to show that $\mathfrak{sl}(n, F)$ is closed under the operation. Let $X, Y \in \mathfrak{sl}(n, F)$. Then $\text{trace}(X) = \text{trace}(Y) = 0$. It remains to show that $\text{trace}([X, Y]) = 0$.

$$\begin{aligned}\text{trace}([X, Y]) &= \text{trace}(XY - YX) \\ &= \text{trace}(XY) - \text{trace}(YX) \\ &= 0,\end{aligned}$$

by properties of trace. So indeed $\mathfrak{sl}(n, F)$ is a Lie subalgebra of $\mathfrak{gl}(n, F)$.

It is interesting to note that it was unnecessary for X, Y to have $\text{trace}(X) = \text{trace}(Y) = 0$ in order for $\text{trace}([X, Y]) = 0$ in the above example. In fact, for all

$X, Y \in \mathfrak{gl}(n, F)$, one has $[X, Y] \in \mathfrak{sl}(n, F)$. We call $\mathfrak{sl}(n, F)$ the derived subalgebra of $\mathfrak{gl}(n, F)$. In general, one has the following definition.

Definition 2.5. We define the derived Lie subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ as

$$\mathfrak{g}_0 = [\mathfrak{g}, \mathfrak{g}] = \operatorname{span}\left(\{[X, Y] : X, Y \in \mathfrak{g}\}\right).$$

Definition 2.6. [6] Within a Lie algebra $\mathfrak{g}$, elements $X, Y \in \mathfrak{g}$ are said to commute if $[X, Y] = 0$. If $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$, we call the Lie algebra abelian.

Note that in $\mathfrak{gl}(n, F)$, one has

$$[X, Y] = 0 \iff XY = YX.$$

This matches the definiton of commuting in the usual sense.

2.2 Linear Transformations and Bilinear Forms

Many Lie algebras are defined with respect to bilinear forms.

Definition 2.7. [9] Where F is a field, a bilinear form on F^n is a map

$$\langle \cdot, \cdot \rangle : F^n \times F^n \to F,$$

which is separately linear in each variable.

At this point, it is necessary to introduce some important propositions and definitions regarding linear transformations and bilinear forms. To begin with, we introduce a proposition which is more natural for Lie groups, but necessary for our thesis. Throughout this section, F is a field.

Proposition 2.8. *Let $\langle \cdot, \cdot \rangle$ be a bilinear form on F^n (F a field) and J the associated matrix with respect to the usual basis (so has entries $J_{i,j} = \langle e_i, e_j \rangle$). Further, suppose J is invertible (i.e., the form is nondegenerate). If we write u and v as column vectors, then $\langle u, v \rangle = u^T Jv$, where u^T is the transpose of u.*

Proof. For scalar coefficients, let $u = u_1e_1 + \cdots + u_ne_n$ and $v = v_1e_1 + \ldots v_ne_n$. Then we have the following:

$$\begin{aligned} \langle u, v \rangle &= \langle u_1e_1 + \cdots + u_ne_n, v_1e_1 + \ldots v_ne_n \rangle \\ &= u_1\langle e_1, v_1e_1 + \ldots v_ne_n \rangle + \cdots + u_n\langle e_n, v_1e_1 + \ldots v_ne_n \rangle \\ &= \sum_{i=1}^{n} u_i\langle e_i, v_1e_1 + \ldots v_ne_n \rangle \\ &= \sum_{i=1}^{n} u_i(v_1\langle e_i, e_1 \rangle + \cdots + v_n\langle e_i, e_n \rangle) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} u_iv_j\langle e_i, e_j \rangle. \end{aligned}$$

To see that this is equivalent to the matrix multiplication of u^TJv, observe that u^TJ is equal to the row vector with jth entries given by the following sum:

$$\sum_{i=1}^{n} u_i\langle e_i, e_j \rangle.$$

Now, when this vector is multiplied on the right, by the vector v, the result is the following sum:

$$\sum_{j=1}^{n}\sum_{i=1}^{n} u_i\langle e_i, e_j \rangle v_j.$$

This result is equivalent to our result for the calculation of $\langle u, v \rangle$. Therefore, we have shown $\langle u, v \rangle = u^TJv$, as desired. □

Proposition 2.9. *Let $A : F^n \longrightarrow F^n$ be a linear transformation. Define the adjoint* *transformation* $A^* : F^n \longrightarrow F^n$ *by*

$$\langle Au, v\rangle = \langle u, A^*v\rangle \in F^n.$$

for all u, v. *Then* $A^T J = JA^*$.

Proof. We identify the linear transformations $A, A^* : F^n \to F^n$ with their matrices with respect to the basis $\{e_1, \ldots, e_n\}$. Then by Proposition 2.7, one has

$$\langle Au, v\rangle = (Au)^T Jv = u^T A^T Jv,$$

whereas

$$\langle u, A^*v\rangle = u^T JA^*v.$$

As

$$\langle Au, v\rangle = \langle u, A^*v\rangle$$

holds for all $u, v \in F^n$, we conclude that $A^T J = JA^*$, as desired. □

Definition 2.10. The linear transformation A preserves the bilinear form if

$$\langle Au, Av\rangle = \langle u, v\rangle$$

for all $u, v \in F^n$.

Proposition 2.11. *The linear transformation* A *preserves the bilinear form* $\langle \cdot, \cdot\rangle$ *if and only if* $J = A^T JA$.

Proof. A preserves the bilinear form if and only if for all $u, v \in F^n$,

$$\langle Au, Av\rangle = \langle u, v\rangle \iff (Au)^T J(Av) = u^T Jv \quad \text{(Proposition 2.8)}$$

$$\iff u^T A^T J A v = u^T J v \qquad \text{(properties of matrix transposition)}$$

$$\iff J = A^T J A.$$

□

2.3 Classical Lie Algebras

Equipped with the propositions regarding bilinear forms, we introduce two additional Lie algebras, the symplectic and special orthogonal algebras. These algebras, together with $\mathfrak{sl}(n, F)$, are commonly referred to as classical Lie algebras.

Example 2.12. We define the special orthogonal algebra as follows:

$$\mathfrak{s}o(n, F) = \{X \in \mathfrak{s}l(n, F) \,|\, X^T J_n + J_n X = 0\}$$

with the $n \times n$ matrix J_n with entries defined by

$$J_{i,j} = \begin{cases} 1 \text{ if } i + j = n + 1 \\ 0 \text{ otherwise,} \end{cases}$$

(the anti-diagonal matrix). This is a Lie subalgebra of $\mathfrak{sl}(n, F)$.

Proof. To begin with, it is easy to check that $\mathfrak{so}(n, F)$ is a linear subspace of $\mathfrak{sl}(n, F)$. Note that the zero matrix clearly satisfy the condition $X^T J_n + J_n X = 0$, so the space is nonempty. Let $X, Y \in \mathfrak{so}(n, F)$ and $a, b \in F$. Then,

$$\begin{aligned} (aX + bY)^T J_n + J_n(aX + bY) &= (aX^T + bY^T)J_n + J_n(aX + bY) \\ &= aX^T J_n + bY^T J_n + J_n aX + J_n bY \\ &= a(X^T J_n + J_n X) + b(Y^T J_n + J_n Y) \end{aligned}$$

$$= a(0) + b(0)$$
$$= 0.$$

Finally, we show closure under the Lie bracket. Let $X, Y \in \mathfrak{g} = \mathfrak{so}(n, F)$. We need to show that $[X, Y]^T J_n + J_n[X, Y] = 0$. We calculate this as follows:

$$\begin{aligned}
[X, Y]^T J_n + J_n[X, Y] &= (XY - YX)^T J_n + J_n(XY - YX) \\
&= (XY)^T J_n - (YX)^T J_n + J_n XY - J_n YX \\
&= Y^T X^T J_n - X^T Y^T J_n + J_n XY - J_n YX \\
&= Y^T(X^T J_n + J_n X) - Y^T J_n X - X^T Y^T J_n + J_n XY - J_n YX \\
&= -(Y^T J_n + J_n Y)X - X^T Y^T J_n + J_n XY \\
&= -X^T Y^T J_n + J_n XY \\
&= (X^T J_n + J_n X)Y - X^T J_n Y - X^T Y^T J_n \\
&= -X^T(Y^T J_n + J_n Y) \\
&= -X^T(0) \\
&= 0.
\end{aligned}$$

□

Example 2.13. The symplectic Lie algebra, given by

$$\mathfrak{sp}(2n, F) = \{X \in \mathfrak{gl}(2n, F) \mid X^T J' + J'X = 0\},$$

where the $2n \times 2n$ matrix J' is defined by

$$\begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix},$$

is a Lie subalgebra of $\mathfrak{gl}(2n, F)$.

Proof. One easily checks that $\mathfrak{sp}(2n, F)$ is a linear subspace of $\mathfrak{gl}(2n, F)$, as in Example 2.12. It suffices then to verify closure under the Lie bracket. Let $X, Y \in \mathfrak{sp}(2n, F)$. We need to show that $[X, Y]^T J' + J'[X, Y] = 0$. Then it suffices to check that

$$(XY - YX)^T J' + J'(XY - YX) = 0.$$

By a property of the transpose of a matrix, the left side reduces to

$$Y^T X^T J' - X^T Y^T J' + J'XY - J'YX.$$

Since $Y^T J' + J'Y = 0$ and $X^T J' + J'X = 0$, this is equal to:

$$\begin{aligned} &Y^T(-J'X) - X^T(-J'Y) + J'XY - J'YX \\ &= -(Y^T J')X + (X^T J')Y + J'XY - J'YX \\ &= (-Y^T J' + J'Y)X + (X^T J' + J'X)Y \\ &= 0. \end{aligned}$$

□

Remark 2.14. Traditionally, one defines the symplectic Lie algebra as lying in the general linear Lie algebra. However, one can further show that the symplectic Lie algebra even lies in the special linear Lie algebra. For the case of $\mathfrak{sp}(2n, \mathbb{C})$, take $X \in \mathfrak{sp}(2n, \mathbb{C})$ and write

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}, \quad J' = \begin{pmatrix} 0 & -J \\ J & 0 \end{pmatrix}.$$

We have the following:

$$
\begin{aligned}
X^T J' + J'X = 0 &\Rightarrow X_1^T J = -JX_4 \\
&\Rightarrow X^T = -JX_4 J \qquad\qquad \text{as } J = J^{-1} \\
&\Rightarrow \text{trace}(X_1) = -\text{trace}(X_4) \\
&\Rightarrow \text{trace}(X) = 0.
\end{aligned}
$$

Chapter 3: Structure Theory for Lie Algebras

Structure theory involves characterizing elements of Lie algebras, as well as identifying their important traits. To introduce Lie algebra structure theory, we first give some basic definitions and work out examples for $\mathfrak{gl}(3,\mathbb{C})$. We then move on to demonstrate the concepts using the special linear Lie algebra over two different fields, as well as the special orthogonal and symplectic Lie algebras of various dimensions over the complex numbers.

3.1 Introduction with Concrete Examples

In Section 2.1, we introduced the notion of an abelian Lie algebra in Definition 2.6. This is a structural property. Consider $\mathfrak{h}$, the subspace of $\mathfrak{gl}(n,F)$, which we define below:

$$\mathfrak{h} = \{\text{diagonal matrices in } \mathfrak{gl}(n,F)\}.$$

It is a rather unsurprising fact that $\mathfrak{h}$ is abelian. We demonstrate this with the concrete and manageable example of $\mathfrak{gl}(3,\mathbb{C})$.

Example 3.1. For $\mathfrak{g} = \mathfrak{gl}(3,\mathbb{C})$, $\mathfrak{h}$ is an abelian Lie subalgebra.

First note that $0 \in \mathfrak{h}$. Let $X, Y \in \mathfrak{h}$. It suffices to show that $[X,Y] = 0$. As X and Y are diagonal matrices, they commute in the usual sense, one has

$$\begin{aligned} XY = YX \Rightarrow XY - YX &= 0 \\ \Rightarrow [X,Y] &= 0. \end{aligned}$$

Definition 3.2. ([8] p. 24) An ideal for a Lie algebra $\mathfrak{g}$ is an absorbing subalgebra $\mathfrak{q}$ (for any $X \in \mathfrak{g}$ and and $I \in \mathfrak{q}$, $[X,I] \in \mathfrak{q}$) .

We now introduce a new structural classification, called nilpotency.

Definition 3.3. [6] For any Lie algebra $\mathfrak{g}$, let $\mathfrak{g}^j$ be the sequence of ideals defined inductively by $\mathfrak{g}^0 = \mathfrak{g}$ and

$$\mathfrak{g}^{j+1} = [\mathfrak{g}, \mathfrak{g}^j] = \operatorname{span}\left(\{[X,Y] : X \in \mathfrak{g}, Y \in \mathfrak{g}^j\}\right),$$

for $j \in \mathbb{N}$. These algebras, called the upper central series of $\mathfrak{g}$, are composed of linear combinations of the Lie brackets. A Lie algebra $\mathfrak{g}$ is said to be nilpotent if $\mathfrak{g}^j = \{0\}$ for some j.

In order to give an example of a nilpotent Lie algebra, we introduce another subspace of $\mathfrak{gl}(n, F)$:

$$\mathfrak{n} = \{\text{strict upper triangular matrices in } \mathfrak{gl}(n,F)\}.$$

Again, it is helpful to consider the more manageable, specific example of $\mathfrak{gl}(3, \mathbb{C})$.

Example 3.4. For $\mathfrak{g} = \mathfrak{gl}(3, \mathbb{C})$, $\mathfrak{n}$ is a nilpotent Lie subalgebra.

Proof. We need to show that some $\mathfrak{n}^j = \{0\}$ for some $j \in \mathbb{N}$. So we begin calculating iteratively what elements of the ideals in the upper central sequence look like. By definition, we have $\mathfrak{n}^0 = \mathfrak{n}$, which is clearly nonzero. We then calculate the general form of an element of $\mathfrak{n}^1$. By definition, $\mathfrak{n}^1$ is linear combinations of the commutator $[\mathfrak{n}, \mathfrak{n}^0] = [\mathfrak{n}, \mathfrak{n}]$. For any two elements $X, Y \in \mathfrak{n}$, the commutator is given by:

$$\begin{aligned}
[X,Y] &= XY - YX \\
&= \begin{pmatrix} 0 & x_1 & x_2 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & y_1 & y_2 \\ 0 & 0 & y_3 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & y_1 & y_2 \\ 0 & 0 & y_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & x_1 & x_2 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} 0 & 0 & x_1y_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & y_1x_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & x_1y_3 - y_1x_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus elements of $\mathfrak{n}^1$ are just linear combinations of matrices of the form

$$\begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with $c \in \mathbb{C}$. Note that since

$$\mathfrak{n}^1 = [\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{n},$$

$\mathfrak{n}$ is a Lie subalgebra of $\mathfrak{g}$.

Now, $\mathfrak{n}^2$ has elements which are linear combinations of the commutator, $[\mathfrak{n}, \mathfrak{n}^1]$. We calculate for any $X \in \mathfrak{n}, Y \in \mathfrak{n}^1$,:

$$\begin{aligned} [X, Y] &= XY - YX \\ &= \begin{pmatrix} 0 & x_1 & x_2 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & x_1 & x_2 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= 0. \end{aligned}$$

Therefore, we have that $\mathfrak{n}$ is nilpotent, since $\mathfrak{n}^2$ is equal to zero. □

Before moving to the next structure theoretic concept, it is helpful to note that $\mathfrak{h}$ for $\mathfrak{gl}(3, \mathbb{C})$ is also a nilpotent Lie algebra. This is true, since

$$\mathfrak{h}^1 = [\mathfrak{h}, \mathfrak{h}] = \{0\}.$$

Recall now, Definition 2.5 of a derived subalgebra. In general, one can define a series of derived Lie subalgebras, as follows.

Definition 3.5. [6] For any Lie algebra $\mathfrak{g}$, we inductively define a sequence of subalgebras

$$\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \ldots$$

in the following way:

$$\mathfrak{g}_0 = \mathfrak{g}$$

and for $n > 0$

$$\mathfrak{g}_{n+1} = [\mathfrak{g}_n, \mathfrak{g}_n] := \operatorname{span}\left(\{[X, Y] : X, Y \in \mathfrak{g}_n\}\right).$$

These subalgebras are called the derived series of $\mathfrak{g}$.

The definition of the derived series is similar to that of the upper central series. However, in the derived series, each subalgebra is composed of the products of the Lie bracket of elements from the subsequent derived subalgebra. This is different from the upper central series, where each subalgebra is composed of the Lie bracket of elements of the original Lie algebra and the previous subalgebra.

Remark 3.6. Following the notational convention for derived series, for $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$ we may write

$$\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}].$$

Definition 3.7. [6] A Lie algebra $\mathfrak{g}$ is called solvable if in the derived series of subalgebras for $\mathfrak{g}$, $\mathfrak{g}_j = \{0\}$ for some $j \in \mathbb{N} \cup \{0\}$.

For the next example, we use the subspace of $\mathfrak{gl}(n, F)$ defined below:

$$\mathfrak{b} = \{\text{upper triangular matrices in } \mathfrak{gl}(n, F)\}.$$

The subalgebra $\mathfrak{b}$ is solvable. Again, we demonstrate this concretely with $\mathfrak{gl}(3, \mathbb{C})$.

Example 3.8. For $\mathfrak{g} = \mathfrak{gl}(3, \mathbb{C})$, $\mathfrak{b}$ is a solvable Lie subalgebra.

Proof. Set $\mathfrak{b}_n$ as the derived sequence of $\mathfrak{b}$. We must show that $\mathfrak{b}_j = \{0\}$ for some j. In fact, it is true that $\mathfrak{b}_3 = 0$. We first claim that elements of $\mathfrak{b}_1 = [\mathfrak{b}_0, \mathfrak{b}_0]$ are strict upper triangular matrices in $\mathfrak{g}$. For any elements $X, Y \in \mathfrak{b}_0 = \mathfrak{b}$, write

$$X = \begin{pmatrix} x_1 & x_2 & x_3 \\ 0 & x_4 & x_5 \\ 0 & 0 & x_6 \end{pmatrix}, Y = \begin{pmatrix} y_1 & y_2 & y_3 \\ 0 & y_4 & y_5 \\ 0 & 0 & y_6 \end{pmatrix} \in \mathfrak{b}_0.$$

Then, we calculate $[X, Y] = XY - YX$ as follows:

$$\begin{aligned} &[X, Y] = \\ &\begin{pmatrix} x_1y_1 & x_1y_2 + x_2y_4 & x_1y_3 + x_2y_5 + x_3y_6 \\ 0 & x_4y_4 & x_4y_5 + x_5y_6 \\ 0 & 0 & x_6y_6 \end{pmatrix} \\ &- \begin{pmatrix} x_1y_1 & x_2y_1 + x_4y_2 & x_3y_1 + x_5y_2 + x_6y_3 \\ 0 & x_4y_4 & x_5y_4 + x_6y_5 \\ 0 & 0 & x_6y_6 \end{pmatrix} = \\ &\begin{pmatrix} 0 & ((x_1 - x_4)y_2 + x_2(y_4 - y_1)) & ((x_1 - x_6)y_3 + x_2y_5 + x_5y_2 + x_3(y_6 - y_1)) \\ 0 & 0 & (x_4 - x_6)y_5 + x_5(y_6 - y_4) \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Note that $\mathfrak{b}_1$ is equivalent to the subspace $\mathfrak{n}$, which is nonzero. Additionally, we have shown now that $\mathfrak{b}$ is closed under the commutator, so is a Lie subalgebra of $\mathfrak{g}$. Next, we calculate a generic element of $\mathfrak{b}_2 = [\mathfrak{b}_1, \mathfrak{b}_1]$. For any elements $X, Y \in \mathfrak{b}_1$, write

$$X = \begin{pmatrix} 0 & a_1 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & b_1 & b_2 \\ 0 & 0 & b_3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then,

$$[X, Y] = XY - YX$$

$$= \begin{pmatrix} 0 & 0 & a_1b_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & b_1a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & a_1b_3 - b_1a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Finally, we compute the form of a generic element of $\mathfrak{b}_3 = [\mathfrak{b}_2, \mathfrak{b}_2]$. Consider any two elements of $\mathfrak{b}_2$:

$$X = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then,

$$[X, Y] = XY - YX$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As $\mathfrak{b}_3 = \{0\}$, $\mathfrak{b}$ is indeed solvable. □

Remark 3.9. Consider the basis for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$:

$$u = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.1}$$

Then,

$$[u, v] = h$$
$$[v, h] = 2v$$
$$[u, h] = -2u.$$

Thus $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, and $\mathfrak{g} = \mathfrak{s}l(2, \mathbb{C})$ is not solvable.

However, $\mathfrak{g} = \mathfrak{s}l(2, \mathbb{Z}_2)$ is solvable. Using the basis,

$$u = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$[u, v] = h$$

$$[v, h] = 0$$

$$[u, h] = 0,$$

which are all diagonal matrices, so $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}$, which we have shown is solvable.

Remark 3.10. In general, nilpotent implies solvable. However, solvable Lie algebras are not necessarily nilpotent [6].

Recall Definition 3.2, and consider the following classification of Lie algebras.

Definition 3.11. ([7] p. 668) Semi-simple Lie algebras are those for which only trivial ideals are solvable.

The requirement for semi-simplicity is often a necessary hypothesis in theorems. Semi-simple Lie algebras work nicely in terms of representations, as is shown in Theorem 4.29.

Definition 3.12. [2] A Lie algebra $\mathfrak{g}$ is called reductive if its derived subalgebra is semi-simple.

Remark 3.13. The Lie algebra $\mathfrak{gl}(n, F)$ is reductive, as its derived subalgebra is $\mathfrak{sl}(n, F)$, which is semi-simple. Note however that $\mathfrak{gl}(n, F)$ is not semi-simple.

Remark 3.14. The derived subalgebra for $\mathfrak{g}$ semi-simple is $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Thus, semi-simple Lie algebras are necessarily reductive.

3.2 The Cartan Subalgebra

Having seen examples of abelian, solvable, and nilpotent subalgebras of $\mathfrak{gl}(3, \mathbb{C})$, we move now to the general situation. The subspace $\mathfrak{h}$, defined for $\mathfrak{gl}(n, F)$, is actually an example of an important Lie subalgebra, called the Cartan subalgebra.

Definition 3.15. [11] Given $\mathfrak{g}$ a Lie algebra and $\mathfrak{h}$ a Lie subalgebra, we call $\mathfrak{h}$ a Cartan subalgebra if the following two conditions are met:

1. $\mathfrak{h}$ is nilpotent, and
2. $[X, H] \in \mathfrak{h}$ ($H \in \mathfrak{h}$ and $X \in \mathfrak{h}$) implies that $X \in \mathfrak{g}$.

In general, we have the following powerful result (Theorem 2.9 [8]):

Theorem 3.16. *[8] Any finite-dimensional complex Lie algebra* $\mathfrak{g}$ *has a Cartan subalgebra.*

In general then, a Cartan subalgebra exists for all the major Lie algebras discussed in this thesis.

3.3 Lie Algebra Direct Sum

There is a notion of combining Lie algebras, which parallels with the familiar idea of direct products of groups. We formally define what we call a Lie algebra direct sum, below:

Definition 3.17. (p. 52, [6]) If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Lie algebras, the (external) direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is the vector space direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, with bracket given by

$$[(X_1, X_2), (Y_1, Y_2)] = ([X_1, Y_1], [X_2, Y_2]).$$

For a Lie algebra $\mathfrak{g}$ with subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, if $\mathfrak{g}$ is the vector space direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ and $[X_1, X_2] = 0$ for all $X_1 \in \mathfrak{g}_1$ and $X_2 \in \mathfrak{g}_2$, we say that $\mathfrak{g}$ decomposes as the Lie algebra (internal) direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ and write $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

Note that although there may be potential ambiguity with the notation of $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, as the vector space direct sum is written identically, this is the convention and we adhere to it in this thesis. Recall the previously characterized subspaces $\mathfrak{h}$ (diagonal matrices), $\mathfrak{n}$ (strict upper triangular matrices), and $\mathfrak{b}$ (upper triangular matrices) of $\mathfrak{g} = \mathfrak{gl}(3, \mathbb{C})$. In fact, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, as a vector-space direct sum, but not a Lie algebra direct sum. We demonstrate this important distinction in the following example.

Example 3.18. $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ as the vector space direct sum for $\mathfrak{g} = \mathfrak{gl}(3, \mathbb{C})$.

Proof. We have that $\mathfrak{b}$ is the direct sum of $\mathfrak{h}$ and $\mathfrak{n}$ as vector spaces, since for any $X \in \mathfrak{h}$, $Y \in \mathfrak{n}$:

$$\begin{aligned} X + Y &= \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} + \begin{pmatrix} 0 & y_1 & y_2 \\ 0 & 0 & y_3 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} x_1 & y_1 & y_2 \\ 0 & x_2 & y_3 \\ 0 & 0 & x_3 \end{pmatrix} \in \mathfrak{b} \end{aligned}$$

with $\mathfrak{h} \cap \mathfrak{n} = 0$.

However, this is not a Lie algebra direct sum, as $[X, Y] \neq 0$ for all $X \in \mathfrak{h}$ and $Y \in \mathfrak{n}$, as is calculated below:

$$\begin{aligned} [X, Y] &= XY - YX \\ &= \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} \begin{pmatrix} 0 & y_1 & y_2 \\ 0 & 0 & y_3 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & y_1 & y_2 \\ 0 & 0 & y_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} 0 & x_1y_1 & x_1y_2 \\ 0 & 0 & x_2y_3 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & y_1x_2 & y_2x_3 \\ 0 & 0 & y_3x_3 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & (x_1 - x_2)y_1 & (x_1 - x_3)y_2 \\ 0 & 0 & (x_2 - x_3)y_3 \\ 0 & 0 & 0 \end{pmatrix} \\
&\neq 0.
\end{aligned}
$$

Therefore although $\mathfrak{b}$ decomposes as the vector space direct sum of $\mathfrak{h}$ and $\mathfrak{n}$, it does not decompose as the Lie algebra direct sum. □

3.4 Subspace Structure for the Classical Lie Algebras

We now characterize the subalgebras $\mathfrak{h}$ and $\mathfrak{n}$ for the classical Lie algebras,

$$\mathfrak{so}(7, \mathbb{C}),\ \mathfrak{so}(6, \mathbb{C}),\text{ and } \mathfrak{sp}(6, \mathbb{C}).$$

As subalgebras of $\mathfrak{gl}(7, \mathbb{C})$ and $\mathfrak{gl}(6, \mathbb{C})$ respectively, the Cartan subalgebra $\mathfrak{h}$ for each is comprised of diagonal matrices and the nilpotent subalgebra $\mathfrak{n}$ is comprised of strict upper triangular matrices. We begin with the special orthogonal Lie algebra of dimension seven.

Example 3.19. Consider $\mathfrak{g} = \mathfrak{so}(7, \mathbb{C})$. Recall from Example 2.12 that $\mathfrak{g}$ is a subalgebra of $\mathfrak{sl}(7, \mathbb{C})$. Thus all $X \in \mathfrak{g}$ must have $\text{trace}(X) = 0$ and must satisfy the identity

$$X^T J_7 + J_7 X = 0.$$

We first characterize $\mathfrak{h}$ for $\mathfrak{g} = \mathfrak{so}(7, \mathbb{C})$. Consider X in the Cartan subalgebra for

$\mathfrak{gl}(7,\mathbb{C})$. Then

$$X = \begin{pmatrix} x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_7 \end{pmatrix}.$$

Note that since X is a diagonal matrix, $X^T = X$. Additionally,

$$J_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

One has

$$\begin{aligned} X^T J_7 + J_7 X = 0 &\iff X^T J_7 = -J_7 X \\ &\iff X J_7 = -J_7 X \\ &\iff \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & x_4 & 0 & 0 & 0 \\ 0 & 0 & x_5 & 0 & 0 & 0 & 0 \\ 0 & x_6 & 0 & 0 & 0 & 0 & 0 \\ x_7 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -x_7 \\ 0 & 0 & 0 & 0 & 0 & -x_6 & 0 \\ 0 & 0 & 0 & 0 & -x_5 & 0 & 0 \\ 0 & 0 & 0 & -x_4 & 0 & 0 & 0 \\ 0 & 0 & -x_3 & 0 & 0 & 0 & 0 \\ 0 & -x_2 & 0 & 0 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ &\iff x_1 = -x_7,\ x_2 = -x_6,\ x_3 = -x_5,\ x_4 = 0. \end{aligned}$$

The subalgebra $\mathfrak{h}$ then is characterized the following way, which also satisfies the condition that $\text{trace}(X) = 0$:

$$\mathfrak{h} = \left\{ \begin{pmatrix} x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{C} \right\}$$

By a parallel procedure, we have that the subalgebra $\mathfrak{n}$ of strict upper triangular matrices is characterized for $\mathfrak{g}$ as:

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 & x_5 & 0 \\ 0 & 0 & x_7 & x_8 & x_9 & 0 & -x_5 \\ 0 & 0 & 0 & x_{12} & 0 & -x_9 & -x_4 \\ 0 & 0 & 0 & 0 & -x_{12} & -x_8 & -x_3 \\ 0 & 0 & 0 & 0 & 0 & -x_7 & -x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} : x_1, x_2, x_3, x_4, x_5, x_7, x_8, x_9, x_{12} \in \mathbb{C} \right\}.$$

Similarly, if we consider the six-dimensional special orthogonal Lie algebra, we characterize the subspaces in the following way.

Example 3.20. For $\mathfrak{g} = \mathfrak{so}(6, \mathbb{C})$, matrices must satisfy the equality

$$X^T J_n + J_n X = 0,$$

so we characterize

$$\mathfrak{h} = \left\{ \begin{pmatrix} x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{C} \right\},$$

and additionally,

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 & 0 \\ 0 & 0 & x_6 & x_7 & 0 & -x_4 \\ 0 & 0 & 0 & 0 & -x_7 & -x_3 \\ 0 & 0 & 0 & 0 & -x_6 & -x_2 \\ 0 & 0 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \ : \ x_1, x_2, x_3, x_4, x_6, x_7 \in \mathbb{C} \right\},$$

where both characterizations necessarily result in trace zero.

Example 3.21. For $\mathfrak{g} = \mathfrak{sp}(6, \mathbb{C})$, matrices must satisfy

$$X^T J' + J' X = 0.$$

Therefore, we characterize

$$\mathfrak{h} = \left\{ \begin{pmatrix} x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_1 \end{pmatrix} \ : \ x_1, x_2, x_3 \in \mathbb{C} \right\}$$

and additionally,

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & x_6 & x_7 & x_8 & x_4 \\ 0 & 0 & 0 & x_{10} & x_7 & x_3 \\ 0 & 0 & 0 & 0 & -x_6 & -x_2 \\ 0 & 0 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \ : \ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10} \in \mathbb{C} \right\}.$$

These characterizations are useful in subsequent examples, but also provide some intuition on the structure of these various Lie algebras. When considering characterizations of Lie algebras, it is interesting to note that the structure of seemingly unrelated Lie algebras may be identical. Consider the the following example for a concrete illustration of this fact.

Example 3.22. $\mathfrak{s}p(2,\mathbb{C}) = \mathfrak{s}l(2,\mathbb{C})$.

Proof. The subalgebra $\mathfrak{s}l(2,\mathbb{C})$ of $\mathfrak{gl}(2,\mathbb{C})$ consists of the elements which have trace equal to zero. Thus, we have the following structure:

$$\mathfrak{sl}(2,\mathbb{C}) = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1. \end{pmatrix} \ : \ x_1, x_2, x_3 \in \mathbb{C} \right\}.$$

But this is also the structure of $\mathfrak{s}p(2,\mathbb{C})$. We must have that for all $X \in \mathfrak{s}p(2,\mathbb{C})$, the following equation is satisfied:

$$X^T J' + J' X = 0,$$

Let $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$:

$$\begin{aligned} X^T J' + J' X = 0 &\Longrightarrow \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = 0 \\ &\Longrightarrow \begin{pmatrix} x_3 & -x_1 \\ x_4 & -x_2 \end{pmatrix} + \begin{pmatrix} -x_3 & -x_4 \\ x_1 & x_2 \end{pmatrix} = 0 \\ &\Longrightarrow \begin{pmatrix} 0 & -x_1 - x_4 \\ x_4 + x_1 & 0 \end{pmatrix} = 0 \\ &\Longrightarrow x_4 = -x_1. \end{aligned}$$

Thus, we necessarily have the same structure for $\mathfrak{sp}(2,\mathbb{C})$ as for $\mathfrak{sl}(2,\mathbb{C})$. □

3.5 Roots and Root Spaces

To motivate the general definition of roots and root spaces, we determine the roots of $\mathfrak{g} = \mathfrak{gl}(3,\mathbb{C})$.

Example 3.23. Consider $\mathfrak{g} = \mathfrak{g}l(3,\mathbb{C})$. Recall that we have the vector space direct sum $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, where

$$\mathfrak{h} = \left\{ \left(\begin{array}{ccc} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{array} \right) \middle| \; a, b, c \in \mathbb{C} \right\}, \qquad \mathfrak{n} = \left\{ \left(\begin{array}{ccc} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{array} \right) \middle| \; x, y, z \in \mathbb{C} \right\}.$$

Observe that the Lie bracket

$$\left[\left(\begin{array}{ccc} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{array} \right), \left(\begin{array}{ccc} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{array} \right) \right]$$

is calculated as follows:

$$\begin{aligned}
&= \left(\begin{array}{ccc} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{array} \right) \left(\begin{array}{ccc} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{array} \right) - \left(\begin{array}{ccc} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{array} \right) \left(\begin{array}{ccc} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{array} \right) \\
&= \left(\begin{array}{ccc} 0 & ax & ay \\ 0 & 0 & bz \\ 0 & 0 & 0 \end{array} \right) - \left(\begin{array}{ccc} 0 & bx & cy \\ 0 & 0 & cz \\ 0 & 0 & 0 \end{array} \right) \\
&= \left(\begin{array}{ccc} 0 & (a-b)x & (a-c)y \\ 0 & 0 & (b-c)z \\ 0 & 0 & 0 \end{array} \right)
\end{aligned}$$

Notice that if we write

$$\mathfrak{n}_1 = \left\{ \left(\begin{array}{ccc} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \middle| \; x \in \mathbb{C} \right\},$$

$$\mathfrak{n}_2 = \left\{ \left(\begin{array}{ccc} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \middle| \; y \in \mathbb{C} \right\},$$

$$\mathfrak{n}_3 = \left\{ \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{array} \right) \middle| \; z \in \mathbb{C} \right\},$$

then $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$ (the direct sum as vector spaces). Additionally, these subspaces are invariant under bracketing with $\mathfrak{h}$. In particular, if

$$h = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \in \mathfrak{h} \text{ and } n_1 = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{n}_1,$$

the calculation above gives that

$$[h, n_1] = \begin{pmatrix} 0 & (a-b)x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (a-b)n_1 \in \mathfrak{n}_1.$$

If we define e_i on $\mathfrak{h}$ by evaluation of the ith diagonal entry (so that $e_1(h) = a$, $e_2(h) = b$, and $e_3(h) = c$), we may rewrite this as

$$[h, n_1] = (e_1 - e_2)(h)n_1.$$

Similarly (in the obvious notation), one has

$$[h, n_2] = (e_1 - e_3)(h)n_2 \text{ and } [h, n_3] = (e_2 - e_3)(h)n_3.$$

We call $e_1 - e_2$, $e_1 - e_3$, and $e_2 - e_3$ the roots for $\mathfrak{g} = \mathfrak{gl}(3, \mathbb{C})$ with $\mathfrak{n}_1$, $\mathfrak{n}_2$, and $\mathfrak{n}_3$ the corresponding root spaces.

Note that all the roots here are nonzero. In general, we do not allow roots to be zero. In the next example, we show another set of roots for $\mathfrak{g} = \mathfrak{gl}(3, \mathbb{C})$.

Example 3.24. Let $\mathfrak{g} = \mathfrak{g}l(3, \mathbb{C})$. We determine the roots and corresponding roots space with respect to $\mathfrak{n}^-$, the subalgebra of strict lower triangular matrices. We first need to calculate the action of $\mathfrak{h}$ on $\mathfrak{n}^-$. So we choose generic elements

$$h = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \in \mathfrak{h} \text{ and } n = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{pmatrix} \in \mathfrak{n}^-,$$

and by similar calculations as those from Example 3.23, we have:

$$[h,n] = \begin{pmatrix} 0 & 0 & 0 \\ (b-a)x & 0 & 0 \\ (c-a)y & (c-b)z & 0 \end{pmatrix}.$$

It is clear that we have the vector space direct sum of $\mathfrak{n}^- = \mathfrak{n}_1^- \oplus \mathfrak{n}_2^- \oplus \mathfrak{n}_3^-$, for

$$\mathfrak{n}^-{}_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \, x \in \mathbb{C} \right\},$$

$$\mathfrak{n}^-{}_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y & 0 & 0 \end{pmatrix} \middle| \, y \in \mathbb{C} \right\},$$

$$\mathfrak{n}^-{}_3 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \middle| \, z \in \mathbb{C} \right\}.$$

These subspaces are invariant under the action of $\mathfrak{h}$. In particular, if

$$h = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \in \mathfrak{h} \text{ and } n_1^- = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{n}^-{}_1,$$

the calculation above tells us

$$[h, n_1^-] = \begin{pmatrix} 0 & 0 & 0 \\ (b-a)x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (b-a)n_1^- \in \mathfrak{n}^-{}_1.$$

Therefore, one has the following:

$$[h, n_1] = (e_2 - e_1)(h)n_1$$

$$[h, n_2] = (e_3 - e_1)(h)n_2$$

$$[h, n_3] = (e_3 - e_2)(h)n_3.$$

Note that $e_2 - e_1$, $e_3 - e_1$, and $e_3 - e_2$ are the roots here; $\mathfrak{n}_1^-$, $\mathfrak{n}_2^-$, and $\mathfrak{n}_2^-$ are the corresponding root spaces.

In general, we have the following definition of dual space.

Definition 3.25. (p. 50, [9]) The dual space V^* of a finite dimensional vector space V is the space of linear maps from V to $\mathbb{C}$, called linear functionals.

Specifically, in this case then, the dual space $\mathfrak{h}^*$ represents the linear functionals which map from $\mathfrak{h}$ to $\mathbb{C}$.

We now introduce the formal definition of roots and root spaces.

Definition 3.26. [6] For $\mathfrak{g}$ a Lie algebra, with a Cartan subalgebra $\mathfrak{h}$, a nonzero element $\alpha \in \mathfrak{h}^*$ is called a root if there exists some nonzero $X \in \mathfrak{g}$ with

$$[H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}.$$

We now move to the classical Lie algebras for more examples of roots and root spaces.

Example 3.27. Let $\mathfrak{g} = \mathfrak{so}(7, \mathbb{C})$. In Example 3.19, we characterized the subalgebras $\mathfrak{h}$ and $\mathfrak{n}$. We begin here by determining how an element $h \in \mathfrak{h}$ acts on an element $n \in \mathfrak{n}$, by calculating the Lie bracket $[h, n]$:

$$= hn - nh$$

$$= \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a \end{pmatrix} \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 & x_5 & 0 \\ 0 & 0 & x_7 & x_8 & x_9 & 0 & -x_5 \\ 0 & 0 & 0 & x_{12} & 0 & -x_9 & -x_4 \\ 0 & 0 & 0 & 0 & -x_{12} & -x_8 & -x_3 \\ 0 & 0 & 0 & 0 & 0 & -x_7 & -x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$
- \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 & x_5 & 0 \\ 0 & 0 & x_7 & x_8 & x_9 & 0 & -x_5 \\ 0 & 0 & 0 & x_{12} & 0 & -x_9 & -x_4 \\ 0 & 0 & 0 & 0 & -x_{12} & -x_8 & -x_3 \\ 0 & 0 & 0 & 0 & 0 & -x_7 & -x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & ax_1 & ax_2 & ax_3 & ax_4 & ax_5 & 0 \\ 0 & 0 & bx_7 & bx_8 & bx_9 & 0 & -bx_5 \\ 0 & 0 & 0 & cx_{12} & 0 & -cx_9 & -cx_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & cx_7 & cx_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & bx_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
- \begin{pmatrix} 0 & bx_1 & cx_2 & 0 & -cx_4 & -bx_5 & 0 \\ 0 & 0 & bx_7 & 0 & -cx_9 & 0 & ax_5 \\ 0 & 0 & 0 & 0 & 0 & bx_9 & ax_4 \\ 0 & 0 & 0 & 0 & cx_{12} & bx_8 & ax_3 \\ 0 & 0 & 0 & 0 & 0 & bx_7 & ax_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & ax_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & (a-b)x_1 & (a-c)x_2 & ax_3 & (a+c)x_4 & (a+b)x_5 & 0 \\ 0 & 0 & 0 & bx_8 & (b+c)x_9 & 0 & (b+a)(-x_5) \\ 0 & 0 & 0 & cx_{12} & 0 & (c+b)(-x_9) & (c+a)(-x_4) \\ 0 & 0 & 0 & 0 & c(-x_{12}) & b(-x_8) & a(-x_3) \\ 0 & 0 & 0 & 0 & 0 & (b-c)(-x_7) & (a-c)(-x_2) \\ 0 & 0 & 0 & 0 & 0 & 0 & (a-b)(-x_1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
$$

We now can write $\mathfrak{n}$ as the direct sum of matrices:

$$
\mathfrak{n} = \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 & x_5 & 0 \\ 0 & 0 & x_7 & x_8 & x_9 & 0 & -x_5 \\ 0 & 0 & 0 & x_{12} & 0 & -x_9 & -x_4 \\ 0 & 0 & 0 & 0 & -x_{12} & -x_8 & -x_3 \\ 0 & 0 & 0 & 0 & 0 & -x_7 & -x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3 \oplus \mathfrak{n}_4 \oplus \mathfrak{n}_5 \oplus \mathfrak{n}_7 \oplus \mathfrak{n}_8 \oplus \mathfrak{n}_9 \oplus \mathfrak{n}_{12}
$$

which are the matrices based off $\mathfrak{n}$, but where all elements except for the numerically indicated element are set equal to 0. Note that these are the root spaces, and the corresponding roots are: $(e_1 - e_2) : \mathfrak{n}_1, (e_1 - e_3) : \mathfrak{n}_2, (e_1) : \mathfrak{n}_3, (e_1 + e_3) : \mathfrak{n}_4, (e_1 + e_2) : \mathfrak{n}_5, (e_2 - e_3) : \mathfrak{n}_7, (e_2) : \mathfrak{n}_8, (e_2 + e_3) : \mathfrak{n}_9, (e_3) : \mathfrak{n}_{12},$

Example 3.28. Next, consider $\mathfrak{g} = \mathfrak{sp}(6,\mathbb{C})$. In Example 3.21, we characterized $\mathfrak{h}$ and $\mathfrak{n}$. We now calculate the form of $[\mathfrak{h},\mathfrak{n}]$. For $h \in \mathfrak{h}$ and $n \in \mathfrak{n}$, we have that $[h,n] = hn - nh$ is:

$$= \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 & 0 \\ 0 & 0 & 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 & 0 & -a \end{pmatrix} \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & x_6 & x_7 & x_8 & x_4 \\ 0 & 0 & 0 & x_{10} & x_7 & x_3 \\ 0 & 0 & 0 & 0 & -x_6 & -x_2 \\ 0 & 0 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$- \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & x_6 & x_7 & x_8 & x_4 \\ 0 & 0 & 0 & x_{10} & x_7 & x_3 \\ 0 & 0 & 0 & 0 & -x_6 & -x_2 \\ 0 & 0 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 & 0 \\ 0 & 0 & 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 & 0 & -a \end{pmatrix}$$
$$= \begin{pmatrix} 0 & ax_1 & ax_2 & ax_3 & ax_4 & ax_5 \\ 0 & 0 & bx_6 & bx_7 & bx_8 & bx_4 \\ 0 & 0 & 0 & cx_{10} & cx_7 & cx_3 \\ 0 & 0 & 0 & 0 & cx_6 & cx_2 \\ 0 & 0 & 0 & 0 & 0 & bx_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$- \begin{pmatrix} 0 & bx_1 & cx_2 & -cx_3 & -bx_4 & -ax_5 \\ 0 & 0 & cx_6 & -cx_7 & -bx_8 & -ax_4 \\ 0 & 0 & 0 & -cx_{10} & -bx_7 & -ax_3 \\ 0 & 0 & 0 & 0 & bx_6 & ax_2 \\ 0 & 0 & 0 & 0 & 0 & ax_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & (a-b)x_1 & (a-c)x_2 & (a+c)x_3 & (a+b)x_4 & (2a)x_5 \\ 0 & 0 & (b-c)x_6 & (b+c)x_7 & (2b)x_8 & (b+a)x_4 \\ 0 & 0 & 0 & (2c)x_{10} & (c+b)x_7 & (c+a)x_3 \\ 0 & 0 & 0 & 0 & (b-c)(-x_6) & (a-c)(-x_2) \\ 0 & 0 & 0 & 0 & 0 & (a-b)(-x_1) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can write $\mathfrak{n}$ as the direct sum of matrices:

$$\begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & x_6 & x_7 & x_8 & x_4 \\ 0 & 0 & 0 & x_{10} & x_7 & x_3 \\ 0 & 0 & 0 & 0 & -x_6 & -x_2 \\ 0 & 0 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3 \oplus \mathfrak{n}_4 \oplus \mathfrak{n}_5 \oplus \mathfrak{n}_6 \oplus \mathfrak{n}_7 \oplus \mathfrak{n}_8 \oplus \mathfrak{n}_{10},$$

which are the matrices based off $\mathfrak{n}$, but with all elements except for the numerically indicated element set equal to zero. Thus the root spaces and corresponding roots are given by $\mathfrak{n}_1 : (e_1 - e_2), \mathfrak{n}_2 : (e_1 - e_3), \mathfrak{n}_3 : (e_1 + e_3), \mathfrak{n}_4 : (e_1 + e_2), \mathfrak{n}_5 : (2e_1), \mathfrak{n}_6 : (e_2 - e_3), \mathfrak{n}_7 : (e_2 + e_3), \mathfrak{n}_8 : (2e_2), \mathfrak{n}_{10} : (2e_3)$.

Example 3.29. Finally, consider $\mathfrak{g} = \mathfrak{so}(6, \mathbb{C})$, noting the characterizations for $\mathfrak{h}$ and $\mathfrak{n}$, given in Example 3.20. We begin by calculating $[\mathfrak{h}, \mathfrak{n}]$. For any $h \in \mathfrak{h}$ and $h \in \mathfrak{n}$, $[h, n]$ is as follows:

$$= hn - nh$$

$$= \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 & 0 \\ 0 & 0 & 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 & 0 & -a \end{pmatrix} \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 & 0 \\ 0 & 0 & x_6 & x_7 & 0 & -x_4 \\ 0 & 0 & 0 & 0 & -x_7 & -x_3 \\ 0 & 0 & 0 & 0 & -x_6 & -x_2 \\ 0 & 0 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$- \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 & 0 \\ 0 & 0 & x_6 & x_7 & 0 & -x_4 \\ 0 & 0 & 0 & 0 & -x_7 & -x_3 \\ 0 & 0 & 0 & 0 & -x_6 & -x_2 \\ 0 & 0 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 & 0 \\ 0 & 0 & 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 & 0 & -a \end{pmatrix}$$

$$= \begin{pmatrix} 0 & ax_1 & ax_2 & ax_3 & ax_4 & 0 \\ 0 & 0 & bx_6 & bx_7 & 0 & -bx_4 \\ 0 & 0 & 0 & 0 & -cx_7 & -cx_3 \\ 0 & 0 & 0 & 0 & cx_6 & cx_2 \\ 0 & 0 & 0 & 0 & 0 & bx_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$- \begin{pmatrix} 0 & bx_1 & cx_2 & -cx_3 & -bx_4 & 0 \\ 0 & 0 & cx_6 & -cx_7 & 0 & ax_4 \\ 0 & 0 & 0 & 0 & bx_7 & ax_3 \\ 0 & 0 & 0 & 0 & bx_6 & ax_2 \\ 0 & 0 & 0 & 0 & 0 & ax_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & (a-b)x_1 & (a-c)x_2 & (a+c)x_3 & (a+b)x_4 & 0 \\ 0 & 0 & (b-c)x_6 & (b+c)x_7 & 0 & (b+a)(-x_4) \\ 0 & 0 & 0 & 0 & (c+b)(-x_7) & (c+a)(-x_3) \\ 0 & 0 & 0 & 0 & (b-c)(-x_6) & (a-c)(-x_2) \\ 0 & 0 & 0 & 0 & 0 & (a-b)(-x_1) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can write $\mathfrak{n}$ as the direct sum of matrices:

$$\begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 & 0 \\ 0 & 0 & x_6 & x_7 & 0 & -x_4 \\ 0 & 0 & 0 & 0 & -x_7 & -x_3 \\ 0 & 0 & 0 & 0 & -x_6 & -x_2 \\ 0 & 0 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3 \oplus \mathfrak{n}_4 \oplus \mathfrak{n}_6 \oplus \mathfrak{n}_7,$$

which are the matrices which are based on $\mathfrak{n}$, but where all elements except for the numerically indicated element are set equal to 0. Thus the root spaces with their roots are given by $\mathfrak{n}_1 : (e_1 - e_2), \mathfrak{n}_2 : (e_1 - e_3), \mathfrak{n}_3 : (e_1 + e_3), \mathfrak{n}_4 : (e_1 + e_2), \mathfrak{n}_6 : (e_2 - e_3), \mathfrak{n}_7 : (e_2 + e_3)$.

3.6 Roots and the Borel Subalgebra

In preceding examples, we have characterized a special Lie algebra subspace, called $\mathfrak{b}$. This space in general is referred to as the Borel subalgebra. We define this now formally for semi-simple Lie algebras. This definition may be generalized to include reductive Lie algebras, such as the helpful concrete example of $\mathfrak{gl}(3, \mathbb{C})$.

Definition 3.30. [8] A Borel subalgebra of a complex semi-simple Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{h}$ is the Cartan subalgebra and $\mathfrak{n}$ is the vector space direct sum of all the positive root spaces.

For example, for $\mathfrak{g} = \mathfrak{g}l(3, \mathbb{C})$ with $\mathfrak{b}$ and $\mathfrak{h}$ as before, the root spaces look like

$$\begin{pmatrix} * & e_1 - e_2 & e_1 - e_3 \\ -e_1 + e_2 & * & e_2 - e_3 \\ -e_1 + e_3 & -e_2 + e_3 & * \end{pmatrix}.$$

As the Borel subalgebra, $\mathfrak{b}$, is upper triangular, the positive roots are $e_1 - e_2$, $e_1 - e_3$, and $e_2 - e_3$. I.e., the positive roots lie in the Borel subalgebra.

Alternatively, positive roots may be determined by lexicographical ordering on a basis for the dual space $\mathfrak{h}^*$. For an ordered basis

$$\mathcal{B} = \{v_1, \ldots, v_n\},$$

a root

$$\alpha = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n,$$

is positive if $x_1 > 0$ and negative if $x_1 < 0$. In the case that $x_1 = 0$, the sign of the next nonzero coefficient determines whether α is positive. In this construction, the positive roots determine the Borel subalgebra instead of the other way around.

In the example of $\mathfrak{g}l(3, \mathbb{C})$ above, the ordering $e_1 > e_2 > e_3$ determines $e_1 - e_2$, $e_1 - e_3$, $e_2 - e_3$ as positive roots. In this case, the corresponding Borel subalgebra takes the form of upper triangular matrices.

We introduce now the notion of a simple root:

Definition 3.31. [8] A positive root which cannot be expressed as a positive linear combination of other positive roots is called simple.

In the case of $\mathfrak{g}l(3, \mathbb{C})$, we have

$$e_1 - e_3 = (e_1 - e_2) + (e_2 - e_3),$$

so $e_1 - e_3$ is not simple. The roots $e_1 - e_2$ and $e_2 - e_3$ are simple. In this example and in the additional examples for classical Lie algebras, the number of positive roots is small. The simple roots are found by eliminating the roots which can be written as linear combinations of other positive roots. It is then easy to verify that the remaining positive roots are simple.

We now determine the positive and simple roots for the classical Lie algebras

$\mathfrak{sp}(6,\mathbb{C})$, $\mathfrak{so}(7,\mathbb{C})$, and $\mathfrak{so}(6,\mathbb{C})$. For the following examples, consider the upper triangular Borel subalgebra.

Example 3.32. In Example 3.27, we determined that the positive roots for $\mathfrak{so}(7,\mathbb{C})$ are given by $e_1-e_2, e_1-e_3, e_1, e_1+e_3, e_1+e_2, e_2-e_3, e_2, e_2+e_3$, and e_3. Further, the simple roots are given by

$$e_1-e_2,\ e_2-e_3, \text{ and } e_3.$$

It is possible to express the remaining positive roots as linear combinations of the simple roots:

$$\begin{aligned}
e_1-e_3 &= (e_1-e_2)+(e_2-e_3)\\
e_1 &= (e_1-e_2)+(e_2-e_3)+(e_3)\\
e_1+e_3 &= (e_1-e_2)+(e_2-e_3)+2(e_3)\\
e_1+e_2 &= (e_1-e_2)+2(e_2-e_3)+2(e_3)\\
e_2 &= (e_2-e_3)+e_3\\
e_2+e_3 &= (e_2-e_3)+2(e_3).
\end{aligned}$$

Example 3.33. Next consider $\mathfrak{sp}(6,\mathbb{C})$. By Example 3.28, we know that the positive roots are given by $e_1-e_2, e_1-e_3, e_1+e_3, e_1+e_2, 2e_1, e_2-e_3, e_2+e_3, 2e_2$, and $2e_3$. The simple roots, which cannot be written by any linear combination of roots are

$$e_1-e_2, e_2-e_3, 2e_3.$$

The remaining positive roots can be written as follows:

$$e_1-e_3 = (e_1-e_2)+(e_2-e_3)$$

$$
\begin{aligned}
e_1 + e_2 &= (e_1 - e_3) + (e_2 - e_3) + (2e_3) \\
e_2 + e_3 &= (e_2 - e_3) + (2e_3) \\
e_1 + e_3 &= (e_1 - e_2) + (e_2 - e_3) + (2e_3) \\
2e_2 &= 2(e_2 - e_3) + (2e_3) \\
2e_1 &= 2(e_1 - e_2) + 2(e_2 - e_3) + 2(e_3).
\end{aligned}
$$

Example 3.34. Finally, consider $\mathfrak{s}o(6, \mathbb{C})$. By Example 3.29, we know that the positive roots are given by $e_1 - e_2, e_1 - e_3, e_1 + e_3, e_1 + e_2, e_2 - e_3$ and $e_2 + e_3$. The simple roots are:

$$e_1 - e_2, e_2 - e_3, e_2 + e_3.$$

The remaining roots can be written as:

$$
\begin{aligned}
e_1 - e_3 &= (e_1 - e_2) + (e_2 - e_3) \\
e_1 + e_3 &= (e_1 - e_2) + (e_2 + e_3) \\
e_1 + e_2 &= (e_1 - e_3) + (e_2 + e_3).
\end{aligned}
$$

Example 3.35. For $\mathfrak{gl}(3, \mathbb{C})$, consider the alternative ordering of roots, determined by the ordering:

$$e_2 > e_3 > e_1.$$

This changes the “shape” of the corresponding Borel subalgebra. As the positive roots here are $-e_1 + e_2, e_2 - e_3, -e_1 + e_3$, elements of $\mathfrak{b}$ have the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & b \\ c & 0 & 0 \end{pmatrix}.$$

3.7 Properties of Roots and Root Spaces

Now that we have discussed some concrete examples of roots and their corresponding root spaces, we introduce a basic property.

Proposition 3.36. *Let α be a root for $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ be the root space for α, i.e.,*

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \,|\, [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}.$$

If β is also a root, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. That is, if X is in the root space for α and Y is in the root space for β, then $[X, Y]$ is in the root space for $\alpha + \beta$. (If $\alpha + \beta$ is not a root, this implies $[X, Y] = 0$.)

Proof. For α and β, roots of $\mathfrak{g}$, we have the following defined root spaces:

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \,|\, [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$$

and

$$\mathfrak{g}_\beta = \{X \in \mathfrak{g} \,|\, [H, X] = \beta(H)X \text{ for all } H \in \mathfrak{h}\}.$$

Let $X \in \mathfrak{g}_\alpha$ and let $Y \in \mathfrak{g}_\beta$. We want to show that $[X, Y] \in \mathfrak{g}_{\alpha+\beta}$. Note that we define

$$\mathfrak{g}_{\alpha+\beta} = \{X \in \mathfrak{g} \,|\, [H, X] = (\alpha + \beta)(H)X \text{ for all } H \in \mathfrak{h}\}.$$

For $H \in \mathfrak{h}$, we have the following:

$$\begin{aligned}
[H, [X, Y]] &= -[[H, Y], X] + [[H, X], Y] && \text{(Jacobi identity / anti-symmetry)} \\
&= -[\beta(H)Y, X] + [\alpha(H)X, Y] && (X \in \mathfrak{g}_\alpha \text{ and } Y \in \mathfrak{g}_\beta) \\
&= [X, \beta(H)Y] + [\alpha(H)X, Y] && \text{(anti-symmetry of the Lie bracket)}
\end{aligned}$$

$$= \beta(H)[X,Y] + \alpha(H)[X,Y] \qquad \text{(Lie bracket a bilinear form)}$$

$$= (\alpha+\beta)(H)[X,Y].$$

□

3.8 Bilinear Forms

Let $\mathfrak{g}$ be a matrix Lie algebra. Then one has the following trace form on $\mathfrak{g} \times \mathfrak{g}$:

$$B(X,Y) = \text{trace}(XY).$$

This is an **ad**-invariant bilinear form, i.e.,

$$B([X,Y],Z) + B(Y,[X,Z]) = 0$$

for all $X, Y \in \mathfrak{g}$. Further, B is a non-degenerate form when restricted to $\mathfrak{h} \times \mathfrak{h}$ (for $\mathfrak{h}$ the Cartan subalgebra of diagonal matrices for $\mathfrak{g}$), i.e., if for all $H' \in \mathfrak{h}$, one has $B(H,H') = 0$, then $H = 0$. As a consequence, there is a vector space isomorphism

$$\phi : \mathfrak{h} \to \mathfrak{h}^*$$
$$H \mapsto \langle H, \cdot \rangle.$$

This gives rise to a corresponding bilinear form on $\mathfrak{h}^* \times \mathfrak{h}^*$:

$$\langle \lambda, \mu \rangle = B(\phi^{-1}(\lambda), \phi^{-1}(\mu))$$

for all $\lambda, \mu \in \mathfrak{h}^*$. Note that is positive definite on $\text{span}_{\mathbb{R}}\{\alpha | \alpha \text{ is a root}\}$. In fact, in the notation of the examples, $\langle e_i, e_j \rangle = \delta_{i,j}$.

To work more generally, note that for a representation (π, V) of $\mathfrak{g}$, one can define a bilinear form

$$B_\pi(X, Y) = \text{trace}(\pi(X)\pi(Y))$$

(so that the form above corresponds to the standard representation π), which is **ad**-invariant. If one chooses $\pi = \mathbf{ad}$ (see Definition 4.3), the result is called the Killing form, which is nondegenerate. One gets a corresponding Killing form on $\mathfrak{h} \times \mathfrak{h}$ as above. This is also positive definite when restricted to the span over $\mathbb{R}$ of the roots.

3.9 Weyl Groups

Before we introduce the idea of a Weyl group, another definition and related proposition are needed.

Definition 3.37. (p. 148, [8]) For α a root, the root reflection s_α is the complex linear map $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$, which is given by

$$s_\alpha(v) = v - 2\frac{\langle \alpha, v \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

for all $v \in \mathfrak{h}^*$, where $\langle \cdot, \cdot \rangle$ is a bilinear form on $\mathfrak{h}^* \times \mathfrak{h}^*$.

Proposition 3.38. *For a root α, the root reflection s_α satisfies $s_\alpha^2 = id$.*

Proof. Let $x \in \mathfrak{h}^*$. We want to show that $s_\alpha^2(x) = x$. We calculate this below:

$$\begin{aligned}
s_\alpha^2(x) &= s_\alpha(s_\alpha(x)) \\
&= s_\alpha(x) - \frac{2 < \alpha, s_\alpha(x) >}{< \alpha, \alpha >}\alpha \\
&= x - \frac{2 < \alpha, x >}{< \alpha, \alpha >}\alpha - \frac{2 < \alpha, x - \frac{2<\alpha,x>}{<\alpha,\alpha>}\alpha >}{< \alpha, \alpha >}\alpha \\
&= x - \frac{2 < \alpha, x >}{< \alpha, \alpha >}\alpha - \frac{2(< \alpha, x > - \frac{2<\alpha,x>}{<\alpha,\alpha>} < \alpha, \alpha >)}{< \alpha, \alpha >}\alpha
\end{aligned}$$

$$= x - \frac{2 < \alpha, x >}{< \alpha, \alpha >}\alpha - \frac{2(< \alpha, x > -2 < \alpha, x >)}{< \alpha, \alpha >}\alpha$$
$$= x - \frac{2 < \alpha, x >}{< \alpha, \alpha >}\alpha + \frac{2 < \alpha, x >}{< \alpha, \alpha >}\alpha$$
$$= x,$$

as desired. □

At this point, it is possible to define the Weyl group:

Definition 3.39. The Weyl group, W, is the subgroup of $GL(\mathfrak{h}^*)$ (the general linear group over $\mathfrak{h}^*$), generated by the root reflections of the simple roots of a Lie algebra $\mathfrak{g}$:

$$W = \langle s_\alpha \,|\, \alpha \text{ simple} \rangle.$$

The product in the group is composition, denoted $\circ$, where for $s_{\alpha_1}, s_{\alpha_2} \in W$ and $v \in \mathfrak{h}^*$, one has $s_{\alpha_1} \circ s_{\alpha_2}(v) = s_{\alpha_1}(s_{\alpha_2}(v))$.

Example 3.40. For $\mathfrak{g} = \mathfrak{g}l(3, \mathbb{C})$, denote the simple roots as $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2 - e_3$. We calculate the action of $s_1 = s_{\alpha_1}$ and $s_2 = s_{\alpha_2}$ on a general element $ae_1 + be_2 + ce_3 \in \mathfrak{h}^*$ $(a, b, c \in \mathbb{C})$. We calculate the action of s_1 as follows:

$$s_{\alpha_1}(ae_1 + be_2 + ce_3) = (ae_1 + be_2 + ce_3) - \frac{2 < e_1 - e_2, ae_1 + be_2 + ce_3 >}{< e_1 - e_2, e_1 - e_2 >}(e_1 - e_2)$$
$$= ae_1 + be_2 + ce_3 - \frac{2(a-b)}{2}(e_1 - e_2)$$
$$= ae_1 + be_2 + ce_3 - ae_1 + ae_2 + be_1 - be_2$$
$$= be_1 + ae_2 + ce_3.$$

We calculate the action of s_2 as follows:

$$
\begin{aligned}
&s_{\alpha_2}(ae_1 + be_2 + ce_3) \\
&\qquad = ae_1 + be_2 + ce_3 - 2\frac{< e_2 - e_3, ae_1 + be_2 + ce_3 >}{< e_2 - e_3, e_2 - e_3 >}(e_2 - e_3) \\
&\qquad = ae_1 + be_2 + ce_3 - \frac{2(b-c)}{2}(e_2 - e_3) \\
&\qquad = ae_1 + be_2 + ce_3 - be_2 + be_3 + ce_2 - ce_3 \\
&\qquad = ae_1 + ce_2 + be_3.
\end{aligned}
$$

Write a general element as $v = ae_1 + be_2 + ce_3 \in \mathfrak{h}^*$ as $v = (a, b, c)$, the above calculations can then be summarized as follows:

$$
\begin{aligned}
s_1(a, b, c) &= (b, a, c) \\
s_2(a, b, c) &= (a, c, b).
\end{aligned}
$$

By Proposition 3.38,

$$
\begin{aligned}
s_1^2(a, b, c) &= (a, b, c) \\
s_2^2(a, b, c) &= (a, b, c).
\end{aligned}
$$

Additionally, it is true that,

$$
\begin{aligned}
s_2 \circ s_1(a, b, c) &= s_2(b, a, c) = (b, c, a) \\
s_1 \circ s_2(a, b, c) &= s_1(a, c, b) = (c, a, b) \\
s_1 \circ s_2 \circ s_1 &= s_2 \circ s_1 \circ s_2 = (c, b, a).
\end{aligned}
$$

Thus,

$$W = \{1, s_1, s_2, s_2 \circ s_1, s_1 \circ s_2, s_1 \circ s_2 \circ s_1 = s_2 \circ s_1 \circ s_)\}.$$

At this point, it should be clear to the reader that W in this case is isomorphic to the group S_3 of permutations of an ordered set of three objects.

Example 3.41. Now consider $\mathfrak{g} = \mathfrak{s}p(4, \mathbb{C})$. In this case, the simple roots are given by

$$\alpha_1 = e_1 - e_2 \text{ and } \alpha_2 = 2e_2.$$

The action of $s_1 = s_{\alpha_1}$ and $s_2 = s_{\alpha_2}$, on a general element $ae_1 + be_2 \in \mathfrak{h}^*$, can be calculated as in the previous example.

$$\begin{aligned} s_1(ae_1 + be_2) &= ae_1 + be_2 - \frac{2\langle ae_1 + be_2, e_1 - e_2\rangle}{\langle e_1 - e_2, e_1 - e_2\rangle}(e_1 - e_2) \\ &= ae_1 + be_2 - 2\frac{a-b}{2}(e_1 - e_2) \\ &= be_1 + ae_2 \end{aligned}$$

and

$$\begin{aligned} s_2(ae_1 + be_2) &= ae_1 + be_2 - \frac{2\langle ae_1 + be_2, 2e_2\rangle}{\langle 2e_2, 2e_2\rangle}(2e_2) \\ &= ae_1 + be_2 - 2\frac{2b}{4}(2e_2) \\ &= ae_1 - be_2. \end{aligned}$$

For ease of notation, denote the generic vector as $v = (a, b)$. We have then that the following unique changes result from applying the root reflections to v :

$$\begin{aligned} s_1^2(a,b) = s_2^2(a,b) &= (a,b) \\ s_1(a,b) &= (b,a) \\ s_2(a,b) &= (a,-b) \\ s_2 \circ s_1(a,b) &= (b,-a) \\ s_1 \circ s_2(a,b) &= (-b,a) \\ s_1 \circ (s_2 \circ s_1)(a,b) &= (-a,b) \\ s_2 \circ (s_1 \circ s_2)(a,b) &= (-b,-a) \\ s_1 \circ (s_2 \circ (s_1 \circ s_2))(a,b) = s_2 \circ (s_1 \circ (s_2 \circ s_1)))(a,b) &= (-a,-b). \end{aligned}$$

Thus for $\mathfrak{sp}(4,\mathbb{C})$,

$$W = \{1, s_1, s_2, s_2 \circ s_1, s_1 \circ s_2, s_1 \circ (s_2 \circ s_1), s_2 \circ s_1 \circ s_2, s_1 \circ s_2 \circ s_1 \circ s_2 = s_2 \circ s_1 \circ s_2 \circ s_1\}.$$

This Weyl group is isomorphic to the group which consists of the permutations and sign changes on a set of two elements.

Example 3.42. For $\mathfrak{g} = \mathfrak{s}o(4,\mathbb{C})$, the simple roots are given by $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_1 + e_2$. Denote the root reflections as $s_1 = s_{\alpha_1}$ and $s_2 = s_{\alpha_2}$. The action of these root reflections on an element $ae_1 + be_2 \in \mathfrak{h}^*$ is calculated below:

$$\begin{aligned} s_1(ae_1+be_2) &= (ae_1+be_2) - 2\frac{\langle (ae_1+be_2),(e_1-e_2)\rangle}{\langle (e_1-e_2),(e_1-e_2)\rangle}(e_1-e_2) \\ &= (ae_1+be_2) - 2\frac{a-b}{2}(e_1-e_2) \\ &= be_1 + ae_2 \end{aligned}$$

and

$$
\begin{aligned}
s_2(ae_1 + be_2) &= (ae_1 + be_2) - 2\frac{\langle (ae_1 + be_2), (e_1 + e_2)\rangle}{\langle (e_1 + e_2), (e_1 + e_2)\rangle}(e_1 + e_2) \\
&= (ae_1 + be_2) - 2\frac{a + b}{2}(e_1 + e_2) \\
&= -be_1 - ae_2.
\end{aligned}
$$

Note then that the Weyl group is given by

$$W = \{1, s_1, s_2, s_1 \circ s_2 = s_2 \circ s_1\}.$$

This group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, a Klein 4 group.

Chapter 4: Representation Theory for Lie Algebras

4.1 Introduction to Representation Theory

We move now from structure theory, which helps to understand what Lie algebras look like, to a new way of viewing Lie algebras. Lie algebra representations consider the actions of elements from the Lie algebra on some vector space.

Definition 4.1. [6] For a complex Lie algebra $\mathfrak{g}$ and V a finite-dimensional complex vector space, a complex, finite-dimensional representation (π, V) is a Lie algebra homomorphism

$$\pi : \mathfrak{g} \to \mathfrak{gl}(V),$$

where $X \in \mathfrak{g}$ is mapped to an operator $\pi(X) \in \mathfrak{gl}(V)$. In particular, π is a linear mapping which preserves the Lie bracket operation. That is, for all $X, Y \in \mathfrak{g}$, π satisfies the following equality:

$$\pi([X, Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X).$$

Remark 4.2. The definition for a real, finite-dimensional representation of a real Lie algebra on a finite-dimensional vector space is parallel to the complex one given in Definition 4.1.

One important example is the adjoint representation.

Definition 4.3. [6] The adjoint representation is a map

$$\mathbf{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}), \;\; \mathbf{ad}(X) = \mathbf{ad}_X$$

where the linear operator $\mathbf{ad}_X$ acts on $Y \in \mathfrak{g}$ by the bracket:

$$\mathbf{ad}_X(Y) = [X, Y].$$

Remark 4.4. Using properties of the Lie bracket, one can easily check that **ad** is a representation for a Lie algebra $\mathfrak{g}$. Indeed, for any $X, Y \in \mathfrak{g}$,

$$\begin{aligned} \mathbf{ad}\left([X,Y]\right)(Z) &= [[X,Y],Z] \\ &= -[[Y,Z],X] - [[Z,X],Y]] \\ &= [X,[Y,Z]] - [Y,[X,Z]] \\ &= (\mathbf{ad}(X)\mathbf{ad}(Y) - \mathbf{ad}(Y)\mathbf{ad}(X))(Z), \end{aligned}$$

for any $Z \in \mathfrak{g}$.

For any Lie algebra $\mathfrak{g}$, one always has the trivial representation, given by

$$\pi : \mathfrak{g} \to \mathfrak{gl}(1, \mathbb{C}) \quad \pi(X) = 0.$$

This representation is not rich, yet is useful in examples.

For sake of concreteness, we show various standard representations for $\mathfrak{gl}(n, \mathbb{C})$. As noted above, the trivial representation is a one-dimensional representation for $\mathfrak{gl}(n, \mathbb{C})$. Another one-dimensional representation is given by trace.

Example 4.5. The trace representation is the map

$$\mathbf{tr} : \mathfrak{gl}(n, \mathbb{C}) \to \mathfrak{gl}(1, \mathbb{C}), \;\; \mathbf{tr}(X) = \text{trace}(X).$$

Here we identify $\mathfrak{gl}(1, \mathbb{C})$, the space of 1×1 matrices with $\mathbb{C}$ itself. Below, we provide proof that this map is indeed a Lie algebra representation.

Proof. Let $X = [x_{ij}], Y = [y_{ij}] \in \mathfrak{g}l(n, \mathbb{C})$. It suffices to show that $\mathbf{tr}$ respects the Lie bracket:

$$\mathbf{tr}([X, Y]) = \mathbf{tr}(X)\mathbf{tr}(Y) - \mathbf{tr}(Y)\mathbf{tr}(X).$$

One has

$$\mathbf{tr}(X)\mathbf{tr}(Y) - \mathbf{tr}(Y)\mathbf{tr}(X) = 0,$$

as the operators $\mathbf{tr}(X)$ and $\mathbf{tr}(Y)$ are just complex numbers. Indeed also,

$$\mathbf{tr}([X, Y]) = \text{trace}(XY - YX) = \mathbf{tr}(XY) - \mathbf{tr}(YX) = 0,$$

as $\mathbf{tr}(XY) = \mathbf{tr}(YX)$ from Linear Algebra. Therefore $\mathbf{tr}$ indeed respects the Lie bracket and is a one-dimensional representation for $\mathfrak{gl}(n, \mathbb{C})$. □

Example 4.6. Let $\mathfrak{g} = \mathfrak{g}l(2, \mathbb{C})$. A representation on $V = \mathbb{C}^3$ is given by

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a & 2b & 0 \\ c & a + d & b \\ 0 & 2c & 2d \end{pmatrix}.$$

By restriction, this also gives a representation of $\mathfrak{s}l(2, \mathbb{C})$.

Proof. It suffices to show that π respects the Lie bracket for $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$. For $X_1, X_2 \in \mathfrak{g}$ defined as

$$X_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; X_2 = \begin{pmatrix} w & x \\ y & z \end{pmatrix},$$

the following must hold:

$$\pi([X, Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X).$$

By calculation:

$$\begin{aligned}
\pi([X,Y]) &= \pi(XY - YX) \\
&= \pi\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} - \begin{pmatrix} w & x \\ y & z \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big) \\
&= \pi\Big(\begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix} - \begin{pmatrix} wa + xc & wb + xd \\ ya + zc & yb + zd \end{pmatrix} \Big) \\
&= \pi\Big(\begin{pmatrix} by - xc & ax + bz - wb - xd \\ cw + dy - ya - zc & cx - yb \end{pmatrix} \Big) \\
&= \begin{pmatrix} 2(by - xc) & 2(ax + bz - wb - xd) & 0 \\ cw + dy - ya - zc & 0 & ax + bz - wb - xd \\ 0 & 2(cw + dy - ya - zc) & 2(cx - yb) \end{pmatrix}.
\end{aligned}$$

And equivalently:

$$\begin{aligned}
&\pi(X)\pi(Y) - \pi(Y)\pi(X) \\
&= \pi\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big)\pi\Big(\begin{pmatrix} w & x \\ y & z \end{pmatrix} \Big) - \pi\Big(\begin{pmatrix} w & x \\ y & z \end{pmatrix} \Big)\pi\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big) \\
&= \begin{pmatrix} 2a & 2b & 0 \\ c & a+d & b \\ 0 & 2c & 2d \end{pmatrix} \begin{pmatrix} 2w & 2x & 0 \\ y & w+z & x \\ 0 & 2y & 2z \end{pmatrix} - \begin{pmatrix} 2w & 2x & 0 \\ y & w+z & x \\ 0 & 2y & 2z \end{pmatrix} \begin{pmatrix} 2a & 2b & 0 \\ c & a+d & b \\ 0 & 2c & 2d \end{pmatrix} \\
&= \begin{pmatrix} 4aw + 2by & 4ax + 2b(w+z) & 2bx \\ 2cw + (a+d)y & 2cx + (a+d)(w+z) + 2by & (a+d)x + 2bz \\ 2cy & 2c(w+z) + 4dy & 2cx + 4dz \end{pmatrix} \\
&\quad - \begin{pmatrix} 4wa + 2xc & 4wb + 2x(a+d) & 2xb \\ 2ya + (w+z)c & 2yb + (w+z)(a+d) + 2xc & (w+z)b + 2xd \\ 2yc & 2y(a+d) + 4zc & 2yb + 4zd \end{pmatrix} \\
&= \begin{pmatrix} 2(by - xc) & 2ax - 2wb + 2bz - 2xd & 0 \\ cw - ya + dy - zc & 0 & bz - xd + ax - wb \\ 0 & 2(cw + yd - cz - ya) & 2(cx - yb) \end{pmatrix}.
\end{aligned}$$

Thus the Lie bracket is indeed respected, and π is a Lie algebra representation for $\mathfrak{gl}(2,\mathbb{C})$ on $\mathbb{C}^3$, as desired. □

Definition 4.7. [8] For (π, V) a finite-dimensional representation of a Lie algebra $\mathfrak{g}$, a subspace W of V is called invariant if $\pi(X)w \in W$ for all $w \in W$ and all $X \in \mathfrak{g}$.

Definition 4.8. (p. 26, [2]) A representation π is irreducible (simple) if there are no proper, invariant subspaces (subrepresentations).

Example 4.9. The adjoint representation for $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$ is reducible.

Proof. Consider the following basis for $\mathfrak{g}$:

$$z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It suffices to calculate the action of $\mathbf{ad}_z, \mathbf{ad}_h, \mathbf{ad}_u$, and $\mathbf{ad}_v$ on each of the basis vectors. We choose to take the order of basis elements to be z, h, u, v. To begin with, we find that $\mathbf{ad}_z$ acts on basis vectors in the following way:

$$\mathbf{ad}_z(z) = [z, z] = 0,$$

$$\mathbf{ad}_z(h) = [z, h] = zh - hz = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0,$$

$$\mathbf{ad}_z(u) = [z, u] = zu - uz = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0,$$

$$\mathbf{ad}_z(v) = [z, v] = zv - vz = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0.$$

So the operator that z is assigned to in the adjoint representation is the zero matrix:

$$\mathbf{ad}_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now $\mathbf{ad}_h$ acts in the following way:

$$\mathbf{ad}_h(h) = [h, h] = 0,$$

$$\mathbf{ad}_h(z) = [h, z] = -[z, h] = 0,$$

$$\mathbf{ad}_h(u) = [h, u] = hu - uh = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2u,$$

$$\mathbf{ad}_h(v) = [h, v] = hv - vh = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} = -2v,$$

Thus

$$\mathbf{ad}_h = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

$\mathbf{ad}_u$ acts as

$$\mathbf{ad}_u(z) = [u, z] = -[z, u] = 0,$$

$$\mathbf{ad}_u(h) = [u, h] = -[h, u] = -2u,$$

$$\mathbf{ad}_u(u) = [u, u] = 0,$$

$$\mathbf{ad}_u(v) = [u, v] = uv - vu = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h.$$

Therefore

$$\mathbf{ad}_u = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Finally we calculate the action of $\mathbf{ad}_v$:

$$\mathbf{ad}_v(z) = [v, z] = -[z, v] = 0,$$

$$\mathbf{ad}_v(h) = [v,h] = -[h,v] = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} = 2v,$$

$$\mathbf{ad}_v(u) = [v,u] = -[u,v] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -h,$$

$$\mathbf{ad}_v(v) = [v,v] = 0.$$

So

$$\mathbf{ad}_v = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

In order to see that **ad** is a reducible representation of $\mathfrak{g}$, consider

$$V_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

and

$$V_2 = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Note that $\mathbf{ad}_X(v_1) = 0$ for any basis vector X and any $v_1 \in V_1$. Additionally, for any $v_2 = \begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix} \in V_2$, we have that

$$ad_z\left(\begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$ad_h\left(\begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2b \\ -2c \end{pmatrix},$$

$$ad_u\left(\begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ c \\ -2a \\ 0 \end{pmatrix},$$

$$ad_v\left(\begin{pmatrix}0\\a\\b\\c\end{pmatrix}\right) = \begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&0&0&0\\0&2&0&0\end{pmatrix}\begin{pmatrix}0\\a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\-b\\0\\2a\end{pmatrix}.$$

Note that the resulting vectors all lie in V_2. Thus the subspace is invariant. □

It is possible to build up from one or more defined representations to create another new representation, via a direct sum. Note that one can take the direct sum of a representation with itself, or take the direct sum of distinct representations.

Proposition 4.10. *Define the direct sum of two representations,*(π_1, V_1) *and* (π_2, V_2), *for a Lie algebra* $\mathfrak{g}$ *as follows:*

$$[(\pi_1 \oplus \pi_2)(X)](v_1, v_2) = (\pi_1(X)v_1, \pi_2(X)v_2).$$

The direct sum of two representations, $\pi_1 \oplus \pi_2$, *so defined is itself a representation of* $\mathfrak{g}$, *on the vector space* $V_1 \oplus V_2$.

Proof. We need to show that $\pi_1 \oplus \pi_2$ respects the Lie bracket: i.e., that

$$(\pi_1 \oplus \pi_2)([X, Y]) = (\pi_1 \oplus \pi_2)(X)(\pi_1 \oplus \pi_2)(Y) - (\pi_1 \oplus \pi_2)(Y)(\pi_1 \oplus \pi_2)(X).$$

For all $v_1 \in V_1$, $v_2 \in V_2$, and $X, Y \in \mathfrak{g}$, one has

$$\begin{aligned}(\pi_1 \oplus \pi_2)([X, Y])(v_1, v_2) &= (\pi_1([X, Y]v_1, \pi_2([X, Y])v_2)\\ &= ((\pi_1(X)\pi_1(Y) - \pi_1(Y)\pi_1(X))v_1,\\ &(\pi_2(X)\pi_2(Y) - \pi_2(Y)\pi_2(X))v_2)\\ &= ((\pi_1(X)\pi_1(Y) - \pi_1(Y)\pi_1(X)),\\ &(\pi_2(X)\pi_2(Y) - \pi_2(Y)\pi_2(X)))(v_1, v_2)\end{aligned}$$

$$= ((\pi_1 \oplus \pi_2)(X)(\pi_1 \oplus \pi_2)(Y) -$$

$$(\pi_1 \oplus \pi_2)(Y)(\pi_1 \oplus \pi_2)(X))(v_1, v_2),$$

as desired. □

Even for the most trivial examples of Lie algebras, there are multiple representations. However, some representations are essentially the same, up to isomorphism. We call such representations equivalent.

Definition 4.11. [6] Two representations (π_1, V_1), (π_2, V_2) of a Lie algebra $\mathfrak{g}$ are equivalent ($\pi_1 \cong \pi_2$) if there exists some invertible linear map

$$\varphi : V_1 \to V_2$$

such that for all $X \in \mathfrak{g}$ and all $v \in V_1$,

$$\varphi(\pi_1(X)v) = \pi_2(X)\varphi(v).$$

Remark 4.12. The equivalence of representations is itself an equivalence relation, as it is reflexive, symmetric, and transitive. Let (π_1, V_1), (π_2, V_2), and (π_3, V_3) be representations of the Lie algebra $\mathfrak{g}$.

- Reflexive: To show $(\pi_1, V_1) \cong (\pi_1, V_1)$, take the invertible, linear map

  $$\varphi : V_1 \to V_1$$

  such that $\varphi(v) = v$. Then for any $X \in \mathfrak{g}$ and $v \in V_1$, we have

  $$\varphi(\pi_1(X)v) = \pi_1(X)\varphi(v),$$

  as desired.

- Next, we show that the equivalence of representations is symmetric. Suppose that $(\pi_1, V_1) \cong (\pi_2, V_2)$. Then there is an isomorphism

$$\varphi : V_1 \to V_2 \qquad \varphi(\pi_1(X)v_1) = \pi_2(X)\varphi(v_1)$$

for all $X \in \mathfrak{g}$ and $v_1 \in V$. Then let $\varphi' : V_2 \to V_1$ be the isomorphism $\varphi' = (\varphi)^{-1}$. Consequently, for any $v_2 = \varphi(v_1)$ and all $X \in \mathfrak{g}$

$$\begin{aligned}\varphi'(\pi_2(X)v_2) &= \varphi'(\pi_2(X)\varphi(v_1)) \\ &= \varphi'(\varphi(\pi_1(X)v_1) \\ &= \pi(X)v_1 \\ &= \pi_1(X)\varphi'(v_2),\end{aligned}$$

so $(\pi_2, V_2) \cong (\pi_1, V_1)$ and equivalence reprentations are symmetric, as desired.

- Finally, we show that the equivalence of representations is transitive. Suppose that $(\pi_1, V_1) \cong (\pi_2, V_2)$ and $(\pi_2, V_2) \cong (\pi_3, V_3)$. Then there exist the following isomorphisms

$$\varphi_a : V_1 \to V_2 \qquad \varphi_a(\pi_1(X)v_1) = \pi_2(X)\varphi_a(v_1)$$

and

$$\varphi_b : V_2 \to V_3 \qquad \varphi_b(\pi_2(X)v_2) = \pi_3(X)\varphi_b(v_2)$$

(for all $X \in \mathfrak{g}$, $v_1 \in V_1$, and $v_2 \in V_2$). We claim $(\pi_1, V_1) \cong (\pi_3, V_3)$ by the isomorphism:

$$\varphi_c = \varphi_b \circ \varphi_a : V_1 \to V_3.$$

In particular,

$$\begin{aligned}\varphi_c(\pi_1(X)v_1) &= \varphi_b(\varphi_a(\pi_1(X)v_1)) \\ &= \varphi_b(\pi_2(X)\varphi_a(v_1)) \\ &= \pi_3(X)\varphi_b(\varphi_a(v_1)) \\ &= \pi_3(X)\varphi_c(v_1),\end{aligned}$$

which is the isomorphism needed to show that the two desired representations are equivalent.

Therefore, the equivalence of representations of Lie algebras is itself an equivalence relation.

Example 4.13. The adjoint representation for $\mathfrak{gl}(2, \mathbb{C})$, given by Example 4.9, is the direct sum of two irreducible Lie algebra representations.

Proof. In Example 4.9, recall that $V_1 = \text{span}\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right\}$ is invariant (zero) under the adjoint representation. Therefore, the adjoint representation restricted to V_1 is trivial. This clearly coincides with the trivial representation.

Now, consider the complementary subspace to V_1 :

$$V_2 = \text{span}\left\{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\},$$

which is also invariant. Note that V_2 is $\mathfrak{sl}(2,\mathbb{C})$. An **ad**-invariant subspace of V_2 amounts to an ideal in $\mathfrak{sl}(2,\mathbb{C})$. Using $[h,u] = 2u$, $[h,v] = -2v$, and $[u,v] = h$, one can show that $\mathfrak{sl}(2,\mathbb{C})$ has no non-trivial, proper ideals. Thus, the adjoint representation restricted to V_2 is also an irreducible representation of $\mathfrak{g}$.

Therefore, the adjoint representation of our Lie algebra is given by the direct sum of these two subrepresentations.

□

4.2 Weights and Weight Spaces

Roots and root spaces, which are fundamental structural aspects of a Lie algebra, are tied to representations by weights and weight spaces. Below is the definition of a weight for a Lie algebra representation.

Definition 4.14. [6] Let (π, V) be a representation of $\mathfrak{g}$. We define $\lambda \in \mathfrak{h}^*$ as a weight for the representation if there is some nonzero $v \in V$ such that

$$\pi(H)v = \lambda(H)v$$

for all $H \in \mathfrak{h}$. We call v a corresponding weight vector.

These weights necessarily exist. The argument for this is given in [7] (p. 65).

Definition 4.15. [10] We call a weight λ of a representation π highest if $\lambda + \alpha$ is not a weight for π for any positive root α.

Refer to Lemma 4.18 for insight on this definition.

Example 4.16. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Define V_n as the space of homogeneous polynomials of degree n, and let $P_k = x^k y^{n-k}$, $0 \leq k \leq n$, be a basis for V_n. Consider the following Lie algebra representation for $\mathfrak{g}$, where for all

$$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}),$$

$$\pi_n \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} P_k = \alpha(n-2k)P_k - \beta k P_{k-1} - \gamma(n-k)P_{k+1}$$

(with P_{-1} and P_{n+1} interpreted as 0). This Lie algebra representation comes from the corresponding representation for Lie groups. We show that this is a representation of $\mathfrak{sl}(2, \mathbb{C})$, in particular, that it respects the Lie bracket. Indeed, let

$$X_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & -\alpha_1 \end{pmatrix}, X_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & -\alpha_2 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}).$$

We check that the necessary equality,

$$\pi_n([X_1, X_2])P_k = \pi_n(X_1)\pi_n(X_2)P_k - \pi_n(X_2)\pi_n(X_1)P_k,$$

holds. By calculation, $\pi_n([X_1, X_2])P_k$ is defined to be

$$\begin{aligned}
\pi_n(X_1X_2 - X_2X_1)P_k &= \pi_n\Bigg[\begin{pmatrix} \alpha_1\alpha_2 + \beta_1\gamma_2 & \alpha_1\beta_2 - \beta_1\alpha_2 \\ \gamma_1\alpha_2 - \alpha_1\gamma_2 & \gamma_1\beta_2 + \alpha_1\alpha_2 \end{pmatrix} \\
&- \begin{pmatrix} \alpha_2\alpha_1 + \beta_2\gamma_1 & \alpha_2\beta_1 - \beta_2\alpha_1 \\ \gamma_2\alpha_1 - \alpha_2\gamma_1 & \gamma_2\beta_1 + \alpha_2\alpha_1 \end{pmatrix}\Bigg]P_k \\
&= \pi_n\Bigg[\begin{pmatrix} \beta_1\gamma_2 - \beta_2\gamma_1 & 2\alpha_1\beta_2 - 2\alpha_2\beta_1 \\ 2\gamma_1\alpha_2 - 2\alpha_1\gamma_2 & \gamma_1\beta_2 - \gamma_2\alpha_1 \end{pmatrix}\Bigg]P_k \\
&= (\beta_1\gamma_2 - \beta_2\gamma_1)(n-2k)P_k \\
&- (2\alpha_1\beta_2 - 2\alpha_2\beta_1)kP_{k-1} \\
&- (2\gamma_1\alpha_2 - 2\alpha_1\gamma_2)(n-k)P_{k+1}.
\end{aligned}$$

And also, $\pi_n(X_1)\pi_n(X_2)P_k - \pi_n(X_2)\pi_n(X_1)P_k$ is given by

$$\begin{aligned}
&\pi_n(X_1)\,\pi_n\left(\begin{pmatrix}\alpha_2 & \beta_2 \\ \gamma_2 & -\alpha_2\end{pmatrix}\right)P_k - \pi_n(X_2)\,\pi_n\left(\begin{pmatrix}\alpha_1 & \beta_1 \\ \gamma_1 & -\alpha_1\end{pmatrix}\right)P_k \\
&= \pi_n(X_1)\left(\alpha_2(n-2k)P_k - \beta_2kP_{k-1} - \gamma_2(n-k)P_{k+1}\right) \\
&-\pi_n(X_2)\left(\alpha_1(n-2k)P_k - \beta_1kP_{k-1} - \gamma_1(n-k)P_{k+1}\right) \\
&= \alpha_2(n-2k)\pi_n(X_1)P_k - \beta_2k\pi_n(X_1)P_{k-1} - \gamma_2(n-k)\pi_n(X_1)P_{k+1} \\
&-\alpha_1(n-2k)\pi_n(X_2)P_k + \beta_1k\pi_n(X_2)P_{k-1} + \gamma_1(n-k)\pi_n(X_2)P_{k+1} \\
&= \alpha_2(n-2k)(\alpha_1(n-2k)P_k - \beta_1kP_{k-1} - \gamma_1(n-k)P_{k+1}) \\
&-\beta_2k(\alpha_1(n-2(k-1))P_{k-1} - \beta_1(k-1)P_{k-2} - \gamma_1(n-(k-1))P_k) \\
&-\gamma_2(n-k)(\alpha_1(n-2(k+1))P_{k+1} - \beta_1(k+1)P_k - \gamma_1(n-(k+1))P_{k+2}) \\
&-\alpha_1(n-2k)(\alpha_2(n-2k)P_k - \beta_2kP_{k-1} - \gamma_2(n-k)P_{k+1}) \\
&+\beta_1k(\alpha_2(n-2(k-1))P_{k-1} - \beta_2(k-1)P_{k-2} - \gamma_2(n-(k-1))P_k) \\
&+\gamma_1(n-k)(\alpha_2(n-2(k+1))P_{k+1} - \beta_2(k+1)P_k - \gamma_2(n-(k+1))P_{k+2})
\end{aligned}$$

$$= (\alpha_1\alpha_2(n-2k)^2 + \beta_2\gamma_1 k(n-(k-1)) + \beta_1\gamma_2(n-k)(k+1)$$
$$-\alpha_1\alpha_2(n-2k)^2 - \beta_1\gamma_2 k(n-(k-1)) - \beta_2\gamma_1(n-k)(k+1))P_k$$
$$+(-\alpha_2\beta_1(n-2k)k - \alpha_1\beta_2 k(n-2(k-1))$$
$$+\alpha_1\beta_2 k(n-2k) + \alpha_2\beta_1 k(n-2(k-1)))P_{k-1}$$
$$+(-\alpha_2\gamma_1(n-2k)(n-k) - \alpha_1\gamma_2(n-k)(n-2(k+1)))$$
$$+\alpha_1\gamma_2(n-k)(n-2k) + \alpha_1\gamma_1(n-k)(n-2(k+1))))P_{k+1}$$
$$+(\gamma_1\gamma_2(n-k)(n-(k+1)) - \gamma_1\gamma_2(n-k)(n-(k+1))P_{k+2}$$
$$+(\beta_1\beta_2 k(k-1) - \beta_1\beta_2 k(k-1))P_{k-2}$$
$$= (\beta_1\gamma_2((n-k)(k+1) - k(n-(k-1)))$$
$$+\beta_2\gamma_1(k(n-(k-1)) - (n-k)(k+1)))P_k$$
$$+(\alpha_1\beta_2(-k(n-2(k-1)) + k(n-2k)) + \alpha_2\beta_1(-k(n-2k) + k(n-2(k-1)))P_{k-1}$$
$$+(\alpha_2\gamma_1(-(n-2k)(n-k) + (n-k)(n-2(k+1)))$$
$$+\alpha_1\gamma_2(-(n-k)(n-2(k+1)) + (n-2k)(n-k)))P_{k+1}$$
$$= (\beta_1\gamma_2 - \beta_2\gamma_1)((n-k)(k+1) - k(n-(k-1)))P_k$$
$$+(\alpha_1\beta_2 - \alpha_2\beta_2)(k(n-2k) - k(n-2(k-1)))P_{k-1}$$
$$+(\alpha_1\gamma_2 - \alpha_2\gamma_1)((n-2k)(n-k) - (n-k)(n-2(k+1)))P_{k+1}$$
$$= (\beta_1\gamma_2 - \beta_2\gamma_1)(n-2k)P_k - (2\alpha_1\beta_2 - 2\alpha_2\beta_1)kP_{k-1}$$
$$-(2\gamma_1\alpha_2 - 2\alpha_1\gamma_2)(n-k)P_{k+1}.$$

as desired. (In Subsection 4.4.2, we show that this representation is even irreducible.)

Now, to determine the weights, let

$$H = \begin{pmatrix} h & 0 \\ 0 & -h \end{pmatrix} \in \mathfrak{h}.$$

Then,

$$\pi_n(H)P_k = h(n-2k)P_k$$

for any P_k. Thus the eigenvectors

$$\{v_0 = P_0, v_1 = P_1, \ldots, v_n = P_n\}$$

are the weight vectors, and the eigenvalues

$$\lambda_0 = n, \lambda_1 = n-2, \ldots, \lambda_n = -n$$

correspond to the weights.

The highest weight is $\lambda_n = ne_1$ which corresponds to the weight vector

$$\text{span}\{v_n = y^n\}.$$

4.2.1 Roots and Weights for Representations

As mentioned in the previous section, the roots and weights are interrelated. In the case of the adjoint representation, the roots and the weights are equivalent.

Example 4.17. The nonzero weights for the adjoint representation of a Lie algebra $\mathfrak{g}$ are the roots of $\mathfrak{g}$. In general for X an element of the Cartan subalgebra and Y_α an element of the root space for a root α,

$$(\mathbf{ad}(X))(Y_\alpha) = [X, Y_\alpha] = \alpha(X)Y_\alpha.$$

Consequently, Y_α is in the weight space for the weight α, which is essentially the root space decomposition.

Lemma 4.18. *Let $\mathfrak{g}$ be a Lie algebra, with $X_\alpha \in \mathfrak{g}$ in the root space corresponding to a root α and (π, V) a representation of $\mathfrak{g}$ with $v \in V$ having weight λ. Then $\pi(X_\alpha)v$ is a weight vector with weight $\alpha + \lambda$.*

Proof. For $H \in \mathfrak{h}$, one has

$$\begin{aligned}
\pi(H)(\pi(X_\alpha)v) &= \pi(H)\pi(X_\alpha)v \\
&= (\pi([H, X_\alpha]) + \pi(X_\alpha)\pi(H))v \\
&= \pi([H, X_\alpha])v + \pi(X_\alpha)\pi(H)v \\
&= \pi(\alpha(H)X_\alpha)v + \pi(X_\alpha)\pi(H)v \\
&= \alpha(H)\pi(X_\alpha)v + \pi(X_\alpha)\lambda(H)v \\
&= (\lambda + \alpha)(H)\pi(X_\alpha)v,
\end{aligned}$$

so $\pi(X_\alpha)v$ is a weight vector with weight $\alpha + \lambda$. (If $\alpha + \lambda$ is not a weight, then $\pi(X_\alpha)v = 0$.) □

4.3 Schur's Lemma

We introduce the idea of an intertwining operator.

Definition 4.19. [6] Let (π_1, V_1) and (π_2, V_2) be representations of $\mathfrak{g}$. We call

$$L : V_1 \to V_2$$

an intertwining map or intertwining operator if

$$L(\pi_1(X)v_1) = \pi_2(X)L(v_1)$$

for all $X \in \mathfrak{g}$ and all $v_1 \in V_1$.

Note that this definition, along with Definition 4.11, means that a bijective intertwining map between two representations gives an equivalence of the two representations.

In the following Lemma, assume that the field $F = \mathbb{C}$.

Lemma 4.20. *(Schur's Lemma)*

1. *Let (π_1, V_1) and (π_2, V_2) be irreducible representations of $\mathfrak{g}$ and let $L : V_1 \to V_2$ be an intertwining map. Then $L = 0$ or L is invertible.*

2. *Let (π, V) be an irreducible representation of $\mathfrak{g}$ and $L : V \to V$ be an intertwining map. Then L is a scalar multiple of the identity (possibly 0).*

3. *Let (π_1, V_1) and (π_2, V_2) be irreducible representations of $\mathfrak{g}$ and let $L_1, L_2 : V_1 \to V_2$ be nonzero intertwining maps. Then L_2 is a multiple of L_1.*

Proof. 1. As L is intertwining,

$$L[\pi_1(X)v_1] = \pi_2(X)[Lv_1] \text{ for all } X \in \mathfrak{g} \text{ and } v_1 \in V_1.$$

Furthermore,

$$ker(L) = \{v_1 \in V_1 : L(v_1) = 0\},$$

is a subspace of V_1 and

$$image(L) = \{v_2 \in V_2 : v_2 = L(v_1) \text{ for some } v_1 \in V_1\}$$

is a subspace of V_2. We wish to show that $ker(L)$ and $image(L)$ are $\mathfrak{g}-$invariant.

First let $X \in \mathfrak{g}$ and $v_1 \in ker(L)$. Then

$$\begin{aligned} L(\pi_1(X)v_1) &= \pi_2(X)L(v_1) \\ &= \pi_2(X)0 \\ &= 0 \in ker(L), \end{aligned}$$

as needed. Next, for $X \in \mathfrak{g}$ and $L(v_1) = v_2 \in image(L)$, we have

$$\begin{aligned} \pi_2(X)(v_2) &= \pi_2(X)(L(v_1)) \\ &= L(\pi_1(X)v_1) \in image(L), \end{aligned}$$

as needed. Since π_1 and π_2 are irreducible representations, they have no proper, invariant subspaces. Since $ker(L)$ is an invariant subspace of V_1, either $ker(L) = \{0\}$ or V_1. Since $image(L)$ is an invariant subspace of V_2, we must have that $image(L) = \{0\}$ or V_2. So either $L = 0$ or L is invertible.

2. Suppose that (π, V) is an irreducible representation of $\mathfrak{g}$ and that $L : V \to V$ is an intertwining operator. We have then that $L(\pi(X)v) = \pi(X)L(v)$ for all $X \in \mathfrak{g}$ and all $v \in V$. As the field $F = \mathbb{C}$, the operator L has an eigenvector and eigenvalue. Now suppose that $u \in V$ is an eigenvector for L with eigenvalue λ. Then we have that $L(u) = \lambda u$. Thus $(L-\lambda I)(u) = 0$, so $L-\lambda I$ is non-invertible. Additionally, $L - \lambda I$ is an intertwining operator, since

$$\begin{aligned} (L - \lambda I)(\pi(X)v) &= L(\pi(X)v) - \lambda I(\pi(X)v) \\ &= \pi(X)L(v) - \pi(X)\lambda I(v) \\ &= \pi(X)(L(v) - \lambda I(v)) \end{aligned}$$

$$= \pi(X)(L - \lambda I)(v),$$

as desired. As $L - \lambda I$ is a non-invertible intertwining operator, by Part 1 of the lemma, $L - \lambda I = 0$. So $L = \lambda I$ is a scalar operator.

3. Since L_1 is non-zero, by (1), there must be some inverse map

$$L_1^{-1} : V_2 \to V_1.$$

Consider then the composition:

$$L = L_1^{-1} \circ L_2 : V_1 \to V_1.$$

By (2), L is a scalar multiple of the identity map. So as L is bijective, for any $v_1 \in V_1$, we have $L(v_1) = \lambda v_1$ for some nonzero $\lambda \in \mathbb{C}$. Equivalently, $L_1^{-1}(L_2(v_1)) = \lambda v_1$. Therefore, $L_2 = \lambda L_1$.

□

4.4 Concrete Example of the Highest Weight

To motivate the main result of this thesis, the Theorem of the Highest Weight, we demonstrate the theorem with $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Note that throughout this section, we use $\mathfrak{g}$ to refer to $\mathfrak{sl}(2, \mathbb{C})$.

For elements in $\mathfrak{h}^*$ we define dominant and integral elements in the following statements. Refer to Section 3.8 for the definition of the inner product on $\mathfrak{h}^*$.

Definition 4.21. $\lambda \in \mathfrak{h}^*$ is integral if

$$\frac{2\langle \lambda, \alpha \rangle}{|\alpha|^2} \in \mathbb{Z}$$

for any root α.

Definition 4.22. [8] Dominant elements of $\lambda \in \mathfrak{h}^*$ are those such that for any simple root α, $\langle \lambda, \alpha \rangle$ is non-negative.

To begin with, we show that for λ a dominant and integral element of the dual space $\mathfrak{h}^*$, there is an irreducible finite-dimensional representation with λ as a highest weight. In order to do so, it is necessary to determine the dominant integral elements of $\mathfrak{h}^*$.

4.4.1 The Dominant and Integral Elements of the Dual Space

Note that the only positive root for $\mathfrak{g}$ is $\alpha = 2e_1$. Additionally, elements of $\mathfrak{h}^*$ are vectors of the form $\lambda = ae_1$, with $a \in \mathbb{C}$. We proceed to characterize the integral elements of $\mathfrak{h}^*$. Performing the calculation as follows,

$$\begin{aligned} \frac{2\langle \lambda, \alpha \rangle}{|\alpha|^2} &= \frac{2\langle ae_1, 2e_1 \rangle}{|2e_1|^2} \\ &= \frac{2(2a)}{4} \\ &= a, \end{aligned}$$

it is clear that elements in $\mathfrak{h}^*$ are integral when $a \in \mathbb{Z}$. Furthermore, the characterization for a dominant element is as follows:

$$\langle \lambda, \alpha \rangle = \langle ae_1, 2e_1 \rangle = 2a \geq 0,$$

if and only if a is non-negative. Thus, the dominant, integral elements of $\mathfrak{h}^*$ are exactly those of the for $\lambda = ae_1$, where a is a non-negative integer.

4.4.2 Dominant Integral Elements are Highest Weights

Now for $\alpha = ne_1$ dominant and integral, we proceed to show that there is an irreducible finite-dimensional representation for $\mathfrak{g}$ with α as the highest weight. We claim that the representation π_n from Example 4.16 is an irreducible (clearly finite-dimensional) representation with highest weight ne_1.

We first calculate the action of $\pi_n(h)$ on the basis for V_n,

$$\{P_0, \ldots, P_n\}.$$

Let $P_k \in V$, a nonzero vector. The action of $\pi_n(h)$ on P_k is given by

$$\begin{aligned}\pi_n(h)P_k &= \pi_n\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)P_k\\ &= (n-2k)P_k.\end{aligned}$$

The action of $\pi_n(v)$ on P_k is given by

$$\begin{aligned}\pi_n(v)P_k &= \pi_n\left(\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right)P_k\\ &= (k-n)P_{k+1}.\end{aligned}$$

Additionally, the action of $\pi_n(u)$ on P_k is given by

$$\begin{aligned}\pi_n(u)P_k &= \pi_n\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)P_k\\ &= -kP_{k-1}.\end{aligned}$$

To show that π_n is irreducible, suppose that there is an invariant, non-trivial subspace,

V_0. Take a nonzero vector, $Q \in V_0$. Write

$$Q = c_0 P_0 + \dots c_n P_n$$

and suppose k is the largest index for which $c_k \neq 0$. Then

$$Q = c_0 P_0 + \dots c_k P_k.$$

Thus

$$\pi_n(u)^k(Q) = c_k k! P_0,$$

so $P_0 \in V_0$. It now follows from repeated applications of $\pi_n(v)$ that V_0 is the entire space. So this n-dimensional representation for $\mathfrak{sl}(2, \mathbb{C})$ is irreducible.

The weight-spaces for π_n are $\mathbb{C}P_0$, $\mathbb{C}P_1$, $\dots$, $\mathbb{C}P_n$ with weights the eigenvalues $\lambda_k = n - 2k$. So a highest weight vector is given by $P_n = y^n$ and the highest weight is ne_1.

4.4.3 All Irreducible Finite-Dimensional Representations Have a Dominant and Integral Highest Weight

It remains to show that an irreducible finite-dimensional representation (π, V) of $\mathfrak{g}$ has a unique highest weight which is dominant and integral, and that any two irreducible representations with the same highest weight are equivalent.

Let (π, V) be a finite-dimensional, irreducible representation of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Recall the basis, h, u, v for $\mathfrak{sl}(2, \mathbb{C})$ given in Equation 3.1, satisfies

$$[h, u] = 2u,\ [h, v] = -2v, \text{ and } [u, v] = h.$$

As π is a Lie algebra representation, the Lie bracket must be respected. In particular, the following must hold:

$$\pi(h) = \pi\left([u, v]\right) = \pi(uv - vu) = \pi(u)\pi(v) - \pi(v)\pi(u),$$

$$\pi(2u) = \pi([h, u]) = \pi(h)\pi(u) - \pi(u)\pi(h),$$

and

$$\pi(-2v) = \pi([h, v]) = \pi(h)\pi(v) - \pi(v)\pi(h).$$

As V is a complex vector space, the operator $\pi(h) : V \to V$ has at least one (nonzero) eigenvector. Suppose $\vec{w} \in V$ is a (nonzero) eigenvector for $\pi(h)$ with eigenvalue μ say,

$$\pi(h)\vec{w} = \mu\vec{w}.$$

We have also

$$\begin{aligned} \pi(h)\left(\pi(u)\vec{w}\right) &= (\pi([h, u]) + \pi(u)\pi(h))\vec{w} && \text{by Lie bracket identity} \\ &= (\pi(2u) + \pi(u)\pi(h))\vec{w} \\ &= 2\pi(u)\vec{w} + \pi(u)\pi(h)\vec{w} \\ &= 2\pi(u)\vec{w} + \pi(u)\mu\vec{w} && \text{since } \pi(h)\vec{w} \text{ is an eigenvector} \\ &= (\mu + 2)\pi(u)\vec{w}, \end{aligned}$$

and

$$\begin{aligned} \pi(h)\pi(v)\vec{w} &= (\pi([h, v] + \pi(v)\pi(h))\vec{w} \\ &= (\pi(-2v) + \pi(v)\pi(h))\vec{w} \end{aligned}$$

$$= -2\pi(v)\vec{w} + \pi(v)\pi(h)\vec{w}$$

$$= (\mu - 2)\pi(v)\vec{w}.$$

So $\pi(u)\vec{w}$ and $\pi(v)\vec{w}$ are also eigenvectors for $\pi(h)$ (if nonzero), with respective eigenvalues $\mu + 2$ and $\mu - 2$.

Additionally then $\pi(u)^2\vec{w}$ is an eigenvector (if nonzero) for $\pi(h)$ with eigenvalue $\mu + 4$:

$$\begin{aligned}
\pi(h)\left(\pi(u)^2\right)\vec{w} &= \pi(h)\pi(u)\pi(u)\vec{w} \\
&= (\pi[h,u] + \pi(u)\pi(h))\pi(u)\vec{w} \\
&= \pi[h,u]\pi(u)\vec{w} + \pi(u)\pi(h)\pi(u)\vec{w} \\
&= \pi(2u)\pi(u)\vec{w} + \pi(u)(\mu+2)\pi(u)\vec{w} \\
&= 2\pi(u)^2\vec{w} + (\mu+2)\pi(u)^2\vec{w} \\
&= (\mu+4)\pi(u)^2\vec{w}.
\end{aligned}$$

And also,

$$\begin{aligned}
\pi(h)\pi(v)^2\vec{w} &= \pi(h)\pi(v)\pi(v)\vec{w} \\
&= (\pi([h,v]) + \pi(v)\pi(h))\pi(v)\vec{w} \\
&= (\pi(-2v) + \pi(v)\pi(h))\pi(v)\vec{w} \\
&= -2\pi(v)\pi(v)\vec{v} + \pi(v)\pi(h)\pi(v)\vec{w} \\
&= -2\pi(v)^2\vec{v} + \pi(v)(\mu-2)\pi(v)\vec{w} \\
&= (\mu-4)\pi(v)^2\vec{w},
\end{aligned}$$

so $\pi(v)^2\vec{w}$ is an eigenvector for $\pi(h)$ with eigenvalue $\mu - 4$.

In general,

$$\pi(h)(\pi(u))^k\vec{w} = (\mu + 2k)\pi(u)^k\vec{w}$$

and

$$\pi(h)(\pi(v))^k\vec{w} = (\mu - 2k)\pi(v)^k\vec{w}$$

for $k \geq 0$. In particular, if $\pi(u)^k\vec{w}$ is nonzero, then

$$\{\vec{w}, \pi(u)\vec{w}, \ldots, \pi(u)^k\vec{w}\}$$

are eigenvectors for $\pi(h)$ with distinct eigenvalues, hence linearly independent. As V is a finite-dimensional vector space, $\pi(u)^k\vec{w} = 0$ for some k sufficiently large. Let $k_0 \geq 0$ be the largest number for which $\pi(u)^{k_0}\vec{w} \neq 0$, and set

$$\vec{v_0} := \pi(u)^{k_0}\vec{w}, \;\; \lambda := \mu - 2k_0.$$

So $\vec{v_0}$ is a λ-eigenvector for $\pi(h)$ and $\pi(u)\vec{v_0} = 0$. (This will be a highest weight vector for (π, V).) Then for $k \geq 0$, define $\vec{v_k} \in V$ as

$$\vec{v_k} = \pi(v)^k\vec{v_0}.$$

From the calculations above, we have

$$\pi(h)\vec{v_k} = (\lambda - 2k)\vec{v_k}.$$

Hence if $\vec{v_k} \neq 0$, then

$$\{\vec{v_0}, \ldots, \vec{v_k}\}$$

is linearly independent as they are $\pi(h)$ eigenvectors for distinct eigenvalues.

Let $n \geq 0$ be the largest value for which $\vec{v_n} = \pi(v)^n \vec{v_0} \neq 0$. We claim that

$$\mathcal{B} = \{\vec{v_0}, \ldots, \vec{v_n}\}$$

is a basis for V. In fact, we know

$$\pi(h)\vec{v_k} = (\lambda - 2k)\vec{v_k},\ \pi(v)\vec{v_k} = \vec{v_{k+1}}.$$

Additionally, we claim

$$\pi(u)\vec{v}_k = k(\lambda - k + 1)\vec{v}_{k-1},$$

(where $\vec{v_{-1}} = 0 = \vec{v_{n+1}}$) which we prove now by induction. As the base case, for $k = 0$, we have

$$\pi(u)\vec{v}_0 = 0(\lambda - 0 + 1)\vec{v}_{-1} = 0,$$

as stated above. Now, for the induction hypothesis, assume that

$$\pi(u)\vec{v}_k = k(\lambda - k + 1)\vec{v}_{k-1}$$

for some $k \geq 0$. Then

$$\begin{aligned}
\pi(u)v_{k+1} &= \pi(u)\pi(v)\vec{v_k} \\
&= \pi([u,v])\vec{v_k} + \pi(v)\pi(u)\vec{v_k} \\
&= \pi(h)\vec{v_k} + \pi(v)\pi(u)\vec{v_k} \\
&= (\lambda - 2k)\vec{v_k} + \pi(v)(k(\lambda - k + 1)\vec{v}_{k-1}) \quad \text{(by the induction hypothesis)} \\
&= (\lambda - 2k)\vec{v_k} + k(\lambda - k + 1)\vec{v_k}
\end{aligned}$$

$$= (\lambda k - k^2 + k + \lambda - 2k)\vec{v_k}$$
$$= ((k+1)\lambda - k^2 - k)\vec{v_k}$$
$$= (k+1)(\lambda - k)\vec{v_k}$$
$$= (k+1)(\lambda - (k+1) + 1)\vec{v_k},$$

as desired.

So the span of $\mathcal{B}$ is invariant under $\pi(h)$, $\pi(u)$, and $\pi(v)$. Thus it is invariant under $\pi(\mathfrak{g})$. As (V, π) is irreducible, it follows that $\text{span}(\mathcal{B}) = V$. As $\mathcal{B}$ is linearly independent, $\mathcal{B}$ is a basis for V.

As $\vec{v_0}$ is a weight vector, with $\pi(u)\vec{v_0} = 0$, we have $\vec{v_0}$ is a highest weight vector. Moreover, the weight λe_1 for $\vec{v_0}$ has $\lambda = n$. Indeed,

$$\begin{aligned} \text{trace}(\pi(h)) &= \text{trace}([\pi(u)\pi(v) - \pi(v)\pi(u)]) \\ &= 0, \end{aligned}$$

by the fact that $h = [u, v]$ and the fact that trace is commutative. But by definition of trace, we also have

$$\begin{aligned} \text{trace}(\pi(h)) &= \lambda + (\lambda - 2) + \cdots + (\lambda - 2n) \\ &= (n+1)\lambda - n(n+1). \end{aligned}$$

Thus we have

$$0 = (n+1)\lambda - n(n+1)$$

so it is clear that $n = \lambda$. As $\lambda = n$, λe_1 is both dominant and integral. Therefore, the highest weights are of the form $\lambda_n = ne_1$ for $n \in \mathbb{N} \cup \{0\}$.

Therefore, (π, V) has a highest weight which is both dominant and integral, completing this part of the proof.

Remark 4.23. From Section 4.4.3, we have that each irreducible, finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$ has a highest weight, and the highest weight is dominant and integral. As the weights are elements in $\mathbb{Z}e_1$, it follows that the highest weight for an irreducible, finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$ is unique. The weight spaces are one-dimensional.

4.4.4 Two Irreducible Representations of $\mathfrak{sl}(2, \mathbb{C})$ with Same Highest Weight Are Equivalent

To complete the argument, we show that if two irreducible representations for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ have the same highest weight, then the representations are equivalent.

Suppose that (π_1, V_1) and (π_2, V_2) are irreducible representations for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ with highest weight λe_1. Additionally, let the basis for V_1 be $\mathcal{B}_1 = \{\vec{v}_0, \ldots, \vec{v}_n\}$ and the basis for V_2 be $\mathcal{B}_2 = \{\vec{u}_0, \ldots, \vec{u}_m\}$. Choose these bases to consist of the weight vectors as constructed in Section 4.4.3. By Subsection 4.4.3, the highest weight for each of these representations is given by $\dim(V_1) - 1 = \dim(V_2) - 1 = \lambda$. Therefore, we must have that $\lambda = n = m$, so in particular, the bases for each of the representations have the same cardinality.

Define the invertible linear map

$$\varphi : V_1 \to V_2$$

such that for $0 \leq i \leq n$,

$$\varphi(\vec{v}_i) = \vec{u}_i.$$

As,

$$\begin{aligned}\varphi(\pi_1(h)\vec{v}_i) &= \varphi((\lambda - 2i)\vec{v}_i)\\ &= (\lambda - 2i)\varphi(\vec{v}_i)\\ &= (\lambda - 2i)\vec{u}_i\\ &= \pi_2(h)\vec{u}_i\\ &= \pi_2(h)\varphi(\vec{v}_i),\end{aligned}$$

and

$$\begin{aligned}\varphi(\pi_1(u)\vec{v}_i) &= \varphi(i(\lambda - i + 1)\vec{v}_{i-1})\\ &= i(\lambda - i + 1)\varphi(\vec{v}_{i-1})\\ &= i(\lambda - i + 1)\vec{u}_{i-1}\\ &= \pi_2(u)\vec{u}_i\\ &= \pi_2(u)\varphi(\vec{v}_i),\end{aligned}$$

with a similar calculation for v, φ is an intertwining operator. So the representations are equivalent.

4.4.5 Summary of Highest Weight Results

We summarize these results in the following proposition, which we have proved by the constructional work in the preceding sections.

Proposition 4.24. *Let* $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. *The following statements hold:*

1. *Each dominant and integral element of $\mathfrak{h}^*$ is the highest weight for a distinct irreducible, finite-dimensional representation of $\mathfrak{g}$.*

2. *Every irreducible, finite-dimensional representation of $\mathfrak{g}$ has a unique highest weight.*

3. *If two representations have the same highest weight, then the representations are equivalent.*

4.5 Universal Enveloping Algebra

A brief introduction to the universal enveloping algebra is required in order to prove the Theorem of the Highest Weight.

Define the tensor algebra $T(\mathfrak{g})$ as an associative algebra comprised of the direct sum of vector spaces:

$$T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} (\otimes\mathfrak{g})^n = \mathbb{C} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus \ldots.$$

The multiplication operation within $T(\mathfrak{g})$ is given by tensor product, where $T_m \in (\otimes\mathfrak{g})^m$ and $T_n \in (\otimes\mathfrak{g})^n$ combine as:

$$T_m \otimes T_n \in (\otimes\mathfrak{g})^{m+n}.$$

Define $\mathfrak{I} \subset T(\mathfrak{g})$ as the two-sided ideal generated by

$$\langle XY - YX - [X, Y] \mid X, Y \in \mathfrak{g} \rangle.$$

The universal enveloping algebra is the quotient

$$\mathfrak{U}(\mathfrak{g}) = T(\mathfrak{g})/\mathfrak{I}.$$

Consequently, for $X, Y \in \mathfrak{U}(\mathfrak{g})$, one has

$$\begin{aligned}(X+\mathfrak{I})(Y+\mathfrak{I}) &= XY + X\mathfrak{I} + \mathfrak{I}Y + \mathfrak{I} \\ &= XY + \mathfrak{I} && \text{as } \mathfrak{I} \text{ is an ideal} \\ &= YX + [X,Y] + \mathfrak{I} && \text{as } XY - YX + [X,Y] \in \mathfrak{I}\end{aligned}$$

so $[X,Y] + \mathfrak{I} = XY - YX + \mathfrak{I}$ in $\mathfrak{U}(\mathfrak{g})$. It is conventional to drop the $+\mathfrak{I}$ and simply write $[X,Y] = XY - YX$ in $\mathfrak{U}(\mathfrak{g})$.

Following the notation established in [8], we denote the natural map between a Lie algebra $\mathfrak{g}$ and the associated universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ as

$$\iota : \mathfrak{g} \to \mathfrak{U}(\mathfrak{g}).$$

To prove the main result for this thesis, the Theorem of the Highest Weight, we need the following famous theorem which uses the universal enveloping algebra for $\mathfrak{g}$. This theorem builds on the notion of the universal enveloping algebra for a complex Lie algebra.

Theorem 4.25. *[8] (Poincaré-Birkhoff-Witt Theorem) There is an injective map,*

$$\iota : \mathfrak{g} \to \mathfrak{U}(\mathfrak{g}),$$

such that for $\mathfrak{g}$ *a finite-dimensional, complex Lie algebra with the (at least partially) ordered basis*

$$\mathcal{B} = \{X_1, \ldots, X_n\},$$

there is a corresponding basis for $\mathfrak{U}(\mathfrak{g})$ *given by*

$$\widetilde{\mathcal{B}} = \{(\iota X_1)^{j_1}, \ldots, (\iota X_n)^{j_n}\},$$

where the powers are non-negative.

The powerful theorem stated above is useful for showing how ordered basis elements from the Lie algebra map to basis elements within the universal enveloping algebra. A consequence of this theorem ([8] p. 222) is that for a Lie algebra $\mathfrak{g}$ which can be expressed as the direct vector space sum of two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$,

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

the associated universal enveloping algebra can be broken into an associated tensor product of vector spaces

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g}_1) \otimes \mathfrak{U}(\mathfrak{g}_2).$$

Therefore, the ordered basis element from the two subalgebras can appropriately correspond within the ordered basis for $\mathfrak{U}$.

This is particularly powerful when one considers $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$. The following example illustrates how ordered basis elements move past one another within the ordering of basis elements in the universal enveloping algebra.

Example 4.26. For $\mathfrak{g} = \mathfrak{s}l(2, \mathbb{C})$ with ordered basis given as in Equation 3.1, we have the following commutation relations:

$$[u, h] = -2u, \quad [v, h] = 2v, \quad \text{and} \quad [u, v] = h.$$

Also, note that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$ (as a vector-space direct sum), where h is a basis for $\mathfrak{h}$, u is a basis for $\mathfrak{n}^+$, and v is a basis for $\mathfrak{n}^-$. Consider the basis ordering for $\mathfrak{g}$ as $h > u > v$. According to the result of the Poincaré-Birkhoff-Witt Theorem, in $\mathfrak{U}(\mathfrak{g})$, there is an associated basis ordering.

In $\mathfrak{U}(\mathfrak{g})$, it is defined for any $x, y \in \mathfrak{g}$, that $[x, y] = xy - yx$, so as a simple example, we can show that the following relation holds:

$$uh = hu - 2u,$$

by applying the commutation relations from $\mathfrak{g}$. We have that

$$\begin{aligned} uh &= [u, h] + hu \\ &= -2u + hu \\ &= hu - 2u \end{aligned}$$

as desired. So it is possible to reorder the elements u and h, as necessary. Even in more complicated situations, this works. Consider the following case:

$$\begin{aligned} vuh &= v(hu - 2u) \\ &= vhu - 2vu \\ &= ([v, h] + hv)u - 2vu \\ &= 2vu + hvu - 2vu \end{aligned}$$

$$
\begin{aligned}
&= hvu \\
&= h(uv - [u,v]) \\
&= huv - h[u,v] \\
&= huv - h^2.
\end{aligned}
$$

Additionally, we note the adapted universal mapping property for the universal enveloping algebra.

Proposition 4.27. *[8] For the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$, canonical map*

$$\iota : \mathfrak{g} \to U(\mathfrak{g}),$$

any complex associative algebra with identity A, and a linear map

$$\pi : \mathfrak{g} \to A$$

with $\widetilde{\pi}(1) = 1$, there exists a unique algebra homomorphism

$$\widetilde{\pi} : \mathfrak{U}(\mathfrak{g}) \to A$$

which respects the identity and such that for all $X \in \mathfrak{g}$,

$$\widetilde{\pi}(\iota(X)) = \pi(X) \in A,$$

i.e.,

$$
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\iota} & \mathfrak{U}(\mathfrak{g}) \\
 & {\scriptstyle \pi} \searrow & \downarrow {\scriptstyle \widetilde{\pi}} \\
 & & A
\end{array}
$$

4.6 More Properties of Representations

Definition 4.28. [6] A finite-dimensional representation of a Lie algebra is completely reducible if it is isomorphic to the direct sum of a finite number of irreducible representations.

We also have the following powerful theorem by Weyl.

Theorem 4.29. *([2] p. 52) "Every finite-dimensional linear representation of a semi-simple Lie algebra is completely reducible."*

Lie algebras which are not semi-simple do not behave this way. For an illustration of the necessity of inclusion of semi-simple in the hypothesis for the preceding theorem, consider the subspace $\mathfrak{b}$ of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$.

Example 4.30. Consider $\mathfrak{b}$, the Borel subalgebra for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$:

$$\mathfrak{b} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}.$$

We claim that $\mathfrak{b}$ is not semi-simple. Let π be the standard representation of $\mathfrak{g}$ on $\mathbb{C}^2$, restricted to $\mathfrak{b}$, where $\pi(X)$ for $X \in \mathfrak{b}$ acts by matrix multiplication on vectors in $\mathbb{C}^2$. Note then that

$$\begin{aligned} \pi\left(\begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \\ &= \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\in \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}. \end{aligned}$$

So

$$\operatorname{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$$

is an invariant subspace for $\mathfrak{b}$. This implies that $\pi|_{\mathfrak{b}}$ is reducible. But there is no invariant complementary subspace, so we do not have a direct sum. Thus the representation is not completely reducible.

Indeed, any complementary subspace to $\mathbb{C}\begin{bmatrix}1\\0\end{bmatrix}$ has the form $\mathbb{C}\begin{bmatrix}z_0\\w_0\end{bmatrix}$ for some $\begin{bmatrix}z_0\\w_0\end{bmatrix} \in \mathbb{C}^2$ with $w_0 \neq 0$. As $\pi(u)\begin{bmatrix}z_0\\w_0\end{bmatrix} = \begin{bmatrix}z_0 + w_0\\w_0\end{bmatrix}$ does not belong to $\mathbb{C}\begin{bmatrix}z_0\\w_0\end{bmatrix}$, we that $\mathbb{C}\begin{bmatrix}z_0\\w_0\end{bmatrix}$ fails to be $\pi(\mathfrak{b})$-invariant.

Chapter 5: Theorem of the Highest Weight

5.1 Theorem of the Highest Weight

We have motivated this important theorem with an example from $\mathfrak{sl}(2,\mathbb{C})$. Now, we provide the Theorem of the Highest Weight. We rely heavily on Knapp's work [8].

Theorem 5.1. *[8] (Theorem of the Highest Weight) There is a one-to-one correspondence between the irreducible, finite-dimensional representations (π, V) of a semi-simple Lie algebra $\mathfrak{g}$ (up to equivalence) with the dominant integral linear functionals λ on $\mathfrak{h}$, where each λ is the highest weight of π_λ. Additionally, each λ satisfies the following:*

1. *λ only depends on the simple system Π from the real form $\mathfrak{h}_0$ of $\mathfrak{h}$ and not the ordering,*

2. *$dim(V_\lambda) = 1$ for the highest weight,*

3. *for each positive root α, the corresponding root vector E_α annihilates only the elements from V_λ,*

4. *all weights can be written as $\lambda - \sum_{i=1}^{\ell} n_i\alpha_i$ where $n_i \in \mathbb{Z}^+$ and $\alpha_i \in \Pi$, and each weight space V_μ for π_λ has dim $V_{w\mu}$ =dimV_μ for all w in the Weyl group $(W(\Delta))$, and each weight μ has $|\mu| \leq |\lambda|$ with equality only if μ is in the orbit $W(\Delta)\lambda$.*

This high powered theorem is a generalization of our concrete example of the highest weights and representations for $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$.

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